L24/01

Interlacing

If Z is a graph on n vertices, we assume its eigenvalues are $\theta_1(Z) \geq \cdots \geq \theta_n(Z)$

Theorem Let X be a graph on n vertices and let Y be an induced subgraph on m vertices. Then

$$\theta_i(Y) \leq \theta_i(X), \qquad \theta_{m-i}(Y) \geq \theta_{n-i}(X). \qquad \square$$

If $Y = X \setminus a$, we may paraphrase this as saying that between each pair of eigenvalues of X, there is an eigenvalue of Y. We prove this.

Theorem. If $a \in V(X)$, the derivative of $\phi(X \setminus a, t)/\phi(X)$ is negative at all points where it is defined.

Proof We have

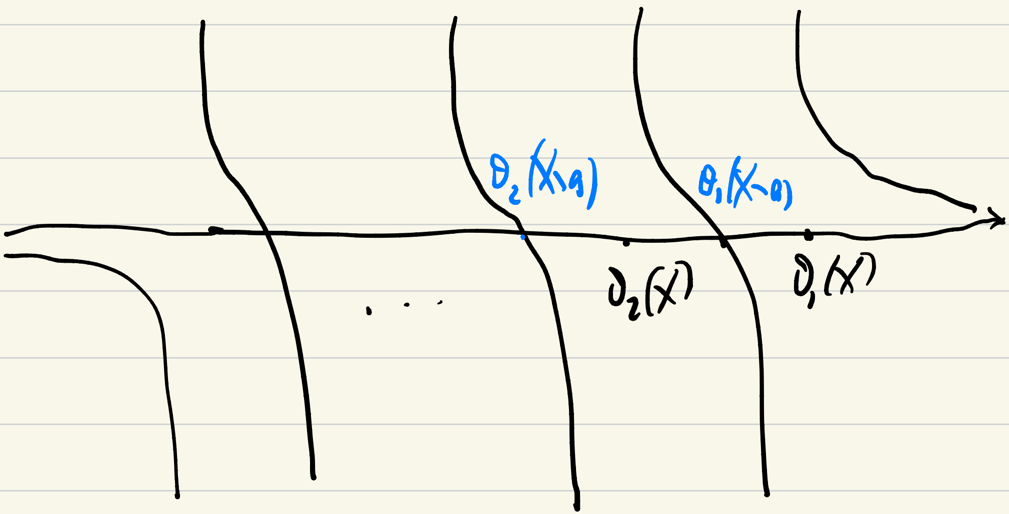
$$\frac{d}{dx}\frac{1}{x-\theta} \le 0$$

$$\frac{\phi(X \setminus a, t)}{\phi(X, t)} = \sum_r \frac{(E_r)_{a,a}}{t-\theta_r}.$$

Since $E_r \succeq 0$, we see that $(E_r)_{a,a} \ge 0$ $\forall r$.

The derivative of $\frac{1}{t-\theta}$ is $-1/(t-\theta)^2$. $\square$

So the graph of $\phi(x-a)/\phi(x)$ has the form



poles lie at zeros of $\phi(x,t)$; zeros at zeros of $\phi(x-a,t)$

Remark: Suppose $\rho = \theta_{max}(X)$, $\tau = \theta_{min}(X)$

Then $\rho I - A \succeq 0$ and since principal submatrices of psd matrices are psd, we see that if Y is an induced subgraph of X then $\theta_{max}(Y) \leq \rho$.

Similarly $A - \tau I \succeq 0$ and so $\theta_{min}(Y) \geq \tau$.

Cospectral Graphs

Two graphs are cospectral if their adjacency matrices are similar. So, for an altogether trivial example, isomorphic graphs are cospectral. (Note: if $X \cong Y$, then it does <u>not</u> follow that $A(X) = A(Y)$.)

