L08/03

Theorem $K_m \to \Omega^b(m) \to \mathcal{HD}_1(m) \to \Omega(m)$

Theorem For any graph X

$$\chi(X) \geq \xi^b(X) \geq \chi_q^{(1)}(X) \geq \xi(X).$$

Let $\Phi(n)$ be the graph with the ± 1-vectors of length n as vertices, adjacent if they are orthogonal.

Then

(a) if n is odd, $\Phi(n)$ is empty

(b) if $n \equiv 2 \pmod 4$, then $\Phi(n)$ is bipartite.

(c) if $4 | n$, then $\omega(X) \leq n$ — the maximal cliques are the Hadamard matrices (of order $n \times n$).

$n \times n$ ± 1 $H^T H = nI$

(d) $\chi_q^{(1)}(\Phi(n)) \leq \xi^b(\Phi(n)) \leq n$,

trust me

(e) $\alpha(\Phi(n)) \leq \frac{2^n}{n}$ (ratio bound)

We aim to prove the following:

Theorem If $4|n$ and $n>8$, then $\chi(\Phi(n))>n$.

From (e) above $\chi(\Phi(n)) \geq n$ and, if equality holds, $\alpha(\Phi(n)) = \frac{2^n}{n}$ and n is a power of two.

If $n = 2^k$ then

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{\otimes k}$$ Sylvester matrix

is a Hadamard matrix; its columns form an n-clique in $\Phi(n)$.

To prove the theorem we need some results on Cayley graphs. (Note that $\Phi(n)$ is a Cayley graph for $\mathbb{Z}_2^n$.)

Lemma If X is Cayley graph on n vertices, $\alpha(X)\omega(X) \le n$.

Proof Suppose S is a coclique in X and a, b are adjacent vertices. If $S^{-1}a \cap S^{-1}b \neq \emptyset$, then $g^{-1}a = h^{-1}b$ for some g, h in S.

Then $hg^{-1} = ba^{-1}$. As $g, h \in S$, we have hg^{-1} is not in the connection set, as $b \sim a$ we have ba^{-1} is in the connection set.

Therefore if S is a coclique in X, the sets $S^{-1}c$ for c in C are pairwise disjoint, and our lemma follows. $\square$

A Cayley graph is normal if its connection set is closed under conjugation, e.g., any Cayley graph for an abelian group.

Corollary If X is a normal Cayley graph and $\alpha(X)\omega(X)=n$, then $\chi(X)=\omega(X)$.

Proof Assume S is a coclique & C is a clique in X. The sets $S^{-1}c$ for c in C are pairwise disjoint. We claim S^{-1} is a coclique. Let D be the connection set. If $g,h\in S^{-1}$ and $g\sim h$ then $hg^{-1}\in D$. Now $g^{-1}(hg^{-1})g=g^{-1}h$ is a conjugate of hg^{-1} and so $g^{-1}h\in D$ and $h^{-1}\sim g^{-1}$.

But $s', t' \in S$. Consequently if $\alpha(X)\omega(X) = n$, the sets $S^{-1}c$ give a proper colouring of X with $|C| = \omega(X)$ colours. $\square$

We have shown that $\chi(\bar{\Phi}(n)) > n$ if n is not a power of two, and if $\chi(\bar{\Phi}(n)) = n$ then $\alpha(\bar{\Phi}(n)) = \frac{2^n}{n}$.

We finish our argument by appealing to a deep result of Frankl & Rödl.

Theorem There is a positive real number ε such that $\alpha(\Phi(4m)) \leq (2-\varepsilon)^n$. $\square$

Thus $\chi(\Phi(4m))$ grows exponentially with m.

Quaternions

The quaternions $\mathbb{H}$ are the 4-dimensional algebra over $\mathbb{R}$, with basis $1, i, j, k$ satisfying

$$i^2 = j^2 = k^2 = -1; \quad ij = k, \ jk = i, \ ki = j.$$

These relations imply the $ji = -k$, $kj = -i$, $ik = -j$.

The quaternions are a skew field.

