L01/03

Equitable partitions

A partition π of $V(X)$ is **equitable** if:

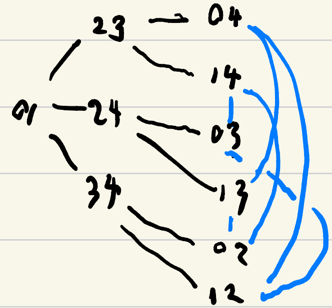
(a) each cell of π induces a regular graph,

(b) the edges joining one cell to another form a biregular bipartite graph.

(We will come to refer to this as "equitable relative to A".)

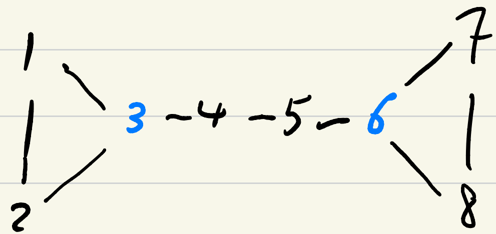
One class of examples is provided by the orbit partitions of a group of automorphisms of X.

2-regular on 6 vxs

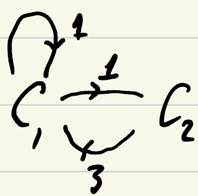


orbit partition

For another:



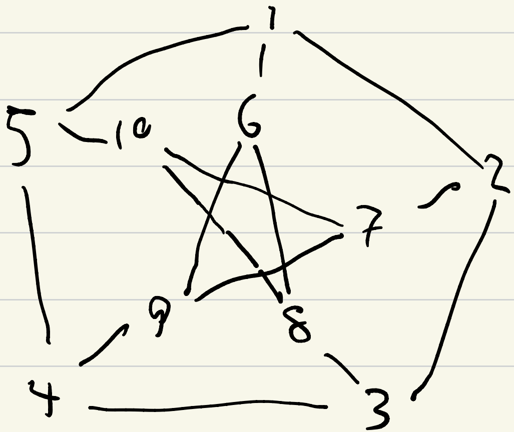
$$\pi = \{\ \{1,2,4,5,7,8\}\ \{3,6\}\ \}$$



And one more. If $a \in V(X)$, we can partition the vertices of X by their distance from a. We call this the ==distance partition== relative to a.

Lemma. If X is a vertex in a strongly regular graph, the distance partition relative to a is equitable.

Proof. Exercise. $\square$



$$\pi = \{\{1\},\{2,5,6\},\{3\ 4\ 7\ 8\ 9\ 10\}\}$$

$$\begin{tikzcd} C_1 \arrow[r, bend left, "3"] & C_2 \arrow[l, bend left, "1"] \arrow[r, bend left, "2"] & C_3 \arrow[l, bend left, "1"] \arrow[loop above, "2"] \end{tikzcd}$$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad M^TM = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Theorem Let X be a graph on n vertices and let π be a partition of $V(X)$ with characteristic matrix M. The following are equivalent:

(a) π is equitable.

(b) There is a matrix B such that $AM = MB$.

(c) $\mathrm{col}(M)$ is A-invariant.

(d) A commutes with the projection $M(M^TM)^{-1}M^T$.

Proof Assume $\pi = \{C_0, C_1, \dots, C_d\}$ and let M be the characteristic matrix of π. Then $(A(Me_j))_i$ is the number of neighbours of vertex i in C_j. We see that π is equitable if & only if $(A(Me_j))_i$ is determined by j and the cell that contains i.

So (a) & (b) are equivalent. The equivalence of (b) and (c) is an exercise in linear algebra.

d) $\Rightarrow$ c) Finally $M(M^TM)^{-1}M^T$ is symmetric,

$$M(M^TM)^{-1}M^TM(M^TM)^{-1}M^T = M(M^TM)^{-1}M^T,$$

and thus $M(M^TM)^{-1}M$ is a projection into $\mathrm{col}(M)$.

As $M(M^TM)^{-1}M^T$ & $(M^TM)^{-1}M^TM = I_{|\pi|}$ have the same non-zero eigenvalues with the same multiplicities $\mathrm{rk}(M(M^TM)^{-1}M^T) = |\pi|$ and therefore $M(M^TM)^{-1}M$ represents orthogonal projection onto $\mathrm{col}(M)$

Now if P is a projection and $AP = PA$, then $\mathrm{Col}(P)$ is A-invariant. So (d) $\Rightarrow$ (c).

Set $\Delta = M^T M$, then $AM\Delta^{-1/2} = MB\Delta^{-1/2} = M\Delta^{-1/2}(\Delta^{1/2} B \Delta^{-1/2})$.

Hence $\Delta^{-1/2} M^T A M \Delta^{-1/2} = \Delta^{-1/2} M^T M \Delta^{-1/2} (\Delta^{1/2} B \Delta^{-1/2}) = \Delta^{1/2} B \Delta^{-1/2}$ and therefore $\Delta^{1/2} B \Delta^{-1/2}$ is symmetric. Also

$AM\Delta^{-1} M^T = AM\Delta^{-1/2}(M\Delta^{-1/2})^T = MB\Delta^{-1} M^T = M\Delta^{-1/2} \cdot \Delta^{1/2} B \Delta^{-1/2} \Delta^{-1/2} M^T$

is symmetric & therefore A & $M\Delta^{-1} M^T$ commute. $\square$

If $\mathcal{A}$ is a matrix algebra, we say π is equitable relative to $\mathcal{A}$ if $\operatorname{col}(M)$ is $\mathcal{A}$-invariant.

