L02/02

Contents

# Cubelike graphs

A cubelike graph is a Cayley graph for $\mathbb{Z}_2^d$.

If $\mathcal{C} = \{e_1, \dots, e_d\}$, then $X(\mathbb{Z}_2^d, \mathcal{C})$ is the $d$-cube.

We view $\mathbb{Z}_2^d$ as a vector space. If $w \in \mathbb{Z}_2^d$, let $A_w : \mathbb{Z}_2^d \to \mathbb{Z}_2^d$ be the linear map that maps $u$ to $u+w$. We note that $A_w^2 = I$, in fact $A_w$ is a permutation with all orbits of length two. Matrices $A_v$ & $A_w$ commute — $A_u A_v = A_{u+v}$.

We can also regard $A_w$ as the adjacency matrix of a graph formed from $2^{d-1}$ vertex-disjoint copies of $K_2$.

**Lemma** $A(X(\mathbb{Z}_2^d, \mathcal{C})) = \sum_{w \in \mathcal{C}} A_w$. $\square$

**Lemma** If $X$ is cubelike, its eigenvalues are integers.

**Proof**

(a) the eigenvalues of $A_w$ are $\pm 1$, with equal multiplicity.

(b) The matrices $\{A_w : w \in G\}$ are normal & commute,

so $\mathbb{Z}_2^d$ has a basis of eigenvectors. If $z$ is one of these eigenvectors

eigenvalue of $A_w$, integer

$$A(x)z = \sum_{w \in G} A_w z = \sum_w \lambda_w z$$

If the eigenvalues of a graph are integers, there is an integer $m$ such that $U(m\pi) = I$. (If $U(t) = \sum_r e^{it\theta_r} E_r$, choose $m$ so $m\theta_r$ is even for all $r$.)

We say that $X$ is periodic if there is a time such that $U(t) = \gamma I$ for some $\gamma$.

norm 1

**Notes:** (a) $|\gamma| = 1$, because $\gamma I$ is unitary. (b) $\gamma^n = \det(\gamma I)$ $= \det(\exp(itA)) = \exp(\mathrm{tr}(itA)) = \exp(0) = 1 \to \gamma$ is an $n$-th root of unity. (c) $P_3$ is periodic, but its eigenvalues are not integers.

**Theorem.** Let $X = X(\mathbb{Z}_2^d, C)$. If $\sum_{u \in C} u = c \neq 0$, $X$ admits perfect state transfer from $v$ to $v+c$ at time $\pi/2$ for all $v$ in $\mathbb{Z}_2^d$.

*Proof.* We have $A = \sum_{w \in C} A_w$.

(a) If $B^2 = I$ then $\exp(itB) = \cos(t) I + i \sin(t) B$

Just use the series expansion.

(c) $U(\frac{\pi}{2}) = i^{|\mathcal{C}|} A_{\mathcal{C}}$

$$U(t) = \exp(it \sum_{w \in \mathcal{C}} A_w) = \prod_{w \in \mathcal{C}} \exp(itA_w)$$

$$= \prod_{w \in \mathcal{C}} (\cos(t) I + i \sin(t) A_w)$$

and hence:

$$U(\tfrac{\pi}{2}) = i^{|\mathcal{C}|} \prod_{w \in \mathcal{C}} A_w = i^{|\mathcal{C}|} A_{\mathcal{C}}. \qquad \square$$

Vertex-transitive graphs

Assume $X$ is vertex-transitive on $n$ vertices and admits pst from vertex 1 to vertex 2

phase factor

at time $\tau$. Then $U(\tau)e_1e_1^T U(-\tau) = e_2e_2^T$, implying that $U(\tau)e_1 = \gamma e_2$, where $\gamma$ is complex, norm one.

So $\overline{U(\tau)}\, e_1 = \bar{\gamma} e_2$ & $U(\tau)e_2 = \gamma e_1$. If $P \in Aut(X)$, then $PA = AP$ and so $PU(t) = U(t)P$ $\forall t$. Hence $\gamma P e_2 = U(\tau) P e_1$.

It follows (we may assume) that $U(\tau) = \gamma \begin{bmatrix} \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \end{bmatrix}$

Further $\gamma^{-1} U(\tau) \in \mathrm{Aut}(X)$, in fact it lies in the centre of $\mathrm{Aut}(X)$. Why?

Thus no pst on Petersen $(\mathrm{Aut}(\mathrm{Pete}) \cong \mathrm{Sym}(5))$.

Perron-Frobenius

The Perron-Frobenius theorem(s) provide important information about the largest eigenvalue of a non-negative, square, real matrix. Since our main concerns lie with symmetric matrices, we work out part of the theorem in this case.

