L 18/03

Declare a 01-matrix in $\mathcal{C}$ to be minimal if it cannot be written as the sum of two non-zero matrices in $\mathcal{C}$. Then the minimal 01-matrices in $\mathcal{C}$ span $\mathcal{C}$ and are linearly independent. $\square$

If $B_1, \ldots, B_m$ is a basis for $\mathcal{C}$, we arrange it so the elements are ordered lexicographically. This justifies the use of 'unique' in the statement of the theorem.

· If $B_1, \ldots, B_m$ is the canonical basis for $\mathcal{B}$, then I is a sum of basis elements (and these are necessarily diagonal).

If I is actually a basis element, then $\mathcal{B}$ is said to be homogeneous. Two claims:

Lemma A commutative coherent algebra is homogeneous □

Lemma If the coherent algebra $\mathcal{B}$ is homogeneous, all matrices in $\mathcal{B}$ have constant row sum. □

If G is a permutation group, then $\mathrm{Comm}(G)$ is homogeneous if & only if G is transitive. If G is transitive, the orbitals of G are the basis for $\mathrm{Comm}(G)$. (Note that even if G is transitive, $\mathrm{Comm}(G)$ need not be commutative.)

A commutative coherent algebra is usually referred to as an association scheme.

The set of elements of the canonical basis of a coherent algebra is known as a **coherent configuration**.

This is a set $\mathcal{B} = \{B_1, \ldots, B_m\}$ of 01-matrices such that

(a) If $i \neq j$, then $B_i \circ B_j = 0$

(b) $B_i^T \in \mathcal{B}$ $\forall i$

(c) $\sum_i B_i = J$

(d) I is a sum of elements in $\mathcal{B}$.

integers

(e) there are scalars $\beta_{i,j}(k)$ such that

$$B_i B_j = \sum_k \beta_{i,j}(k) B_k .$$

Isomorphisms of coherent algebras

An algebra is a ring, and so homomorphisms of algebras are ring homomorphisms. However coherent algebras have extra structure, so more detail is needed.

An isomorphism β from a coherent algebra $\mathcal{C}$ to a coherent algebra $\mathcal{D}$ is a linear map such that:

(a) β is a ring homomorphism

(b) $\beta(A \circ B) = \beta(A) \circ \beta(B)$.

$\mathcal{M}$ algebra of matrices

What are the linear maps $\mathcal{M} \to \mathcal{M}$.

Typical: $M \to A^{-1}MA \quad M \in \mathcal{M}$ similarity

$M \to AMB$ "

$M \to \sum_r A_r M B_r$ "

Like to say: invertible line map $\leftrightarrow$ similarity

Noether-Skolem

Lemma If $\beta : \mathcal{C} \to \mathcal{D}$ is a homomorphism of coherent algebras, then $\beta(J) = J$.

Proof We have $\beta(J) = \beta(J \circ J) = \beta(J) \circ \beta(J)$ and so $\beta(J)$ is a 01-matrix. Further $n\beta(J) = \beta(nJ) = \beta(J^2) = \beta(J)^2$ and so each row of $\beta(J)$ sums to n. Hence $\beta(J) = J$. $\square$

There are two special kinds of isomorphisms of coherent algebras:

(1) Similarity: there is an invertible matrix S such $\beta(M) = S^{-1}MS$, $\forall M$.

(2) Permutation equivalence: as in (1), but with S a permutation matrix.

Whatever version of isomorphism we use, an isomorphism $\beta: \mathcal{C} \to \mathcal{D}$ maps the canonical basis of $\mathcal{C}$ to that of $\mathcal{D}$.

If X & Y are strongly regular graphs with the same parameters their coherent algebras are similar.

$I \quad A \quad J-I-A$

Quantum permutations

A **quantum automorphism** of X is a quantum permutation P such that:

magic unitary

$$P(A(X)\otimes I_d) = (A(X)\otimes I_d)P$$

If Y is a second graph and

$$P(A(X)\otimes I_d) = (A(Y)\otimes I_d)P$$

then X & Y are **quantum isomorphic**.

Lemma If P is a quantum permutation then it is unitary.

Also $P(J \otimes I_d) = J \otimes I_d$. $\square$

Lemma If P and Q are quantum permutations, so are $P \star Q$ & $P \otimes Q$.

If P & Q commute with $A(X)$, so do $P \star Q$ and $P \otimes Q$. $\square$

We define the quantum automorphism group of X to be the set of quantum permutations that commute with $A(X)$.

Theorem If $\mathcal{P}$ is a quantum isomorphism of index d from X to Y and D is a $d \times d$ density matrix, then $\langle \mathcal{P}, D \rangle$ is a fractional isomorphism from X to Y.

If $AR = RB$, then $ARR^T = RBR^T$; as RBR^T is symmetric, A & RR^T commute.

Corollary If X is controllable, its quantum automorphism group is trivial.

Theorem The commutant of a set of quantum permutations of order $n \times n$ is Schur-closed.

Proof It suffices to prove this when the set is a singleton.

The ij-block of $(M \otimes I)P$ is

scalar projection

$$\sum_r M_{ir} P_{rj}$$

and by hypothesis, this is equal to the ij-block of $P(M \otimes I)$:

$$\sum_r P_{ir} M_{rj}$$

Now

$$\sum_r M_{ir} P_{rj} \sum_s N_{is} P_{sj} = \sum_r (M_{ir} N_{ir}) P_{rj}$$

$(M \circ N)_{ir}$

and the right side here is the ij-block of $((M \circ N) \otimes I)P$. Similarly

$$\sum_r M_{rj} P_{ir} \sum_s N_{sj} P_{is} = \sum_r (M_{rj} N_{rj}) P_{ir}$$

Since the left sides of the last two equations are equal, the result follows. □

$(M \circ N) \otimes I \in \mathrm{Comm}(P)$

Lemma Let $\mathcal{P}$ be a quantum permutation. The commutant of $\mathcal{P}$ is $*$-closed.

Proof Since the entries of $\mathcal{P}$ are Hermitian,

$$(\mathcal{P}(M^* \otimes I))_{x,y} = \sum_r P_{xr} M^*_{ry} = \sum (P_{xr} M_{ry})^*$$

$$= (\mathcal{P}(M \otimes I))^*$$

and so if $\mathcal{P}$ and $M \otimes I$ commute

$$((\mathcal{P}(M \otimes I))_{x,y})^* = ((M^* \otimes I)\mathcal{P})_{x,y}.$$

Hence $M^* \otimes I \in \mathrm{Comm}(P)$ if & and only if $M \otimes I \in \mathrm{Comm}(P)$. $\square$

Corollary The commutant of a set of $n \times n$ quantum permutations is a coherent algebra.

(You should show that the commutant is transpose-closed if & only if it is $*$-closed.)

$\mathrm{Aut}(X)$ = perms in Comm(A) ∩ perms in Comm(J)

= perms in Comm(A, J)

= . . . . Coherent algebra generated by A & J

Group Rings