2 11/03

Bounds

We derive some bounds on quantum parameters

We have the following bounds on classical parameters:

(a) If X is k-regular on n vertices, $\alpha(X) \leq \frac{n}{1 - \frac{k}{\tau}}$.

(b) If X is 1-walk regular, $\omega(X) \leq 1 - \frac{k}{\tau}$.

Cvetković

inertia bound

(c) $\alpha(X) \leq \min\{n - n_-(X), n - n_+(X)\}$.

no. of -ve eigenvalues

" " +ve "

(d) For any X, $\chi(X) \geq 1 - \frac{\theta_1}{\theta_{min}}$. Hoffman bound

We prove (c), the inertia bound. Let A be a weighted adjacency matrix for X — so $A_{ij} = 0$ if $ij \notin E(X)$ and A is Hermitian. Let $V(+)$ be the subspace of $\mathbb{R}^n$ ($n = |V(X)|$) spanned by the eigenvectors of A with positive eigenvalues and let $V(-)$ be the span of the eigenvectors with negative eigenvalues. Then $V(+) \cap V(-) = \underset{\sim}{0}$.

Suppose S is a coclique in X and define W to be the span of the vectors e_u for u in S.

Claim $W \cap V(+) = W \cap V(-) = \underline{0}$

Corollary $|S| \leq \min\{n - \dim(V(+)), n - \dim(V(-))\}$

Next, the chromatic number. Let π be a partition of $V(X)$ with c cells and let $P_1, \ldots, P_c$ be the diagonal 01-matrices such that $P_{i,u} = 1$ if & only if the vertex u lies in the i-th cell of π. As $P_i^2 = P_i$, we see that P_i is a projection. Further $\sum_{i=1}^{c} P_i = I$ (and $P_i P_j = 0$ if $i \neq j$).

Next:

Lemma Let π be a partition of $V(X)$ with c cells, and with associated projections $P_1, \dots, P_c$. Then the cells of π are cocliques if & only if

$$\sum_{i=1}^{c} P_i A P_i = 0.$$

□

Now let W be a flat unitary matrix of order $c \times c$ and define matrices $U_1, \dots, U_c$ by

$$U_i := \sum_{j=1}^{c} W_{i,j} P_j$$

Then $U_1, \dots, U_c$ are unitary (also diagonal, but that plays no role).

Lemma $\sum_{i=1}^{c} P_i \otimes P_i = \sum_{i=1}^{c} U_i \otimes U_i^{-1}$ Exercise □

This implies the following

Corollary $\sum_{i=1}^{c} P_i A P_i = \sum_{i=1}^{c} U_i A U_i^{-1}$ Exercise   □

This will give the Hoffman bound on $\chi(X)$

Theorem $\chi(X) \geq 1 - \frac{\theta_1}{\theta_{min}}$.

Proof Assume the partition π determines a c-colouring. Then

$$0 = \sum_i P_i A P_i = \sum_i U_i A U_i^{-1}$$

and therefore

$$A = -\sum_{i=2}^{c} U_1^{-1} U_i A U_i^{-1} U_1$$

Use M_j to denote $-U_1^{-1} U_j A U_j^{-1} U_1$. Then

$$A = M_2 + \dots + M_c$$

and accordingly

$$\theta_1(A) \leq \theta_1(M_2) + \dots + \theta_1(M_c)$$

$$\theta_{max}(A) = \max \frac{x^T A x}{x^T \cdot x}$$

$$\theta_1(A) \leq (c-1)\, \theta_1(M)$$
?

As $M_2, \dots, M_c$ are similar to A, we have $\theta_1(M_j) = -\theta_{min}(A)$

and therefore

$$\theta_1(A) \le -(c-1)\,\theta_{min}(A),$$

which implies that $c \ge 1 - \dfrac{\theta_1(A)}{\theta_{min}(A)}$.

Pete $\quad -2^{(4)} \quad 1^{(5)} \quad 3^{(1)}$

$$1 + \frac{3}{2} = 2\tfrac{1}{2}$$

Elphick & Wocjan

Now for $\chi_q(X)$. Assume $\mathcal{P}$ is a quantum c-colouring of X with index d. Let B_i denote the block diagonal matrix with projections $P_{i1}, \dots, P_{ic}$ as its blocks. We see that B_i is a projection. As before, let W be a flat type-II matrix of order $c \times c$ and define matrices $U_1, \dots, U_c$ by setting $U_i = \sum_{j=1}^{c} W_{i,j} B_j$.

The projections B_i sum to I_{nd}, and are pairwise orthogonal. The matrices $U_1, \ldots, U_c$ are unitary.

Since P is a quantum c-colouring, we have

$$\sum_{i=1}^{c} B_i (A \otimes I_d) B_i = 0,$$

exercise

and so

$$\sum_{i=1}^{c} U_i (A \otimes I_d) U_i^{-1} = 0.$$

Since $\theta_1(A \otimes I_d) = \theta_1(A)$ & $\theta_{max}(-A \otimes I_d) = \theta_{min}(A)$, we conclude that

$$\chi_q(X) \geq 1 - \frac{\theta_1(A)}{\theta_{min}(A)}. \qquad \square$$

Latin square graphs

Let L be an $n \times n$ Latin square. The vertices of the Latin square graph are the n^2 triples

$$(i, j, L_{ij}),$$

with two triples adjacent if they agree on one coordinate. It is strongly regular with parameters $(n^2, 3n-3, n, 6)$.