Homomorphisms

homomorphisms

colourings   automorphisms

$\longleftrightarrow$

quantum homs

quantum colourings   quantum automorphisms

Some useful homomorphisms:

1) $\varphi : X \to K_m$: m-colouring

2) $\varphi : X \to K_{m,k}$: min value of m/k is the fractional chromatic number, χ_f

Kneser graph

3) Ω : unit sphere in $\mathbb{R}^d$ $\quad x \sim y \iff x^T y = 0$ (flat vectors)

4) Ω unit sphere $\quad x \sim y \iff \langle x, y \rangle \le \alpha \quad (-1 < \alpha \le 0) \quad \chi_{vec}$

or $\quad x \sim y \iff \langle x, y \rangle = \alpha \quad \chi_{sv}$

5) $V(Y) = \mathrm{Sym}(d)$, $\alpha \sim \beta \iff \beta\alpha^{-1}$ has no fixed points

Homs into Y?

Quantum homomorphisms

First, we want represent homomorphisms $X \to Y$ by matrices. A homomorphism $f: X \to Y$ is a function from $V(X)$ into $V(Y)$. The sets $\{v \in V(X) : \varphi(v) = y\}$ is the fibre of f at y in $V(Y)$. (It may be empty.) The non-empty fibres form a partition of $V(X)$, we denote it by π_f. Since f is a graph homomorphism induces a coclique in X.

The characteristic matrix of π_f determines f. It is a 01-matrix with each row summing to 1, but not every such matrix comes from a homomorphism.

Lemma Assume $f: X \to Y$ is a map from $V(X)$ to $V(Y)$. Let M be the characteristic matrix of π_f. Then f is a homomorphism if & only if the following hold:

(a) $M\underset{\sim}{1} = \underset{\sim}{1}$.

(b) The support of a column of M is a coclique in X.

(c) If $i \underset{X}{\sim} j$ and $k \underset{Y}{\sim} l$, then $M_{ik} M_{jl} = 0$. $\square$ so $M_{ik} M_{jk} = 0$ too

Lemma If $f: Y \to X$ and $g: Z \to Y$ are homomorphisms and M_f, M_g are the characteristic matrices of π_f & π_g respectively, then $M_g M_f$ is the characteristic matrix of the homomorphism $f \circ g: Z \longrightarrow X$. $\square$

A quantum homomorphism from X to Y is a matrix P with rows indexed by $V(X)$, columns indexed by $V(Y)$, and entries $d \times d$ projections P_{ij} such that

(a) $\sum_j P_{ij} = I_d$.

(b) If $i \underset{X}{\sim} j$ and $k \underset{Y}{\nsim} l$, then $P_{ik} P_{jl} = 0$.

Note entries in the same row are orthogonal projections

We call d the index of the quantum homomorphism.

A quantum homomorphism of index one is a classical homomorphism.

Measurements on a graph

There is an alternative definition of quantum homomorphism which is often useful.

First, a projective measurement is a sequence of projections $P_1, \ldots, P_d$ such that $\sum_i P_i = I$. If the state of the system is given by a density D, the result of the above measurement is an index i in $\{1, \ldots, d\}$ with probability $\langle D, P_i \rangle$.

We note an important property of projective measurements.

Lemma If $P_1, \ldots, P_m$ are projections and $\sum P_i = I$

then $P_i P_j = 0$ when $i \neq j$.

Proof We have

$$I = I^2 = \sum_i P_i \sum_j P_j = \sum_i P_i^2 + \sum_{i \neq j} P_i P_j = I + \sum_{i \neq j} P_i P_j$$

Hence $\sum_{i \neq j} P_i P_j = 0$ and so $\sum_{i \neq j} \mathrm{tr}(P_i P_j) = 0$. As $P_i, P_j \geq 0$

$\mathrm{tr}(P_i P_j) = 0 \Rightarrow P_i P_j = 0$. $\square$

Remark: If $P, Q \geq 0$, then $P = CC^*$, $Q = DD^*$,

$$\operatorname{tr}(PQ) = \operatorname{tr}(CC^*DD^*) = \operatorname{tr}(D^*CC^*D) = \langle C^*D, C^*D \rangle$$

and so if $\operatorname{tr}(PQ) = 0$, then $C^*D = 0$ and

$$PQ = CC^*DD^* = 0.$$

A measurement on a graph Y is a projective measurement $P_1, \dots, P_m$ where $V(Y) = \{1, \dots, m\}$. So it is indexed by $V(Y)$. Each row of a quantum homomorphism from X to Y is a measurement indexed by $V(Y)$.

We say two measurements $(P_i), (Q_i)$ indexed by Y are compatible if the matrix $\begin{bmatrix} P_1 & \cdots & P_m \\ Q_1 & \cdots & Q_m \end{bmatrix}$ is a quantum homomorphism from K_2 to Y.

Equivalently, two measurements are compatible if

$P_i Q_j = 0$ when $i \neq j$.

We define the **measurement graph** of Y to be the graph with the measurements on Y as vertices, with two vertices adjacent if the corresponding measurements are compatible. If the projections are $d \times d$, we denote this graph by $\mathcal{M}(Y, d)$.

Theorem $X \xrightarrow{q} Y$ if & only if $X \to \mathcal{M}(Y, d)$ for some d. $\square$