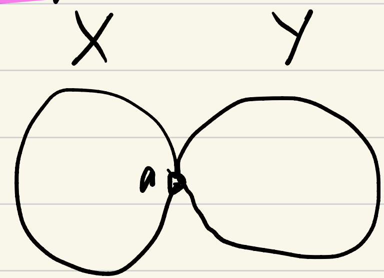
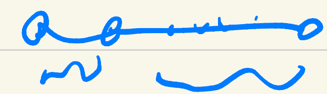
L 26/01

One-sum formula

Z:





$$C_a^{(1)}(Z,t) = C_a^{(1)}(X,t) + C_a^{(1)}(Y,t)$$

$$C_a(Z,t^{-1}) = \frac{t\,\phi(Z\setminus a)}{\phi(Z)}$$

$$1 - \frac{1}{C_a(Z)} = 1 - \frac{1}{C_a(X)} + 1 - \frac{1}{C_a(Y)}$$

$$1 - \frac{\phi(Z)}{t\,\phi(Z\setminus a)} = 1 - \frac{\phi(X)}{t\,\phi(X\setminus a)} + 1 - \frac{\phi(Y)}{t\,\phi(Y\setminus a)}$$

Schwenk

$$\boxed{\phi(Z) = \phi(X)\,\phi(Y\setminus a) + \phi(X\setminus a)\,\phi(Y) - t\,\phi(X\setminus a)\,\phi(Y\setminus a)}$$

Cospectral vertices

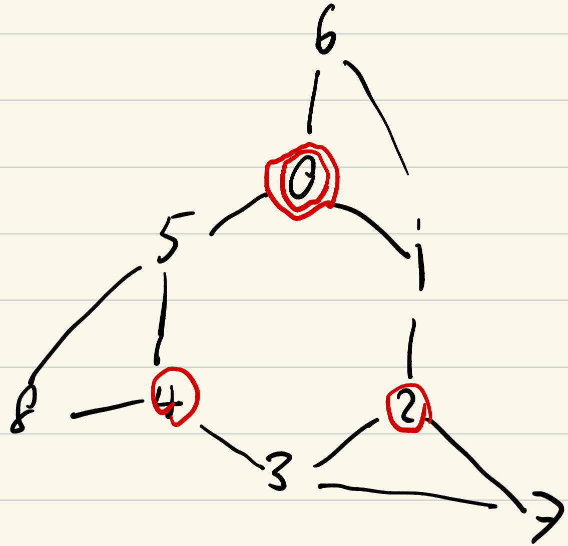
Vertices a & b in X are cospectral if $X \setminus a$ and $X \setminus b$ are cospectral. E.g.

(a) if there is g in $\mathrm{Aut}(X)$ s.t. $ag = b$ then $X \setminus a \cong X \setminus b$. We say a & b are similar.

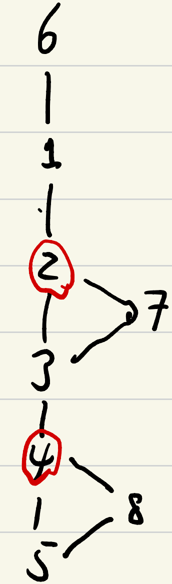
(b) $X \setminus a \cong X \setminus b$, but a & b are not similar.

We say they are pseudosimilar.

(c)



X



$X \setminus 0$

In $X \setminus 0$, vertices 2 & 4 are pseudosimilar.

(d)

Schwenk's tree

$$
\begin{tikzcd}[column sep=small]
 & & & & & 9 \arrow[d, no head] & & \\
1 \arrow[r, no head] & 2 \arrow[r, no head] & 3 \arrow[r, no head] & 4 \arrow[r, no head] & 5 \arrow[r, no head] & 6 \arrow[r, no head] & 7 \arrow[r, no head] & 8
\end{tikzcd}
$$