It eigenvalues are the valency $3n-3$ and the zeros of

$$t^2 - \overset{a-c}{(n-6)}\,t - \overset{k-c}{(3n-9)} = (t+3)(t-(n-3))$$

i.e. -3 and $n-3$ (with multiplicities n^2-3n+2 & $3n-3$)

Hence

$$1 + \frac{k}{-\tau} = 1 + \frac{3n-3}{3} = n$$

and therefore $\chi(X) \geq n$.

Each column of a Latin square graph determines an n-clique, so $\omega(X(L)) \geq n$. exercise: $\omega = n$

Any coclique contains at most one vertex from each column, whence $\alpha(X) \leq n$.

It can be shown that if L is the multiplication table of a group, then $\chi(X) = n \Leftrightarrow \alpha(X) = n$.

However if L is the multiplication table of a cyclic group of even order, then $\alpha(X) < n$, and consequently $\chi(X(L)) > n$.

Question Could L admit a quantum n-colouring?

(See G&R Section 10.4 for missing details.)

Besharati, Goddyn, Mahmoodian, Mortezaeefar conjecture that if L is an $n \times n$ Latin square,

$$\chi(L) = \begin{cases} n+1, & n \text{ odd}; \\ n+2, & n \text{ even}. \end{cases}$$

They also deduce, from a result of Molloy & Reed, that $\chi(L) - n = o(n)$.

The Kneser graph $K_{d:r}$ has $\chi(K_{d:r}) = d-2r+2$ but the Hoffman bound is d/r. We do not know $\chi_q(K_{d:r})$.

Orthogonality graphs & Grassmannians.

We defined the vertices of the orthogonality graph $\Omega(d)$ to be the unit vectors in $\mathbb{C}^d$. Two vectors are adjacent if they are orthogonal. If x & y are unit vectors in $\mathbb{C}^d$ spanning the same line ($x = \eta y$, $\|\eta\|=1$), then $z \perp x \Leftrightarrow z \perp y$.

Lemma $\Omega(d)$ is homomorphically equivalent to the graph on the 1-dimensional subspaces (aka lines) of $\mathbb{C}$, with lines adjacent if they are orthogonal.

exercise $\square$

Note that in the real case, the real case, the map vectors to lines is 2:1.

In practice it is easy to work with lines are than vectors. If we do choose to work with lines, we can represent the line spanned by the unit vector x by the projection xx^*. We have

$$\langle xx^*, yy^* \rangle = tr(xx^*yy^*) = tr(x^*y \cdot y^*x)$$

$$= |\langle x, y \rangle|^2$$

We define the Grassmann graph $\mathcal{G}(d,r)$ to be the graph with the r-dimensional subspaces of $\mathbb{C}^d$ as its vertices, with two subspaces adjacent if & only the corresponding projections are orthogonal. (Recall that if P, Q & C, then $tr(PQ) = \langle P, Q \rangle = 0$ if & only if $PQ = 0_d$)

We have already worked with the Grassmann graph. For suppose P is a rank-r quantum colouring of index d of X and consider the column

$$\begin{matrix} P_{1,u} \\ P_{2,u} \\ \vdots \\ P_{n,u} \end{matrix}$$

Each entry is a $d \times d$ projection of rank r, and hence maps $V(X)$ into $V(\mathcal{G}(d,r))$.

Further, since this is a column of a quantum colouring, if $i \underset{X}{\sim} j$ then $P_{iu} P_{ju} = 0$ and so we have a homomorphism from X into $\mathcal{G}(d, r)$.

There is a finite analog of $\mathcal{G}(d,r)$.

The **Kneser graph** $K_{d:r}$ has a vertices the $\binom{d}{r}$ r-subsets of $\{1,\dots,d\}$; two r-subsets are adjacent if they are disjoint.

If we use the standard basis vectors $\{e_1,\dots,e_d\}$ as our underling set, each r-subset determines an r-dimensional subspace, and disjoint subsets give subspaces with intersection zero. So $K_{d:r} \to \mathcal{G}(d,r)$. We note two properties of $K_{d:r}$ (neither trivial)

(a) If $d \geq 2r+1$, then $\alpha(K_{d:r}) = \binom{d-1}{r-1}$ Erdős-Ko-Rado

(b) If $d \geq 2r+1$, then $\chi(K_{d:r}) = d-2r+2$ Lovász

(We also have $K_{d:1} = K_d$ & $K_{5:2}$ is the Petersen graph.)

Theorem For any graph X, the minimum value of the set $\{ \frac{d}{r} : X \to K_{d:r} \}$ is the fractional chromatic number of X. $\square$

It is not obvious that this minimum value is achieved. (But it is.)

Fractional chromatic number??

Let X be a graph and let B be the matrix with the characteristic vectors of the maximal (by inclusion) cocliques of X. So B is $v \times n$ (and n is large, usually). A colouring of X is given by a 01-vector y of length n, such that

$$By \geq \mathbf{1}$$

and the number of colour classes in $\mathbf{1}^T y$.

So $\chi(X)$ is the value of the integer program

$$\min \ \mathbf{1}^T y$$
$$By \geq \mathbf{1}$$
$$y_i \in \{0, 1\}$$

and $\chi_f(X)$ is the value of the linear program

$$\min \ \mathbf{1}^T y$$
$$By \geq \mathbf{1}$$
$$y \geq 0$$

Clearly $\chi_f(X) \leq \chi(X)$. Computing $\chi_f(X)$ is NP-hard.

If X is vertex transitive, $\chi_f(X) = \dfrac{|V(X)|}{\alpha(X)}$.