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Algebraic states

Gelfond-Schneider Theorem

If α, β are algebraic numbers, $\alpha \neq 0, 1$ and β is irrational,

then α^{β} is transcendental. $\square$

A matrix is algebraic if its entries are algebraic numbers.

Theorem Let D_1 & D_2 be algebraic density matrices. If the Hamiltonian of the quantum walk on X is algebraic and there is perfect state transfer from D_1 to D_2, then the ratio condition holds.

Proof Suppose

$$D_2 = U(t) D_1 U(-t) = \sum_{r,s} e^{it(\theta_r - \theta_s)} E_r D_1 E_s$$

If the Hamiltonian is algebraic, its eigenvalues are algebraic and accordingly the spectral idempotents are algebraic.

We have

$$E_k D_2 E_l = e^{it(\theta_k - \theta_l)} E_k D_1 E_l$$

and therefore $e^{it(\theta_k - \theta_l)}$ is algebraic whenever $E_k D_1 E_l \neq 0$

If $k \neq l$,

$$e^{it(\theta_r - \theta_s)} = \left(e^{it(\theta_k - \theta_l)}\right)^{\frac{\theta_r - \theta_s}{\theta_k - \theta_l}} \qquad (1)$$

By the Gelfond-Schneider theorem, with $\alpha = e^{it(\theta_r - \theta_s)}$ and $\beta = \frac{\theta_r - \theta_s}{\theta_k - \theta_l}$, we have that if β is algebraic & not rational, the right side of (1) is transcendental. Therefore β must be rational, giving us the ratio condition. $\square$

Local uniform mixing

We have local uniform mixing at time t, starting from a state D, if $D(t) = U(t) D U(-t)$ has constant diagonal. (The uniform mixing is local relative to D.)

In most cases, the initial state will be a vertex-state D_a.

We have uniform mixing on X at time t if there is local uniform mixing from each vertex state at time t.

The easiest way to get uniform mixing is to get local uniform mixing (relative to a vertex) on a vertex-transitive graph.

We see that $U(t)e_a e_a^T U(-t)$ has constant diagonal if & only if $U(t)e_a$ is flat. So, local uniform mixing from a iff the a-column of $U(t)$ is flat; uniform mixing if & only if $U(t)$ itself is flat.

$M(t) = U(t) \circ U(-t)$

Lemma We have uniform mixing at time t on X if and only if $U(t)$ is flat or, equivalently, $M(t) = \frac{1}{|V(X)|} J$. $\square$

Lemma Assume X is bipartite. If there is local uniform mixing at time t from vertex a, then $D_a(t)$ is algebraic.

Proof Recall that if X is bipartite, $U(t)$ has the form

$$\begin{bmatrix} C_1(t) & iK(t) \\ iK^T(t) & C_2(t) \end{bmatrix}$$

where $C_1(t)$, $C_2(t)$ are real & symmetric & $K(t)$ is real.

If $n = |V(X)|$ and a column of $U(A)$ is flat, then all entries of this column lie in $\{\pm\frac{1}{\sqrt{n}}, \pm\frac{i}{\sqrt{n}}\}$ — thus they are algebraic numbers (and $D_a(A)$ is algebraic).

Corollary If X is bipartite and admits local uniform mixing from the vertex a, then the eigenvalue support of a satisfies the ratio condition & a is periodic. $\square$

Pretty good state transfer

pgst

There is pretty good state transfer from D_1 to D_2 if D_2 lies in the closure of the Q-orbit. Equivalently, for each $\varepsilon > 0$ there is a time t such that

$$\| U(t) D_1 U(-t) - D_2 \| < \varepsilon.$$

Claims

(a) If there is pgst from D_1 to D_2, there is pgst from D_2 to D_1

(b) If there is pgst from D_1 to D_2, then D_1 & D_2 are strongly cospectral

For one example, see the treatment of P_+ in Chapter 1 of the notes.

We view pst as a special case of pgst. We justify our claims. Let P & Q be density matrices.

(a) Since the complex conjugate of $U(t)PU(-t)-Q$ is $U(-t)PU(t)-Q$

$$\|U(-t)\,P\,U(t)-Q\| = \|P-U(t)\,Q\,U(-t)\|.$$

So we have pgst from P to Q if & only there is pgst from Q to P.

