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Orbits

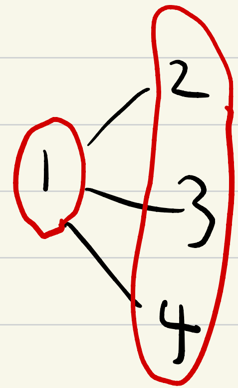
Assume G acts as a group of permutations of a set V. Define elements u & v of V to be G-equivalent if there is g in G such that $ug = v$. Denote this by $u \approx v$. Then

(a) $u \approx u$ reflexive

(b) if $u \approx v$, then $v \approx u$ symmetric

(c) if $u \underset{g}{\approx} v$ & $v \underset{h}{\approx} w$, then $u \approx w$ transitive

X



$$V = \{1, 2, 3, 4\}$$

Then $\approx$ is an equivalence relation. Its equivalence class are the orbits of G on V. We say G is transitive if G has just one orbit.

If $i \in V$, then $G_i = \{g \in G : i^g = i\}$; it is the stabilizer of i in G. It is a subgroup of G.

Assume Ω is an orbit of G acting on V and that $i \in \Omega$. Define

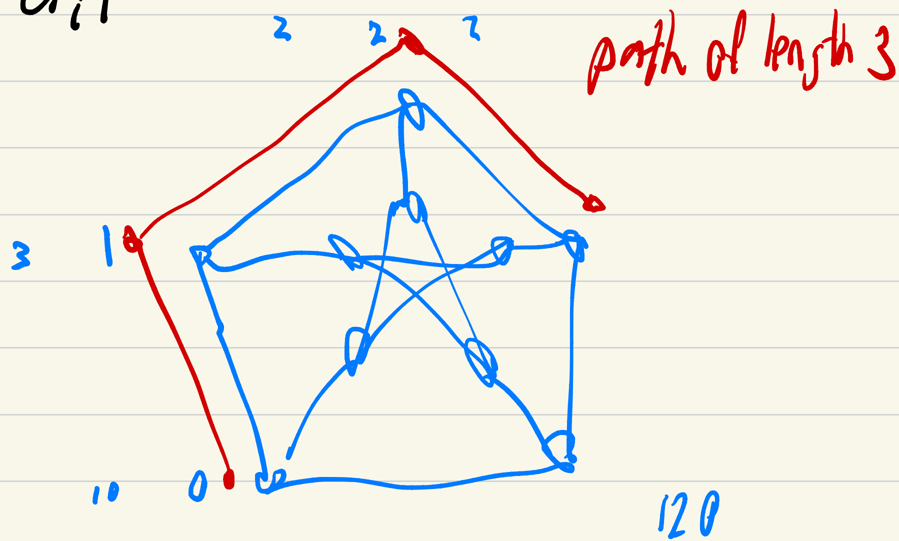
$$G_{i\to j} := \{ g \in G : ig = j \}.$$

Then $G_{i\to j}$ is a coset of G_i (in G) and the cosets $\{ G_{i\to j} : j \in \Omega \}$ partition G.

This leads to the orbit-stabiliser relation:

Theorem If Ω is an orbit of G and $i \in \Omega$, then

$$|\Omega| = |G : G_i|$$



Orbitals

If G is a group of permutations of V, it also gives rise to permutations on the set of k-subsets of V, and on k-tuples of elements of V. In particular there is an action on ordered pairs:

$$g : (u, v) \longmapsto (ug, vg)$$

An orbit of G on $V \times V$ is known as an orbital.

Example Take our permutation group to be $Sym(7)$ acting on the 3-element subsets of $\{0,1,\dots,6\}$. Denote this set of 35 triples by Ω. What are the orbitals?

An **arc** in X is an ordered pair of adjacent vxs.

$u \sim v$ (u,v) $(5,4)$

If Ω is an orbit of G on V, then

$$\{(u,u) : u \in \Omega\}$$

is an orbital. Hence G is transitive on V if and only if the diagonal $\{(u,u) : u \in V\}$ is an orbital.

It is important to note that an orbital on V is a directed graph; further it is vertex and arc-transitive.

Theorem. Let G be a permutation group acting on V and let $\Omega_0, \Omega_1, \ldots, \Omega_d$ be the distinct orbitals of G. Let B_i be the adjacency matrix of Ω_i. Then

(a) If $i \neq j$, then $B_i \circ B_j = 0$

(b) $\sum_i B_i = J$

Exer (c) $B_i^T \in \{B_0, \ldots, B_d\}$

(d) I is a sum of matrices from $\{B_0, \ldots, B_d\}$

(e) For all i & j, $B_iB_j \in \operatorname{span}\{B_0, \dots, B_d\}$

The matrices $B_0, \dots, B_d$ generate a matrix algebra of dim $d+1$ that is Schur-closed and contains J. We call it a coherent algebra.

The group G is transitive on V if & only if

usually $I = B_0$

$I \in \{B_0, \dots, B_d\}$; in this case we say the coherent algebra is homogeneous.

Let G be a group & let C be a subset of G.

The **Cayley digraph** $X(G,C)$ has:

$$V(X) = G$$

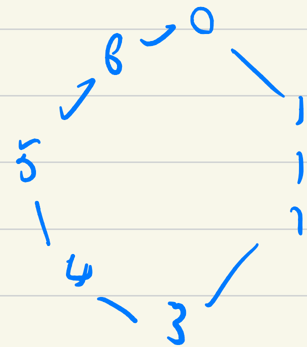
$$\text{arcs}(X) = \{ (g,h) : hg^{-1} \in C \}$$

$(1,c)$ is an arc $\forall c$ in C

- $X(G,\mathcal{C})$ has loops $\iff 1\in\mathcal{C}$

- $X(G,\mathcal{C})$ is a graph

  $\iff \mathcal{C}=\mathcal{C}^{-1}$ $\left(=\{g^{-1} : g\in\mathcal{C}\}\right)$



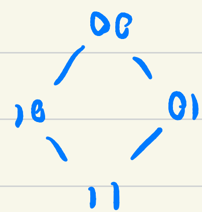
$X(\mathbb{Z}_7, \{1,-1\})$

Examples

(a) A Cayley graph for $\mathbb{Z}_n$ is a circulant

(b) A Cayley graph for $\mathbb{Z}_2^d$ is a cubelike graph

(e.g. Q_d)



$\{(0,1) \quad (1,0)\}$

If $g \in G$, the map

$$e_g : x \mapsto xg$$

is a permutation of G.

fixed-point free if $g \neq 1$.

Theorem For each g in G, the permutation e_g is an automorphism of $X(G, b)$.

Proof If $x, y \in V(X)$, then

$\rho : (x, y) \longmapsto (xg, yg)$

and $yg(xg)^{-1} = ygg^{-1}x^{-1}$. So $x \sim y \Leftrightarrow \rho_g(x) \sim \rho_g(y)$. $\square$

Lemma If $\psi \in \mathrm{Aut}(G)$ and $\mathcal{C} = \mathcal{C}^{\psi} = \{c^{\psi} : c \in \mathcal{C}\}$

then $\psi \in \mathrm{Aut}(X(G, \mathcal{C}))$ and $\psi(1) = 1$. $\square$

If G is abelian and $\mathcal{C} \subseteq G$ that contains an element of order at least three, then the map $x \longmapsto x^{-1}$ is a non-trivial automorphism of $X(G, \mathcal{C})$ that fixes 1 and fixes $\mathcal{C}$ as a set.