If $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ then we define

$$q(x) := x_0 + x_1 i + x_2 j + x_3 k$$

We define

conjugate

$$(x_0 + x_1 i + x_2 j + x_3 k)^* = x_0 - x_1 i - x_2 j - x_3 k$$

This is an anti-automorphism of $\mathbb{H}$. We refer to $2x_0$ as the trace of a quaternion; a quaternion is pure if this is zero. We have

$$q(x)^* q(x) = x_0^2 + x_1^2 + x_2^2 + x_3^2$$

We call $\alpha^* \alpha$ the norm of the quaternion α.

If $a \in H$, we have a linear mapping

$$\mu_a : x \longmapsto ax$$

Define $\mu_0 = I_4$. Then

$$\mu_i : \begin{array}{ccccc} 1 & 0 & i & 0 & 0 \\ i & -1 & 0 & 0 & 0 \\ j & 0 & 0 & 0 & k \\ k & 0 & 0 & -j & 0 \end{array} \quad \to \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}^T$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Similarly

$$\mu_j: \quad \begin{array}{c|cccc} 1 & 0 & 0 & j & 0 \\ i & 0 & 0 & 0 & -k \\ j & -1 & 0 & 0 & 0 \\ k & 0 & i & 0 & 0 \end{array} \quad \to \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^T$$

$$\mu_k \quad \to \quad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}^T$$

Lemma

$$\mu_x = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ -x_1 & x_0 & x_3 & -x_2 \\ -x_2 & -x_3 & x_0 & x_1 \\ -x_3 & x_2 & x_1 & x_0 \end{pmatrix} \qquad \square$$

Looking at the diagonal entries of

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ -x_1 & x_0 & x_3 & -x_2 \\ -x_2 & -x_3 & x_0 & x_1 \\ -x_3 & x_2 & -x_1 & x_0 \end{pmatrix} \begin{pmatrix} y_0 & -y_1 & -y_2 & -y_3 \\ y_1 & y_0 & -y_3 & y_2 \\ y_2 & y_3 & y_0 & -y_1 \\ y_3 & -y_2 & y_1 & y_0 \end{pmatrix}$$

we see that $\mu(x)\mu(y^*)$ has zero diagonal if & only if & only if $x^T y = 0$. Also

$$\mu(x)\mu(x^*) = (x_0^2 + x_1^2 + x_2^2 + x_3^2) I_4$$

and therefore if x is a unit vector in $\mathbb{R}^4$, then μ_x is a real orthogonal matrix.

If $\|x\| = \|y\| = 1$, then $x^T y = 0$ if & only $\mu(x)\mu(y^*)$ is pure, equivalently $x \perp y$ if x & y are adjacent in $\mathrm{AD}_1(4)$.

Let $\Omega_{\mathbb{R}}(n)$ denote the unit vectors in $\mathbb{R}^n$.

Theorem There is a homomorphism $\Omega_{\mathbb{R}}(4) \longrightarrow \mathcal{M}_1(4)$. $\square$

Hence $\chi_q^{(1)}(\Omega_{\mathbb{R}}(4)) = 4$.

Cameron, Newman, Severini, ...

We use G_{13} to denote the orthogonality graph of the following 13 vectors:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & 0 & 1 & -1 & 1 \end{bmatrix}$$

If we add a bottom row of zeros, and a 14th vector $(0\,0\,0\,1)^T$, we get G_{14}.

Facts

(a) $\chi(G_{13}) = 4$ (computer)

(b) $\chi(G_{14}) = 5$ (cone)

(c) $\chi_q^{(1)}(G_{13}), \chi_q^{(1)}(G_{14}) \leq 4$

in fact $\chi_q^{(1)}(G_{13}) = 4$

Mancinska & Roberson

Oddities of quantum colorings

Where does G_{13} come from? It is an Erdős-Rényi graph (but not an Erdős-Renyi random graph).

To construct, choose a finite field $\mathbb{F}$ of odd order q. The vertices of the graph are 1-dimensional subspaces of the 3-dimensional vector space over $\mathbb{F}$. Two 1-dimensional subspaces spanned by vectors x & y are adjacent if $x^T y = 0$.

