L31/01

pst — K_2, P_3, Cartesian power

uniform mixing — Q_d

$a \to b$ pst at time t $\Rightarrow$ pst $b \to a$

pst $\Rightarrow$ periodicity

Bipartite walks & Hadamard matrices

Hadamard matrices: A complex Hadamard matrix is a flat complex $n \times n$ matrix H such that $H^* H = nI$.

A Hadamard matrix is a flat real matrix such that $H^T H = nI$. (So its entries are ± 1.)

H is an $n \times n$ complex Hadamard matrix if and only if $\frac{1}{\sqrt{n}} H$ is a flat unitary matrix

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Examples

(a) If θ is an n-th root of unity, not 1, then the Vandermonde matrix $\left(\theta^{(i-1)(j-1)}\right)_{1\le i,j\le n}$ is a complex Hadamard matrix

(b) H_1 & H_2 are complex Hadamard matrices, so is $H_1 \otimes H_2$.

(c) $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ & $\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$ are Hadamard matrices.

(d) A unitary monomial matrix is a product of a permutation matrix & a diagonal unitary matrix. If M_1 & M_2 are unitary monomial matrices and H is a complex Hadamard matrix, so is $M_1 H M_2$. We say H & $M_1 H M_2$ are equivalent. (If M is complex Hadamard, so is M^*, but in general they are not equivalent.)

Lemma If an $n \times n$ Hadamard matrix exists, then either $n = 2$ or $4 \mid n$. $\square$

Theorem Let X be a bipartite graph on n vertices. Then X admits uniform mixing then $n = 2$ or $4 \mid n$. If, in addition, X is regular, then n is the sum of two integer squares.

Proof (in part) If X is bipartite, we may assume that $U(t) = \begin{pmatrix} C_1 & iS \\ iS^T & C_2 \end{pmatrix}$. If there is uniform mixing at time t, the entries of $U(t)$ are $\pm\frac{1}{\sqrt{n}}$ & $\pm\frac{i}{\sqrt{n}}$.

Now

$$\begin{pmatrix} iI & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} C_1 & iS \\ iS^T & C_2 \end{pmatrix} = \begin{pmatrix} iC_1 & -S \\ iS^T & C_1 \end{pmatrix}\begin{pmatrix} -iI & 0 \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} C_1 & -S \\ S^T & C_1 \end{pmatrix}$$ — real, entries $\pm\frac{1}{\sqrt{n}}$

with all entries $\pm\frac{1}{\sqrt{n}}$. The rest is up to you! $\square$

Periodicity

Theorem Assume D_1 & D_2 are real density matrices and $U(\tau)D_1U(-\tau) = D_2$. Then

(a) $U(\tau)D_2U(-\tau) = D_1$ perfect state transfer

(b) $U(2\tau)D_1U(-2\tau) = D_1$, $U(2\tau)D_2U(-2\tau)$ periodicity $\square$

Note that (b) holds if & only if $U(2\tau)$ commutes with D_1 & D_2.

$$E_k D_1 E_l = e^{it\theta_k} E_k D_2 E_l e^{-it\theta_l} = e^{it(\theta_k - \theta_l)} E_l D_1 E_k$$

Corollary If we have transfer from the real state D_1 to the real state D_2 at time τ, both states are periodic at time 2τ.

Therefore periodicity is a necessary condition for perfect state transfer to occur.

Further if D_1 & D_2 are, say $D_1 = yy^*$, $D_2 = zz^*$

then they are projections. Now if

$$Mww^* = ww^* M \qquad (\|w\|=1)$$

$$Mw = Mww^*w = w\underbrace{w^*Mw}$$

whence w is an eigenvector for M, with eigenvalue w^*Mw.

So pst at τ from D_1 to D_2 occurs if & only if y & z are

eigenvectors for $U(2\tau)$.

