2/16/02

The mixing matrix of a walk is $M(t) = U(t) \circ U(-t)$ &

$$U(t) \circ U(-t) = \sum_{r,s} e^{it(\theta_r - \theta_s)} E_r \circ E_s$$

and consequently

$$\hat{M} = \lim_{T \to \infty} \frac{1}{T} \int_0^T M(t)\, dt = \sum_r E_r \circ E_r$$

$\hat{M}$ is the average mixing matrix

Corollary $\hat{M}$ is a rational matrix

matrix of inner products

Theorem $\hat{M}$ is the Gram matrix of the average states $\hat{D}_a$ (for a in $V(X)$).

$u_1, \ldots, u_m$

$[\langle u_i, u_j \rangle]$

Gram matrix

$z^* M z \geq 0$

Proof.

$$\langle \hat{D}_a, \hat{D}_b \rangle = \mathrm{tr}\left(\sum_r E_r D_a E_r \sum_s E_s D_b E_s \right)$$

$$= \mathrm{tr}\left(\sum_r E_r D_a E_r D_b E_r \right)$$

$$= \mathrm{tr}\left(\sum_r E_r e_a \underline{e_a^T E_r e_b} e_b^T E_r \right)$$

$$= \sum_r (E_r)_{ab} (E_r)_{ba}$$

Since $E_r = E_r^T$, the last sum equals $\sum_r (E_r \circ E_r)_{ab} = \hat{M}_{ab}$. $\square$

Theorem Vertices a & b in X are strongly cospectral if & only if $e_a^T \hat{M} = e_b^T \hat{M}$.

Proof If a & b are strongly cospectral, $\hat{O}_a = \hat{O}_b$ and the claim follows from the previous theorem.

On the other hand, if $(e_a - e_b)^T \hat{M} = 0$, then

$$0 = (e_a - e_b)^T \hat{M} (e_a - e_b) = \sum_r (e_a - e_b)^T E_r^{\circ 2} (e_a - e_b)$$

and, since $E_r^{\circ 2} \succeq 0$, this implies that $(e_a - e_b)^T E_r^{\circ 2} (e_a - e_b) = 0$.

Again, since $E_r^{\circ 2} \succeq 0$, we have $(e_a - e_b)^T E_r^{\circ 2} = 0$. Therefore,

for all r,

$$((E_r)_{aa})^2 = ((E_r)_{ba})^2 = ((E_r)_{bb})^T$$

$$\begin{pmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle y, x \rangle & \langle y, y \rangle \end{pmatrix}$$

Since $(E_r)_{ab} = \langle E_r e_a, E_r e_b \rangle$, by Cauchy-Schwarz, $(E_r) e_a = \pm (E_r) e_b$. $\square$

see notes

It can be shown that if $rk(\hat{M}_X) = 1$, then $X = K_1$ or K_2.

It is an open question whether there are infinitely many graphs X such that $rk(\hat{M}_X) = 2$.

For a discrete quantum walk we have a state space and an initial state D. Given a unitary matrix U, the sequence of states

$$D,\ UDU^{-1},\ U^2DU^{-2},\ \dots$$

forms a discrete walk. As for continuous walks, we measure using a POVM $P_1, \dots, P_m$; the probability we observe the i-th outcome at time k is $\langle U^kDU^{-k}, P_i\rangle$.

Difficulty Unitary matrices are expensive to implement.
The idea is to express U as a product of simple operations.
in general. For us. these simpler operations are reflections.

Reflections on projections

A ==reflection== on a real inner product space U is an endomorphism L that fixes each vector in a subspace U_0 of U, and acts as multiplication by -1 on the orthogonal complement U_1 to U. We see that $L^2 = I$.

So the simplest examples are the diagonal matrices with diagonal entries ± 2.

exercise

Any reflection is orthogonal.

Although we will not need it, we define a reflection on a complex vector space U to be an endomorphism that fixes a subspace U_0 and acts as multiplication by an m-th root of unity on its orthogonal complement for some m.

We present a construction.

Assume $a \in U$ and $\|a\| = 1$. Define $\tau_a : U \to U$ by

$$\tau_{a,\theta}(u) = u - (1-\theta)\langle a, u\rangle a, \quad \theta^k = 1$$

Since τ_a is the sum of two linear maps, it is linear.