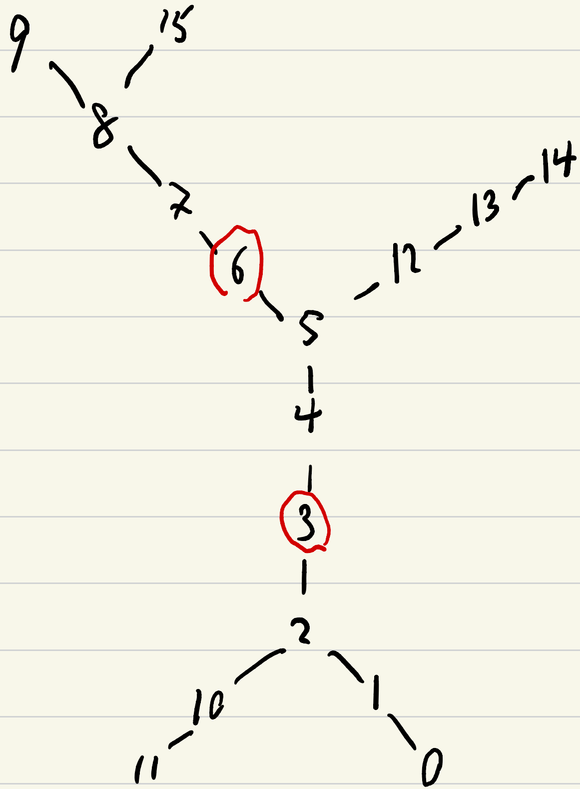
$S \setminus 7$ $\quad t(P_4 P_3 - P_3 P_2) = t(t^3-2t)(t^4-4t^2+2)$

$S \setminus 4$ $\quad (t^3-2t)(t^5-4t^3+2t)$

(e) Any two vertices in a strongly regular graph.

BDM tree

(16 vxs)



Theorem Assume $a, b \in V(X)$. The following are equivalent:

(a) a & b are cospectral

(b) $X \setminus a$ & $X \setminus b$ are cospectral

(c) $(A^k)_{a,a} = (A^k)_{bb}$ $\forall k \geq 0$

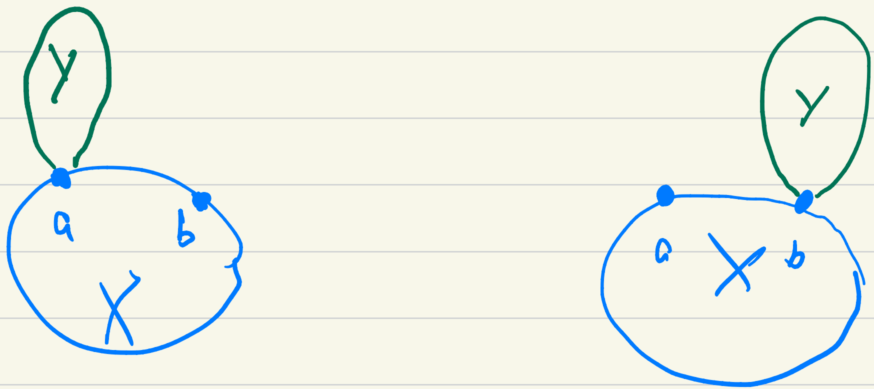
(d) $(E_r)_{aa} = (E_r)_{bb}$ for each idempotent E_r

(e) $(e_a - e_b)^T A^k (e_a + e_b) = 0$ $\forall k \geq 0$

(f There is a symmetric matrix Q s.t. $Q^2 = I$ & $QA = AQ$ & $Qe_a = e_b$. $\square$

Suppose a, b are cospectral vertices in X.

Then so are these two graphs.



We say X is walk-regular if A^k has constant diagonal for all $k \geq 0$.

Remarks

(a) Walk-regular graphs are regular.

(b) If X is walk-regular, so is $\overline{X}$.

(c) Vertex-transitive graphs are walk-regular

(d) If X & Y are walk-regular, so are $X \times Y$ & $X \circ Y$.

X is walk-regular if $I \circ E_r = c_r I$ for all r.

X is 1-walk-regular if it is walk-regular & $A \circ E_r = d_r A$ $\forall r$.

Theorem Assume X is 1-walk regular with valency k and least eigenvalue τ. Then the maximum size $\omega(X)$ a clique in X satisfies $\omega(X) \leq 1 - \frac{k}{\tau}$.