If $M$ is real & square, define

$$Q_M(x) := \frac{x^T M x}{x^T x} \quad (x \neq 0);$$

we call it the Rayleigh quotient of $M$.

If $Mx = \theta x$, then $Q_M(x) = \theta$.

**Theorem** Let $M$ be a symmetric real matrix. Then

$$\max_{\|x\|=1} \{ Q_M(x) \} = \max \{ Q_M(x) : x \neq 0 \}$$

is realized by an eigenvector $z$ of $M$. If $M$ is non-negative, then $z$ is non-negative. If the underlying graph of $M$ is connected, all entries of $z$ are positive and the eigenvalue belonging to $z$ is simple.

$\theta_r : E_r z \neq 0$

**Proof**

(existence of maximum) $R_M(x)$ is a continuous function on the sphere $\{x : \|x\| = 1\}$, hence it attains its maximum.

(maximum $\to$ eigenvector) Assume the unit vector $z$ maximizes $R_M$. Let $h$ be a 'small' vector orthogonal to $z$.

So $\quad \|z+h\|^2 = \langle z,z\rangle + 2\langle h,z\rangle + \langle h,h\rangle \approx \langle z,z\rangle$ up to linear terms

Then

$$R_M(z+h) = (z+h)^T M (z+h)$$

$$= h^T M z$$

$$= z^T M z + h^T M z + z^T M h + \cancel{h^T M h}$$

and if $z$ maximizes $R_M$, then $h^T M z = 0$. So

$h^T z = 0$ implies $h^T M z = 0$, equivalently $h \in z^\perp$

implies $h \in (Mz)^\perp$ and thus $z^\perp \subseteq (Mz)^\perp$ and

hence $Mz \in \langle z \rangle$, i.e. $z$ is an eigenvector.

($M \geq 0 \Rightarrow z \geq 0$) If $N$ is a matrix, then $|N|$ is the matrix

$$|N|_{i,j} = |N_{i,j}| \qquad \forall i,j$$

Since $M \geq 0$,

$$|x^T M x| \leq |x|^T |M| \, |x| = \underbrace{|x|^T M \, |x|}_{R_M(|x|)}$$

Note that $\| \, |x| \, \| = \|x\|$. It follows that if the unit vector $z$ maximizes $R_M$, we may assume $z \geq 0$.

($M \geq 0$, graph of $M$ connected $\Rightarrow z > 0$)

$$f \quad \Downarrow \quad (Af)(u) = \sum_{v \sim u} f(v)$$

If $Mz = \lambda z$, then

$$\lambda z_k = \sum_{\ell} M_{k,\ell} z_\ell .$$

Hence if $z > 0$ and $M_{k,\ell} > 0$, we see that $z_k > 0$.

Now induct.

graph connected

(eigenvalue of maximizer is simple)

Assume $z$ maximizes $R_M$ and $Mz = \lambda z$. If $\lambda$ is not simple, there is an eigenvector $y$ with eigenvalue $\lambda$ such that $y^T z = 0$. Some entry of $y$ must be negative and so there is a constant $c$ such $z + cy$ is non-negative with a zero entry. As $M(z+cy) = \lambda(z+cy)$, the proof of the previous claim produces a contradiction. □

The spectral radius of a square matrix is the maximum absolute value of an eigenvalue. For a symmetric real matrix, this is just the largest eigenvalue (which we often denote by $\rho$). For a connected graph, $\rho$ is simple and we may assume the associated eigenvector is positive. We call this the Perron vector. $\left(Az = \lambda z \Rightarrow A(-z) = \lambda(-z)\right)$

# Size of the eigenvalue support

We know that size of the eigenvalue support of a pure state $e_a e_a^T$ is $|\{\theta_r : E_r e_a \neq 0\}|$. The eigenvalue support always contains $\rho$, so the size is at least one. If it were exactly one, then since $e_a = \sum_r E_r e_a$, we would have to conclude that $e_a$ was the Perron vector (and $|N(x)|=1$).

There is a useful lower bound on the size of the eigenvalue support. This generalizes one of the first results you meet in spectral theory, which we present now.

**Lemma** If $X$ is a graph with diameter $d$, then

$$d+1 \leq \#\text{distinct eigenvalues of } X.$$

**Proof** The adjacency algebra of $X$ is the ring $\mathbb{R}[A]$ of all polynomials in $A$. The spectral idempotents form an orthogonal basis for this algebra, and therefore $\dim \mathbb{R}[A] = \#$ distinct eigenvalues.

On the other side, if $d = \operatorname{diam}(X)$ then the polynomials $(A+I)^0, A+I, \dots (A+I)^d$ are linearly independent. $\square$