L 25/03

Matrices of Idempotents

Let W be an $n \times n$ type-II matrix.

Define

$$Y_{i,j} := \frac{1}{n} W_{i/j} (W_{j/i})^T$$

Clearly $Y_{i,i} = \frac{1}{n} I$ and $(Y_{i,j})^T = Y_{j,i}$. If W is flat, then

$Y_{i,j} = \frac{1}{n} W_{i/j} W_{i/j}^*$ is Hermitian. You might also confirm

that $Y_{i,j}^2 = Y_{i,j}$.

Let $\mathcal{Y}_W$ denote the $n \times n$ (block) matrix with ij-entry $Y_{i,j}$. We call it the matrix of idempotents of W. As $Y_{i,j}^T = Y_{j,i}$, we have $\mathcal{Y}^{(T)} = \mathcal{Y}$. Also

$$Y_{i,j}^{(-)} = n\, W_{j,i} (W_{i,j})^T = n^2\, Y_{j,i}$$

If $\mathcal{Y}^{(T)}$ denotes the partial transpose of $\mathcal{Y}$, then

$$\mathcal{Y}^{(+)} = \frac{1}{n} \mathcal{Y}^{(-)}$$

$$W W^{(-1)T} = nI$$

$$AB = \sum_i A e_i e_i^T B$$

$$\Rightarrow \sum_i W e_i e_i W^{(-1)T} = nI$$

Dual basis If $x_1, \ldots, x_m \in V$, vectors $y_1, \ldots, y_m$ in V form a dual basis if $y_i^T x_j = \delta_{ij}$. So $\{x_1, \ldots, x_m\}$ are linearly independent if & only if there is a dual basis for it. Thus the rows of $W^{(-1)}$ form a dual basis to the rows of W.

$$[y_1 \cdots y_m]^T [x_1 \cdots x_m] = I_m$$

Assume $\dim(V) = n$ and $y_1, \dots, y_n$ is a dual basis

to $x_1, \dots, x_n$. Set $Y = [y_1 \cdots y_n]$, $X = [x_1, \dots, x_n]$. $n = \dim(V)$

Then

$$I = Y^T X = X Y^T = \sum_{i=1}^{n} x_i y_i^T$$

Further

$$x_i y_i^T \cdot x_j y_j^T = \delta_{i,j}\, x_i y_i^T.$$

Thus $\{x_i y_i^T\}$ is a set of pairwise orthogonal

idempotents summing to I.

If $Mx_iy_i^T = x_iy_i^TM$ then $Mx_i = \lambda x_i$ & $y^TM = \mu y_i^T$

So

$$\left.\begin{aligned} tr(Mx_iy_i^T) &= \lambda\, tr(x_iy_i^T) = \lambda \\ tr(x_iy_i^TM) &= \mu\, tr(x_iy_i) = \mu \end{aligned}\right\} \lambda = \mu$$

Lemma If $y_i^Tx_i = 1$ and M commutes with $x_iy_i^T$, then x is a right eigenvector for M & y^T is a left eigenvector. □

Assume $Mx_i = \lambda_i x_i$. Then if M & $x_i y_i^T$ commute for all i,

$$M = MI = \sum_{i=1}^{n} M x_i y_i^T = \sum \lambda_i x_i y_i^T.$$

This gives an analog, for diagonalizable matrices, to spectral decomposition for Hermitian matrices

Theorem If $\mathcal{Y}$ is the matrix of idempotents of an $n \times n$ type-II matrix, each row & column of $\mathcal{Y}$ sums to I_n. If W is flat, then $\mathcal{Y}_W$ is flat and is a quantum permutation.

Let $S: \mathbb{C}^n \otimes \mathbb{C}^n$ be defined by $S(a \otimes b) = b \otimes a$

Then S is a permutation and $S^2 = I$

Lemma If W is type II, then $S \mathcal{Y}_W S = \mathcal{Y}_{W^T}$.

Proof We have

$$n\,(Y_{i,j})_{r,s} = \frac{W_{r,i}}{W_{r,j}} \frac{W_{s,j}}{W_{s,i}} = \frac{W_{r,i}}{W_{s,i}} \frac{W_{s,j}}{W_{r,j}} = \frac{W^T_{i,r}}{W^T_{i,s}} \frac{W^T_{j,s}}{W^T_{j,r}} = n(Y_{r,s})_{i,j}$$

and

$$n\,(Y_{i,j})_{r,s} = (e_i \otimes e_r)\, \mathcal{Y}_W\, (e_j \otimes e_s). \qquad n\,(Y_{r,s})_{i,j} = (e_r \otimes e_i)\, \mathcal{Y}_{W^T} (e_s \otimes e_j)$$

from which the result follows. $\square$

If M & N are matrices of the same order then $[M,N]$ denotes their Lie bracket, given by $[M,N] := MN - NM$. It is bilinear and skew symmetric. (We are only using the Lie bracket to simplify notation; we will not work with Lie algebras as such.)

$MN = NM \iff [M,N] = 0$

Theorem If $\mathcal{Y}$ is the matrix of idempotents of the type-II matrix W, then

$$\mathcal{N}_W = \{M : [I \otimes M, \mathcal{Y}] = 0\},$$

$$\mathcal{N}_{W^T} = \{N : [N \otimes I, \mathcal{Y}] = 0\}.$$

m m $[y_{i,j}]$

Proof We have $[I \otimes M, \mathcal{Y}] = 0$ if and only if $[M, Y_{i,j}] = 0$ for all i & j. Since M commutes with the rank-1 matrix uv^* if and only if u is a right eigenvector of M, it follows that $[M, Y_{i,j}] = 0$ for all i,j if and only if $M \in \mathcal{A}_W$.