Clearly τ_a fixes $a^\perp$. Further

$$\tau_a(a) = a - (1-\theta)\langle a, a\rangle a = \theta a.$$

We say that τ_a is reflection in a hyperplane.

Over $\mathbb{R}$, $\tau_a = u - 2\langle a, u\rangle a$, $\|a\| = 1$.

In the real case, the only useful choice for α is -1,

and then $(T_{a,-1})^2 = 1$, and so it is a real reflection.

If P is an orthogonal projection on U, then

$$(2P-I)^2 = 4P - 4P + I = I$$

and $2P-I$ is a real reflection which fixes $\operatorname{im}(P)$ and

acts as $-I$ on $\ker(P)$ (the orthogonal complement to $\operatorname{Im}(P)$).

If R is a reflection, $\frac{1}{2}(R+I)$ is a projection.

As $\frac{1}{n}J$ is a projection, $\frac{2}{n}J-I$ is a reflection that fixes $\underset{\sim}{1}$ and acts as $-I$ on $\underset{\sim}{1}^{\perp}$. We will refer to $\frac{2}{n}J-I$ as the Grover coin.

Most of our reflections will be constructed from partitions of sets, in the way we describe now. Let V be a (finite set). A partition π of V is a set, each element of which is a subset such that ...

These subsets may be called cells and $|\pi|$ is the number of cells in π. The characteristic matrix M of π is the $|V| \times |\pi|$ matrix with the characteristic vectors of the cells of π as its columns. So M is a 01-matrix and $M\underset{\sim}{1} = \underset{\sim}{1}$. Further, M^TM is diagonal with $(M^TM)_{k,k}$ equal to the size of the k-th cell of π. Note that the columns of M are pairwise orthogonal. If we scale the columns so they each have norm 1, we have the normalized characteristic matrix. If M is the normalized characteristic matrix for π, then $M^TM = I_{|\pi|}$. It follows that

$$(MM^T)^2 = MM^T MM^T = M \cdot I_{|\pi|} M^T = MM^T$$

and MM^T is a projection. If $F(\pi)$ denotes the functions on V that are constant on the cells of π, then MM^T is orthogonal projection onto $F(\pi)$.

We offer two examples.

(c) Let X be a bipartite graph with bipartition (L, R). We define two partitions of $E(X)$.

For the first π_L, the cells are the edges that contain a given vertex from L; for the second partition π_R, the cells are the edges that contain a given vertex from R

We note that $\pi_R \wedge \pi_L$ is the discrete partition (all cells are singletons).

meet

join

The cells of $\pi_L \vee \pi_R$ are the edges in a connected component of X. In practice X is connected &

$\pi_R \vee \pi_L$ is the partition with just one cell.

The discrete walk determined by this pair of partitions is known as a bipartite walk.

(2) For the second example, we construct partitions of the arcs of a graph X. (An arc is an ordered pair of adjacent vertices.) The cells of the first partition are the pairs $\{(a,b), (b,a) : a \sim b\}$ — so each cell is a pair of opposing arcs. The cells of the second partition are the sets $\{(a,u) : u \sim a\}$ — the arcs pointing away from a vertex. The meet of these two partitions is discrete;

if X is connected, the join has just one cell.

The associated discrete walk is the arc-reversal walk.

It is customary to assume X is connected and regular.

Note that there are many classes of discrete walks that do not arise from pairs of partitions.

The arc reversal walk

Assume X is k-regular on n vertices, so $|\mathrm{arcs}(X)| = nk$.

The state space for the arc-reversal walk is the space of complex functions on the arcs of X, i.e., it is $\mathbb{C}^{nk}$. $\mathbb{C}^n \otimes \mathbb{C}^k$

For our first reflection, let R be the permutation matrix on arcs that maps the arc ab to ba. Then $R = R^T$ & $R^2 = I$.

For a second reflection, let G be a $k\times k$ unitary matrix such that $G^2 = I$. In most cases, $G = \frac{2}{k}J - I$, and we call this the Grover coin as noted earlier. Then

$$C = I_n \otimes G$$

is a reflection and the transition matrix $U = RC$.

Our problem is to determine the eigenvalues and eigenvectors of U.