Proof First, $\mathrm{tr}(E_r) = m_r$ and so $(E_r)_{a,a} = \frac{m_r}{n}$

Second $AE_r = \theta_r E_r$ and so

$$\theta_r m_r = \mathrm{tr}(AE_r) = \mathrm{sum}(A \circ E_r) = nk\,(E_r)_{ab} \quad (a \sim b)$$

and $(E_r)_{ab} = \frac{m_r \theta_r}{nk}$. Hence the diagonal entries of $\frac{n}{m_r} E_r$ are 1 and the entries $(E_r)_{a,b}$ $(a \sim b)$ are $\frac{\theta_r}{k}$.

Now $E_r \succcurlyeq 0$ and therefore any principal submatrix of E_r is positive semidefinite.

Assume C is a clique. The submatrix of E_r with rows & columns indexed by vertices in C has the form

$$\underbrace{\left.\begin{bmatrix} 1 & & \theta_{r/k} \\ & \ddots & \\ \theta_{r/k} & & 1 \end{bmatrix}\right\}|C|}_{|C|}$$

?

Since this is psd, the row sums are non-negative.

So $\quad 1 + (c-1)\frac{\tau}{k} \geq 0 \Rightarrow (c-1)\frac{\tau}{k} \geq -1$

$$\Rightarrow c-1 \leq (-1)\frac{k}{\tau}$$

$$\Rightarrow c \leq 1 - \frac{k}{\tau}. \qquad \square$$

Remark (Hoffman) $\chi(X) \geq 1 - \frac{k}{\tau}$. $\square$

Quantum walks

No-clon theorem

$D \longmapsto D \otimes D$ cloning

$\alpha D \longmapsto \alpha^2 D \otimes D$ not linear, oops

$\underbrace{\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{n}$ $\dim = 2^n$

qubits

Ingredients:

(a) state space $\mathbb{C}^n$

(b) states: $D \succeq 0$, $tr(D) = 1$

$P_r = e_r e_r^T$

outcome is a vertex a

prob. of a is $D_{a,a}$

(c) measurement – $D \circ I$

(d) transitions – unitary operators

$P_1, \dots, P_m$ $\quad \sum P_i = I$, $P_i \succeq 0$

$\langle P, D \rangle$

POVM

States

a) pure states: 1-dimensional subspaces, spanned by unit vectors ξ

b) in all cases: density matrices; pure states are rank-1 density matrices $\xi\xi^*$.

$e_a e_a^T$ is a vertex state

$|a\rangle\langle a|$ bracket notation

Transition operators

(a) discrete walk: unitary U, $D \to U^k D U^{-k}$

$k = 0, 1, \ldots$

(b) continuous walk: $U = \exp(itH)$, H hermitian

$\bar{U} = U^{-1}$

Special cases: $H =$

(i) $A = A(X)$ (ii) $\Delta \pm A$ (iii) iS, S real skew symmetric

Laplacian $\qquad U(t) = \exp(-tS)$

signless Laplacian

Example: (continuous walk)

initial state: pure, spanned by z, density zz^*

state at time t: $U(t)zz^*U(-t)$

So the new state is spanned by the vector $U(t)z$; since $U(t)$ is unitary this vector is a unit vector.

If we measure with respect to the standard basis, the outcome is a with probability

$$\langle U(t) z z^* U(-t), e_a e_a^T \rangle$$

$$= tr\left(U(t) z z^* U(-t) e_a e_a^T \right)$$

$$= e_a^T U(t) z \cdot z^* U(-t) e_a$$

$$= |e_a^T U(t) z|^2$$

If our initial states are vertex states, and we start at e_b, the probability we observe a is

$$|e_a^T U(t) e_b|^2 = |U(t)_{a,b}|^2 = U(t) \circ \overline{U(t)}$$

mixing matrix

and $\bar{U}(t) = U(-t)$.

The mixing matrix of a walk is

$$M(t) = U(t) \circ \overline{U(t)} = U(t) \circ U(-t)$$

It is row & column stochastic. Its ab-entry is the probability that a walk starting at b is "on" a at time t.

e.g. $X = K_2$ $\quad H = A(X) = A$

initial state $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$U(t) = \exp(itA) = \sum_m \frac{(it)^m}{m!} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^m$$

$$= \cos(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} c & is \\ is & c \end{pmatrix}$$