For the second claim

$$S(N \otimes I)\mathcal{Y}_W S = (I \otimes N)\mathcal{Y}_{W^T}.$$

□

Corollary If W is type-II, then $\mathcal{N}_W$ is is a commutative coherent algebra (hence homogeneous).

If $\mathcal{Y} = \mathcal{Y}_{W^T}$ and W is flat, then $\mathcal{Y}$ is a quantum automorphism of each graph in $\mathcal{N}_W$. $\square$

We have the following:

Theorem If W is an $n\times n$ type-II matrix and $M\in\mathcal{N}_W$, then

$$\Theta_W(M)\in\mathcal{N}_{W^T}.$$

If $M,N\in\mathcal{N}_W$, then

$$\Theta_W(M\circ N)=\frac{1}{n}\Theta_W(M)\,\Theta_W(N). \qquad \square$$

$$\Theta_W:\mathcal{N}_W\to\mathcal{N}_{W^T} \qquad \Theta_{W^T}:\mathcal{N}_{W^T}\to\mathcal{N}_W$$

Examples Let $V = V_n$ be the Vandermonde matrix.

Then V is type-II and $V_{i/j}$ is a column of V.

(The columns of V are closed under Schur-inverse & Schur multiplication.) If P is the permutation matrix for the n-cycle, then $P \in \mathcal{N}_V$; in <u>fact</u> (exer) $\mathcal{N}_W$ is the algebra of polynomials in P, i.e., the algebra of circulant matrices.

Further, all entries of $\mathcal{Y}$ commute, so this automorphism is classical.

For a second example, suppose H is a $n \times n$ Hadamard matrix. We claim that if W_1 & W_2 are type-II, then

$$\mathcal{N}_{W_1 \otimes W_2} = \mathcal{N}_{W_1} \otimes \mathcal{N}_{W_2} \qquad \text{(exercise)}$$

and so $\dim(H^{\otimes n}) \geq 2^n$.

Lemma If W is a flat type-II matrix and the set of columns of W is closed under Schur multiplication, then all entries of $\mathcal{Y}_W$ commute.

Proof We have $Y_{i,j} = W_{i/j} W_{i/j}^*$ and, for fixed i, the vectors $W_{i/j}$ are orthogonal. $\square$

So if W is a Kronecker product of Vandermonde matrices, the entries of $\mathcal{Y}_W$ commute.

All known Hadamard matrices with $\dim(\mathcal{N}_H) > 2$ are products. We say a Nomura algebra is trivial if its dimension is two. We do have the following:

Lemma If H is an $n \times n$ Hadamard matrix and $\dim(\mathcal{N}_H) > 2$, then $8 \mid n$. $\square$

For all known Hadamard matrices, the entries of the matrix of idempotents commute.

We have a limited supply of type-II matrices that are not flat and have non-trivial Nomura algebra.

There is one family based on Hadamard graphs.

Recall that if H is $n\times n$ Hadamard, the Hadamard graph has adjacency matrix of the form $\begin{bmatrix} 0 & \hat{H} \\ \hat{H} & 0 \end{bmatrix}$. The form matrices

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} J-I & 0 \\ 0 & J-I \end{pmatrix}, \begin{pmatrix} 0 & \hat{H} \\ \hat{H} & 0 \end{pmatrix}, \begin{pmatrix} 0 & J-\hat{H} \\ J-\hat{H} & 0 \end{pmatrix}$$

are the canonical basis of a commutative coherent algebra, and Nomura proved that this algebra contains a type-II matrix W such that $W \in \mathcal{N}_W$.

(Hence $\dim(\mathcal{N}_W) \geq 3$).

There is one more example. The Higman-Sims graph is a strongly regular graph, first found by Mesner. Its parameters are $(100, 22; 0, 6)$. If A is the adjacency matrix of this graph, Jaeger showed that there are scalars α, β, γ such that

$$W = \alpha I + \beta A + \gamma (J - I)$$

is type-II and $W \in \mathcal{N}_W$.

The Higman-Sims graph is interesting because it is triangle-free and its automorphism group contains the Higman-Sims group, a sporadic simple group as a subgroup of index two.

To get more examples in this way, we need more triangle-free strongly regular graphs (not bipartite). But the Higman-Sims graph is the largest known.

We have many examples of type-II matrices arising from combinatorial structures but generally $\dim(\mathcal{N}_W) = 2$.

C^*-algebras

The official definition of quantum automorphisms involves C^*-algebras. We discuss this briefly. The example is the algebra of bounded linear maps on a Hilbert space, e.g., $\mathrm{Mat}_{n\times n}(\mathbb{C})$. The key parts of a C^*-algebra are a Banach algebra and an involution (think conjugate transpose).

An involution on a algebra $\mathcal{A}$ (over $\mathbb{C}$) is a map

$\mathcal{A} \to \mathcal{A}$ such that

(a) $(a^*)^* = a$

(b) $(ab)^* = b^* a^*$

(c) $(a + \gamma b)^* = a^* + \bar{\gamma} b^*$

A Banach algebra is a normed algebra with $\|ab\| \leq \|a\| \, \|b\|$.

A C*-algebra is a Banach algebra with an involution *
such that $\|a^*a\| = \|a\|^2$ for all a. A compact quantum group is a C*-algebra $\mathcal{C}$ with unit and a homomorphism
$\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ such that

(a) $(\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta$

(b) $\Delta(\mathcal{C})(1 \otimes \mathcal{C}) = \Delta(\mathcal{C})(\mathcal{C} \otimes I)$

coproduct

A **projection** in a C^*-algebra is an element a such that $a^2 = a = a^*$. A **quantum permutation** is an $n \times n$ matrix of projections from a C^* algebra such that each row & column sums to the identity.