209/02

Contents

Time to perfect state transfer

Time to perfect state transfer If there is ab-pst at time τ,

$$e_b e_b^T = \sum_{r,s} e^{i\tau(\theta_r-\theta_s)} E_r e_a e_a^T E_s ,$$

implying that $e^{i\tau(\theta_r-\theta_s)} = \pm 1$ ($\forall\, r,s$ in the eigenvalue support) and $e^{2i\tau(\theta_r-\theta_s)} = 1$. Hence a is periodic, with minimum period being the least τ such that $\tau(\theta_r-\theta_s)$ is an integer multiple of π. Note that this happens if & only if $\tau(\overset{\text{spectral radius}}{\theta_0}-\theta_r)$ is an integer multiple of π, for all r.

Since $\theta_r = \frac{a + b_r\sqrt{\Delta}}{2}$ we have $\theta_0 - \theta_r = (b_0 - b_r)\frac{\pi}{2}$. If g is the gcd of $\{(\theta_0 - \theta_r) : r = 1, \ldots\}$ then the time to perfect state transfer is $\frac{2\pi}{g\sqrt{\Delta}}$, which is at most 2π.

Monogamy

Lemma (A Kay) If $a, b, c \in V(X)$ and there is pst from a to b and from a to c, then $b = c$.

Proof The time to perfect state transfer is determined by the eigenvalue support. So if ab-pst occurs at time T the so does ac-pst. Therefore $b = c$. $\square$

Bipartite & regular graphs

Lemma If X is bipartite, the eigenvalue support of a vertex is closed under multiplication by -1.

Proof Let D be the diagonal matrix with diagonal entries ± 1 such that $DAD = -A$. As we saw $DE_\theta D = E_{-\theta}$ and therefore if $E_\theta e_a \neq 0$, then $E_{-\theta} e_a = DE_\theta De_a \neq 0$. $\square$

Lemma Assume X is bipartite. If the vertex a in X is periodic, its eigenvalue support consists either of integers, or of integer multiples of $\sqrt{\Delta}$ for some square-free integer Δ.

Proof Suppose the eigenvalue support S of a contains a non-integer eigenvalue. The each element in S can be written in the form

$\frac{1}{2}(a+b_r\sqrt{\Delta})$.

Since S is closed under multiplication by -1, we have $a = 0$. If $\theta = \frac{b\sqrt{\Delta}}{2} \in S$ then $\theta^2 = \frac{b^2 \Delta}{4}$ is a rational algebraic integer, hence is an integer. As Δ is square-free, it is not divisible by 4. So b is even. $\square$

Lemma Assume X is connected and its spectral radiuse is an integer. If a is a periodic vertex in X, all elements of its eigenvalue support are integers.

Proof Assume by way of contradiction that θ lies in the eigenvalue support S of a and $\theta \notin \mathbb{Z}$. Then θ is a quadratic integer and there is a field automaphism of order two mapping η in S to $\bar{\eta}$ for each η in S

We may assume $\bar{\theta} \neq \theta$.

Then

$$\frac{\theta - \rho}{\bar{\theta} - \rho} \in \mathbb{Q}$$

and, as $\rho \in \mathbb{Q}$,

$$\frac{\bar{\theta} - \rho}{\theta - \rho} = \frac{\theta - \rho}{\bar{\theta} - \rho},$$

Therefore $(\bar{\theta} - \rho)^2 = (\theta - \rho)^2$ and $(\bar{\theta} - \theta)(\bar{\theta} + \theta - 2\rho) = 0$.

Hence $\rho = \frac{1}{2}(\theta + \bar{\theta})$.

Since ϱ is the spectral radius, this implies that $\vartheta = \varrho$. But $\vartheta \notin \mathbb{R}$. $\square$

Controllable vertices

Let X be a graph on n vertices. If $z \in \mathbb{R}^n$ define the walk matrix W_z to be

$$[z \ \ Az \cdots A^{n-1}z]$$

The pair (A, z) is controllable if W_z is invertible.

In the cases of interest to us, z will be the characteristic vector of a subset S of $V(X)$; usually $S = \{u\}$ for some vertex, or $S = V(X)$.

Deep work of Behzad & Touri tells us that

(a) almost all graphs are controllable

(b) for almost graphs X, all pairs $(X, \{v\})$ (for v in $V(X)$)

are controllable.

We say a vertex u is controllable if $(X, \{u\})$ is and X itself is controllable if $(X, V(X))$ is. When $S \subseteq V(X)$ we write the walk matrix as W_S. Its ij-entry is the number of walks in the graph of length $j-1$ that start at the vertex i and end in S.