The Ratio Condition & Eigenvalue Support

$U(t)\, e_a e_a^* U(-t)$ $\qquad$ $\{U(t) e_a : t \geq 0\}$ $\qquad$ $\mathbb{C}^d$

$\uparrow$

poly in A

$U(t)\, e_a e_a^* U(t)$

$$= \sum_{r,s} e^{it(\theta_r - \theta_s)} E_r e_a e_a^* E_s$$

$\{A^k e_a : k \geq 0\}$

$\{E_r e_a : 1 \leq r \leq m\}$

The set of pairs $(\vartheta_k, \vartheta_\ell)$ such that $E_k D E_\ell \neq 0$ is the eigenvalue support of D, an important concept to which we shall return.

Ignoring this for now, if $e^{i\sigma(\vartheta_k - \vartheta_\ell)} = 1$ then $\sigma = 2m_{k,\ell}\pi$ for some integer $m_{k,\ell}$. This leads us to the ratio condition:

Theorem If D is a periodic state and

(a) (θ_k, θ_l) & (θ_r, θ_s) lie in the eigenvalue support of D_1,

(b) $\theta_k \neq \theta_l$.

$i\tau(\theta_k - \theta_l) = 2m_1\pi i$

$i\tau(\theta_r - \theta_s) = 2m_2\pi i$

then

$$\frac{\theta_r - \theta_s}{\theta_k - \theta_l} \in \mathbb{Q}.$$

$\square$

(This is also a necessary condition for pst.)

Example: P_4

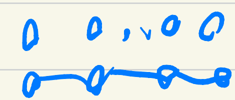
We prove pst does not occur between the end-vertices of P_4

$$\phi(P_4,t) = t^4-3t^2+1 = (t^2-1)^2-t^2 = (t^2-t-1)(t^2+t-1)$$

and the eigenvalues of P_4 are $\frac{1\pm\sqrt{5}}{2}$, $\frac{-1\pm\sqrt{5}}{2}$.

In increasing order, they are

$$\underset{\theta_3}{\frac{-1-\sqrt{5}}{2}},\quad \underset{\theta_2}{\frac{1-\sqrt{5}}{2}},\quad \underset{\theta_1}{\frac{-1+\sqrt{5}}{2}},\quad \underset{\theta_0}{\frac{1+\sqrt{5}}{2}}$$



(Since P_4 is bipartite, the eigenvalues are symmetric about 0.)

No eigenvector for P_4 can be zero on the first or last vertex of P_4 and hence $(E_r e_1 e_1^T E_r)_{3,1} \neq 0$. It follows that the eigenvalue support of the vx 1 consists of all pairs (θ_i, θ_j) $(i \neq j)$. But

$$\frac{\theta_1 - \theta_2}{\theta_1 - \theta_3} = \frac{\sqrt{5}}{1} \notin \mathbb{Q}.$$

Eigenvalue support for pure states

If D is a pure state, say $D = zz^*$ then

$E_r z z^* E_s \neq 0$ if & only $E_r z \neq 0$ and $E_s z = 0$.

So we see that the eigenvalue support of D is

$$\{ (\theta_r, \theta_s) : E_r z \neq 0, E_s z = 0 \}$$

and hence it is determined by the set

$$\{ \theta_r : E_r z \neq 0 \}.$$

$E_r D E_r = 0$

$E_r D^{1/2} D^{1/2} E_s = 0$

$E_r D E_s = 0 \quad \forall s$

$E_r D^{1/2} = 0$

So for pure states, we view this set as the eigenvalue support of $D = \mathfrak{z}\mathfrak{z}^*$.

Lemma The eigenvalue support of the state $\mathfrak{z}\mathfrak{z}^*$ is the set of eigenvalues of the restriction of A to the subspace $\{A^m \mathfrak{z} : m \geq 0\}$. $\square$

Lemma If B is the matrix representing the action of A on this subspace, the eigenvalues of B are simple & $\phi(B,t) \mid \phi(A,t)$. $\square$

Proof of Lemma

$$\operatorname{span}\{A^k g : k \geq 0\} = \operatorname{span}\{E_r g : r = 1, \ldots, d\}$$

and

$$\dim(\operatorname{span}\{A^k g : k \geq 0\}) = |\{\theta_r : E_r g \neq 0\}| . \qquad \square$$

(details for your amusement)