colourings & derangements

Let $\mathcal{P}$ be a quantum homomorphism from X to Y of index d. If M is $d \times d$, define $\langle \mathcal{P}, M \rangle$ to be the $|V(X)| \times |V(Y)|$ matrix with

$$\langle \mathcal{P}, M \rangle_{i,j} = \langle P_{i,j}, M \rangle$$

Usually M will be a density matrix and then we are using the rows of $\mathcal{P}$ as a measurement on M.

Lemma If $\mathcal{P}: X \xrightarrow{q} Y$ and $\mathcal{P}$ has index d & D is a $d \times d$ density matrix, then $\langle \mathcal{P}, D \rangle$ is row-stochastic. $\square$

Lemma Assume $\mathcal{P}$ & Q are quantum homomorphism from X to Y of index d & e respectively, and assume that C & D are density matrices of order $d \times d$ & $e \times e$ respectively. Then $\langle \mathcal{P} \oplus Q, C \oplus D \rangle = \langle \mathcal{P}, C \rangle + \langle Q, D \rangle$. $\square$

Lemma Suppose $\mathcal{P} : X \xrightarrow{q} Y$ and $\mathcal{Q} : Y \to Z$ are quantum homomorphisms of index d and e respectively, and that D_1 & D_2 are density matrices of orders $d \times d$ and $e \times e$ respectively.

Then $\langle \mathcal{P} * \mathcal{Q}, C \otimes D \rangle = \langle \mathcal{P}, C \rangle \langle \mathcal{Q}, D \rangle$. $\square$

Quantum Colourings

A quantum m-colouring of a graph X is a quantum homomorphism from X to K_m. The minimum possible value of m is the quantum chromatic number of X, denoted χ_q. As K_n is vertex transitive, we may assume all entries of the quantum n-colouring have the same rank. If this rank is r, we say we have a quantum rank-r colouring.

Fixing r, the minimum value of m such that X has a quantum m-colouring of rank r is denote $\chi_q^{(r)}(X)$. We have

$$\chi(X) \geq \chi_q^{(1)}(X) \geq \chi_q^{(2)}(X) \geq \cdots \geq \chi_q(X).$$

Lemma $K_m \xrightarrow{q} K_n$ if & only if $m \leq n$.

Proof If $m \leq n$, then $K_m \to K_n$ and thus $K_m \xrightarrow{q} K_n$

So suppose that $\mathcal{P}: K_m \xrightarrow{q} K_n$, and that each projection in $\mathcal{P}$ has rank r. For y in $V(K_m)$, the projections in the y-column of $\mathcal{P}$ are pairwise orthogonal. If the index of $\mathcal{P}$ is d, then $\sum_j P_{y,j} = I_d$ & so $\sum_{i,j} P_{i,j} = m I_d$.

The projections in a column of $\mathcal{P}$ must also be pairwise orthogonal and therefore $\sum_y P_{y,i} \preceq I_d$.

Hence $\sum_{y,i} P_{y,i} \preceq n I_d$, which implies $m \leq n$. $\square$

Lemma If $X \xrightarrow{q} K_2$, then X is bipartite. $\square$

Unitary derangements

A permutation in $Sym(n)$ is an derangement if it has no fixed points. The set of derangements does not contain the identity, but it does contain the inverse of each elements. We use $D(n)$ to denote the Cayley graph for $Sym(n)$ with the derangements as the connection set.

We note the following:

(a) the cosets of a point-stabilizer form n pairwise-disjoint cocliques of size $(n-1)!$

(b) the rows of an $n \times n$ Latin square form a clique of size n; any such clique is maximal.

So $n = \omega(\mathcal{D}(n)) \leq \chi(\mathcal{D}(n)) \leq n$, and hence $\chi(\mathcal{D}(n)) = n$.

Now for the unitary version.

$n = |V(X)|$

Assume $\mathcal{N} : X \to K_m$ is a quantum m-colouring of X with rank r. If N is an entry of $\mathcal{N}$, there there is a

index

$d \times r$ matrix M with pairwise orthonormal columns (i.e $M^*M = I_r$) such that $N = MM^*$. Let $U_{i,j}$ be the $n \times r$ matrix assigned to $\mathcal{N}_{i,j}$.

Consider the matrix

$$\begin{bmatrix} u_{1,1} & \cdots & u_{1,m} \\ \vdots & & \vdots \\ u_{n,1} & \cdots & u_{n,m} \end{bmatrix}, \qquad \begin{matrix} u_1 \\ \vdots \\ u_m \end{matrix} \qquad u_1^* u_2 \quad \begin{matrix} 0 & & \\ & 0 \cdot & \\ & & \cdot 0 \end{matrix}$$

which has order $nd \times mr = nd \times d$. The vectors in a row of this matrix form a $d \times d$ unitary matrix; let U_i denote the unitary matrix coming from the i-th row.