As just defined this gives a $(q+1)$-regular graph on $\frac{q^3-1}{q-1} = q^2+q+1$ vertices, but there are $q+1$ vertices with loops. We delete the loops. If $q=3$, we get G_{13}.

The Erdős-Rényi graphs have no 4-cycles and, given this, have many edges.

Bounds

We derive some bounds on quantum parameters

We have the following bounds on classical parameters:

(a) If X is k-regular on n vertices, $\alpha(X) \leq \frac{n}{1 - \frac{k}{\tau}}$.

(b) If X is 1-walk regular, $\omega(X) \leq 1 - \frac{k}{\tau}$.

inertia bound

(c) $\alpha(X) \leq \min\{n - n_-(X), n - n_+(X)\}$.

no. of -ve eigenvalue

" " +ve "

(d) For any X, $\chi(X) \geq 1 - \frac{\theta_1}{\theta_{min}}$. Hoffman bound

We prove (c), the inertia bound. Let A be a weighted adjacency matrix for X — so $A_{ij} = 0$ if $ij \in E(X)$ and A is Hermitian. Let $V(+)$ be the subspace of $\mathbb{R}^n$ $|V(X)|$ spanned by the eigenvectors of A with positive eigenvalues and let $V(-)$ be the span of the eigenvectors with negative eigenvalues. Then $V(+) \cap V(-) = \underset{\sim}{0}$.

Suppose S is a coclique in X and define W to be the span of the vectors e_u for u in S.

Claim $W \cap V(+) = W \cap V(-) = \underline{0}$

Corollary $|S| \leq \min\{n - \dim(V(+)), n - \dim(V(-))\}$

Next, the chromatic number. Let π be a partition of $V(X)$ with c cells and let $P_1, \ldots, P_c$ be the diagonal 01-matrices such that $P_{i,u} = 1$ if & only if the vertex u lies in the i-th cell of π. As $P_i^2 = P_i$, we see that P_i is a projection. Further $\sum_{i=1}^{c} P_i = I$

We also have

Lemma Let π be a partition of $V(X)$ with c cells, and with associated projections $P_1, \ldots, P_c$. Then the cells of π are cocliques if & only if

$$\sum_{i=1}^{c} P_i A P_i = 0.$$

$\square$

Now let W be a flat unitary matrix of order $c \times c$ and define matrices $U_1, \dots, U_c$ by

$$U_i := \sum_{j=1}^{c} W_{i,j} P_j$$

Then $U_1, \dots, U_c$ are unitary (also diagonal, but that plays no role).

Exercise

Lemma $\sum_{i=1}^{c} P_i \otimes P_i = \sum_{i=1}^{c} U_i \otimes U_i^{-1}$ $\square$

This implies the following

Corollary $\sum_{i=1}^{c} P_i A P_i = \sum_{i=1}^{c} U_i A U_i^{-1}$ $\square$

This will give the Hoffman bound on $\chi(X)$

Theorem $\chi(X) \geq 1 - \frac{\theta_1}{\theta_{min}}$.

Proof Assume the partition π determines a c-colouring. Then

$$0 = \sum_i P_i A P_i = \sum_i U_i A U_i^{-1}$$

and therefore

$$A = -\sum_{i=2}^{c} U_1^{-1} U_i A U_i^{-1} U_1$$

Use M_j to denote $-U_1^{-1} U_j A U_j^{-1} U_1$. Then

$$A = M_2 + \dots + M_c$$

and accordingly

$$\theta_1(A) \leq \theta_1(M_2) + \dots + \theta_c(M_c)$$

As $M_2, \ldots, M_c$ are similar to A, we have $\theta_1(M_j) = -\theta_{min}(A)$

and therefore

$$\theta_1(A) \leq -(c-1)\,\theta_{min}(A),$$

which implies that $c \geq 1 - \dfrac{\theta_1(A)}{\theta_{min}(A)}$.