(b) Assume A is real. We have

$$U(t)PU(-t) = \sum_{r,s} e^{it(\theta_r-\theta_s)} E_r P E_s,$$

$$Q = \sum_{r,s} E_r Q E_s,$$

and hence if $U(t)PU(-t)$ is close to Q, then

$$e^{it(\theta_k-\theta_l)} E_k P E_l, \quad E_k Q E_l$$

are close. It follows that $E_k P E_l = \pm E_k Q E_l$ $\forall k,l$ and $E_k P E_k = E_k Q E_k$.

So P & Q are strongly cospectral.

For the proof of the following, see the notes:

Theorem There is pgst between the end vertices of P_n if and only if $n = p-1$ or $2p-1$ (p prime) or $n = 2^k - 1$ $\square$

So P_{11} is the smallest path which does not have pgst between its end-vertices.

Averages

Recall that

$$D(t) = U(t) D U(-t).$$

We define the average state $\hat{D}$ by

$$\hat{D} = \lim_{T \to \infty} \frac{1}{T} \int_0^T U(t) D U(-t) \, dt$$

Does it exist? We have $U(t) D U(-t) = \sum_{r,s} e^{it(\theta_r - \theta_s)} E_r D E_s$

and if $r \neq s$, then $\lim_{T \to \infty} \frac{1}{T} \int_0^T e^{it(\theta_r - \theta_s)} dt = 0$. So

Lemma $\hat{D} = \sum_r E_r D E_r$. $\square$

So $\hat{D} \succcurlyeq 0$ and $tr(\hat{D}) = \sum tr(E_r D E_r) = \sum_r tr(D E_r^2)$

$$= \sum_r tr(D E_r)$$

$$= tr(D \sum E_r)$$

$$= tr(DI)$$

$$= tr(D)$$

$$= 1.$$

Note that the terms in $\sum_r E_r D E_r$ are pairwise orthogonal.

Also if $\hat{D}_1 = \hat{D}_2$, then

$$E_k D_1 E_k = E_k \left(\sum_r E_r D_1 E_r\right) = E_k \left(\sum_r E_r D_2 E_r\right) = E_k D_2 E_k .$$

Thus if the sums are equal, the terms are equal.

Further:

Theorem If D_1 and D_2 are strongly cospectral, then $\hat{D}_1 = \hat{D}_2$.

exercise

If D_1 & D_2 are pure states and $\hat{D}_1 = \hat{D}_2$, then D_1 & D_2 are strongly cospectral. □

We have $\hat{\hat{D}} = \hat{D}$, so the map $D \to \hat{D}$ is idempotent

and linear. Also

$$\langle D_1, \hat{D}_2 \rangle = \mathrm{tr}\left(D_1 \sum_r E_r D_2 E_r\right) = \sum_r \mathrm{tr}\left(D_1 E_r D_2 E_r\right) = \sum_r \mathrm{tr}\left(E_r D_1 E_r D_2\right)$$

$\hat{D}_1, D_2 = \dots = \hat{D}_1, \hat{D}_1$

$$= \mathrm{tr}\left(\sum E_r D_1 E_r D_2\right)$$

$$= \langle \hat{D}_1, \hat{D}_2 \rangle$$

and therefore the map $D \to \hat{D}$ is self-adjoint.

Hence it is a projection. Onto what?

If $DA = AD$, then $E_r D = D E_r$ and

$$\hat{D} = \sum_r E_r D E_r = \sum_r D E_r^2 = \sum_r D E_r = D \sum E_r = D$$

As $\hat{D}$ & E_r commute for each r, we have:

Lemma The map $D \to \hat{D}$ is orthogonal projection onto the commutant of A. $\square$

Lemma If $D(t) := U(t) D U(-t)$, then

$$\|D(t) - \hat{D}\| = \|D - \hat{D}\| \quad \forall t$$

Proof

$$\|D(t) - \hat{D}\| = \|U(t) D U(-t) - \hat{D}\| = \|U(t)(D - \hat{D})U(-t)\|$$

$$= \|D - \hat{D}\| \qquad \square$$

We also note that

(Pythagoras)

$$\|D - \hat{D}\|^2 = \|D\|^2 - \|\hat{D}\|$$