Thus $U_i = [M_{i,1} \cdots M_{i,m}]$.

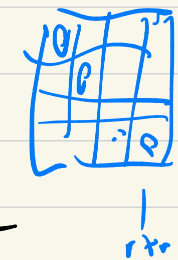
Lemma If i & j are adjacent in X, then $M_{i,r}^* M_{j,r} = 0$ for $r = 1, \dots, m$.

Proof If $i \underset{X}{\sim} j$, then $0 = N_{i,r} N_{j,r} = M_{i,r} M_{i,r}^* M_{j,r} M_{j,r}^*$

and so $0 = M_{i,r}^* \cdot 0 \cdot M_{j,r} = M_{i,r}^* M_{i,r} M_{i,r}^* M_{j,r} M_{j,r}^* M_{j,r} = M_{i,r}^* M_{j,r}$. $\square$

It follows that $U_i^* U_j$ is a $d \times d$ unitary matrix with $m = d/r$ diagonal $r \times r$ blocks of zero.

A unitary derangement of index r is a unitary matrix U such that $U \circ (I_m \otimes J_r) = 0$. Thus it is $mr \times mr$ with diagonal $r \times r$ blocks zero. The set of $d \times d$ unitary derangements of index r does not contain I and is closed under inversion. We use $\mathcal{UD}_r(d)$ to denote the Cayley graph on $U(d)$ with the index-r unitary derangements as connection set.



Since $Sym(n) \leq U(n)$, a derangement in $Sym(n)$ is a unitary derangement of index one.

$|V(X)| = mr$

Theorem X admits a rank-r quantum m-colouring if and only if $X \to UD(mr)$. $\square$

We next derive bounds on $\chi_q^{(1)}(X)$.

Let $\Omega(m)$ denote the graph with the unit vectors in $\mathbb{C}^m$ as its vertices, with two vectors adjacent if they are orthogonal. This is the orthogonality graph. The subgraph of $\Omega(m)$ induced by the flat unit vectors is $\Omega^b(m)$.

We need a preliminary lemma.

Lemma Assume W is a flat unitary matrix and D_1 & D_2 are diagonal matrices, all of order $m \times m$. Then

$$\langle D_1, W^* D_2 W \rangle = tr(D_1)\, tr(D_2) \qquad \square$$

Theorem $K_m \to \Omega^b(m) \to \mathcal{HD}_1(m) \to \Omega(m)$

Proof. For the first arrow, we note that the columns of any flat unitary matrix form a clique in $\Omega(m)$.

For the second, if $z \in \mathbb{C}^m$ define the diagonal matrix D_z by $(D_z)_{i,i} = z_i$. Let W be a flat unitary matrix. If z is flat, the map $z \to D_z W$ takes vertices of $\Omega^b(m)$ to unitary matrices.

Consider the matrix

$$Q = (D_y W)^* D_z W.$$

Then $Q_{i,j} = tr\langle e_i e_i^T, W^* D_y^* D_z W\rangle$ and applying the lemma

(with $D_1 = e_i e_i^T$ & $D_2 = D_y^* D_z$) yields that

$$tr\langle e_i e_i^T, W^* D_y^* D_z W\rangle = tr\langle D_1, W^* D_y^* D_z W\rangle = 1 \cdot tr(D_y^* D_z) = \langle y, z\rangle.$$

$Q_{ij} = 0 \iff \langle y, z\rangle$ and there Q is a derangement $\iff \langle y, z\rangle = 0$.

Finally we have $\langle Me_1, Ne_1 \rangle = (M^* N)_{1,1}$ and so

$\Phi \mapsto \Phi_{e_1}$ is the required homomorphism. $\square$

The minimal value of m such that $X \to \Omega(m)$

is the orthogonal rank of X, denoted $\underset{\cdots}{\xi}(X)$. The minimal

value of m such that $X \to \Omega^b(m)$ is denoted $\xi^b(X)$.

The homomorphisms we have derived imply three quarters

of the following:

Theorem For any graph X

$$\chi(X) \geq \xi^{b}(X) \geq \chi_q^{(1)}(X) \geq \xi(X).$$

Proof The previous theorem yields all but the last inequality. For this, if $\gamma: X \to K_m$, the rows of the characteristic matrix of the partition determined by γ are standard basis vectors. Hence we have a homomorphism from X to the subgraph of $\Omega^{b}(X)$ induced by the std basis vectors.