Lemma If (X, S) is controllable, then any automorphism of X that fixes S is the identity.

Proof Assume P is a permutation matrix that commutes with A. If e_S is the characteristic vector of S, then P fixes S if & only if $Pe_S = e_S$. Hence

$$PA^r e_S = A^r P e_S = A^r e_S$$

and therefore $PW_S = W_S$.

Thus if W_ρ is invertible, $P = I$. $\square$

Corollary If X is controllable, $\mathrm{Aut}(X) = \langle 1 \rangle$.

Lemma Let $S = \{v_1, \dots, v_m\}$ be a set of vectors in $\mathbb{R}^n$ and assume $Q_{i,j} = v_i v_j^T$. Then S spans a subspace of dimension k if & only if the matrices $Q_{i,j}$ span a subspace of dimension k^2 in $Mat_{n \times n}(\mathbb{R})$.

Proof We show that the vectors $v_1, \dots, v_m$ are linearly independent if & only if the matrices $v_i v_j^T$ are linearly independent.

If $v_1, \dots, v_m$ are linearly dependent, the matrices $v_i w^T$ ($w \in S$) are linearly dependent.

So we assume $v_1, \dots, v_m$ are linearly independent.

Then there is a dual basis $w_1, \dots, w_m$ in $\mathbb{R}$, i.e., a set of vectors $w_1, \dots, w_m$ such that $v_i^T w_j = \delta_{i,j}$.

$$\begin{bmatrix} v_1^T \\ \vdots \\ v_m^T \end{bmatrix} [w_1 \cdots w_m] = I_m$$

If there were scalars $a_{i,j}$ such that

$$\sum_{i,j} a_{i,j} v_i v_j^T = 0$$

then

$$0 = w_r^T \Big(\sum_{i,j} a_{i,j} v_i v_j^T \Big) w_s = \sum_{i,j} a_{i,j} w_r^T v_i v_j^T w_s = a_{r,s}$$

for all r & s $\square$

Corollary Assume S is a subset of $V(X)$ with characteristic vector z. Then (X, S) is controllable if & only if the matrices $A^i z z^T A^j$ $(0 \le i, j < n)$ form a basis for $\mathrm{Mat}_{n \times n}(\mathbb{R})$. $\square$

Corollary if S is controllable $\langle A, zz^T \rangle = \mathrm{Mat}_{n \times n}(\mathbb{R})$

$S = V(X)$ $zz^T = J$

We aim to prove that controllable vertices cannot be periodic. For this, we need:

Lemma Let X be a graph on n vertices, $n \geq 3$. If σ is the minimum distance between two eigenvalues of X, then

$$\sigma^2 < \frac{12}{n+1}.$$

Proof Assume that the eigenvalues of X in non-increasing order are $\theta_1 \geq \cdots \geq \theta_n$. Define

$$M = A \otimes I - I \otimes A.$$

signed Cartesian product

The eigenvalues of M are $\theta_i - \theta_j$ $(1 \leq i, j \leq n)$.

Now

$$M^2 = A^2 \otimes I + I \otimes A^2 - 2 A \otimes A$$

$tr(A \otimes B) = tr(A)\, tr(B)$

If $m = |E(X)|$, then

$$\sum_{i,j} (\theta_i - \theta_j)^2 = tr(M^2) = 2n\, tr(A^2) = 4mn$$

As $\theta_i - \theta_j \geq (i-j)\sigma$, we have $\sum_{i,j} (\theta_i - \theta_j)^2 \geq \sigma^2 \sum (i-j)^2 = \frac{n^2(n^2-1)}{6} \sigma^2$

Therefore

$$\sigma^2 \frac{n^2(n-1)^2}{6} \leq 4mn = 2n^2(n-1)$$

This proves $\sigma^2 \leq \frac{12}{n+1}$.

To get the strict inequality, note that if equality holds, then $m = \binom{n}{2}$ and $X = K_n$, but $\sigma(K_n) = 0$ (if $n \geq 3$). $\square$

Theorem Assume X is a connected graph on at least four vertices. If a in $V(X)$ is controllable, it is not periodic.

Proof Assume $n = |V(X)|$. Assume by way of contradiction that a is controllable. Then the walk matrix

$$[e_a, Ae_a, \dots, A^{n-1}e_a]$$

is invertible. Since the vectors $E_r e_a$ form a basis for the column space, the eigenvalue support of a has size n, and consists of all eigenvalues of X.

Assume a is controllable, $\sigma \geq 1$ and therefore (by the above lemma), $n \leq 10$. Now use a computer. $\square$

Remark A vertex a is controllable if and only if $\phi(X \setminus a, t)$ and $\phi(X, t)$ are coprime Exercise