Lemma Graphs X & Y are cospectral if & only if their characteristic polynomials are equal. $\square$

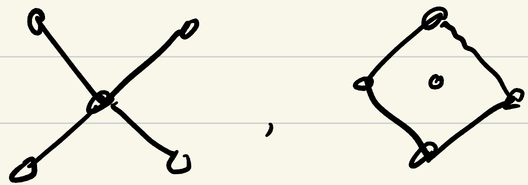
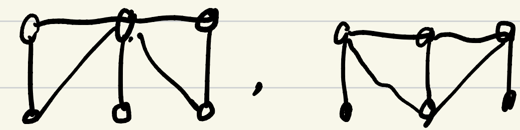
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ t^3

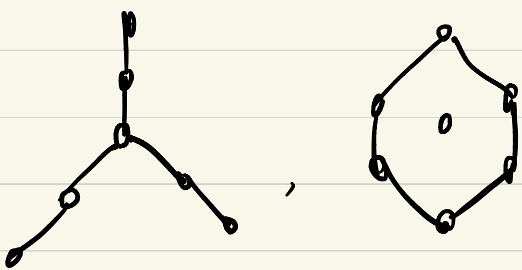
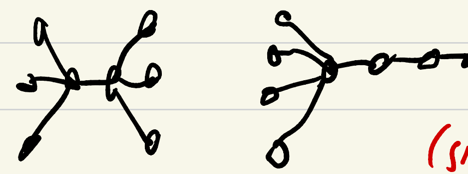
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$1 - 2 - 3$

$1 - 3 - 2$

Examples of cospectral graphs, not isomorphic.

a)  ; b) 

c)  ; d) 

(also their complements)

(smallest pair of cospectral trees)

Latin square graphs

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{pmatrix}$$

triples $(i, j, L_{i,j})$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 3 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 0 \\ & \vdots & \end{pmatrix}$$

orthogonal array

$n \times n$

n^2 triples

Latin square graphs: $(3n-3)$-regular graph on n^2 vertices

Strongly regular graphs

A graph is strongly regular if there are constants k, a, c such that:

(a) X is k-regular $3n-3$

(b) ~~two~~ adjacent vertices have exactly $\overset{n}{a}$ common neighbors

(c) two distinct non-adjacent vertices have exactly c 6 common neighbors

(d) X is not complete or empty.

Examples

primitive

X & $\bar{X}$ connected

(a) mK_n & $\overline{mK_n}$, when $m, n \geq 2$.

(b) Latin square graphs.

(c) 2×2 matrices over a finite field, matrices adjacent if & only if their difference has rank one.

(d) Line graphs of K_n $(n \geq 4)$ and $K_{n,n}$ $(n \geq 2)$.

? vertices of $L(X)$ are the edges of X

edges are adjacent if they have exactly one common vertex

Theorem. A graph X on n vertices is strongly regular if & only if there are parameters k, a, c such that

$$A^2 = kI + aA + c(\overbrace{J-I-A}^{A(\bar{X})}) \qquad \square$$

We often use this in the form

$$\boxed{A^2-(a-c)A-(k-c)I = cJ}$$

Assume X is strongly regular with parameters (n, k, a, c)

Then $\underset{\sim}{1}$ is an eigenvector with eigenvalue k, and

if X is connected, k is a simple eigenvalue.

If z is eigenvector for A with eigenvalue λ:

and $\lambda \neq k$, then $\underset{\sim}{1}^T z = 0$. So $Jz = 0$ and

$$
\begin{aligned}
0 = cJz &= (A^2 - (a-c)A - (k-c))z \\
&= (\lambda^2 - \lambda(a-c) - (k-c))z
\end{aligned}
$$

Hence λ is a zero of the quadratic

$$t^2 - (a-c)t - (k-c)$$

i.e, $\lambda = \frac{1}{2}\left[a-c \pm \sqrt{(a-c)^2+4k-4c}\right]$

Lemma If $A = A(X)$ & $B = A(Y)$, then the following are equivalent:

(a) X & Y are cospectral

(b) $\operatorname{tr}(A^k) = \operatorname{tr}(B^k)$ $\quad (k = 0, 1, \ldots)$

(c) $\operatorname{tr}((I - tA)^{-1}) = \operatorname{tr}((I - tB)^{-1})$

generating function for closed walks

If c_k (temporarily) denotes the number of closed walks of length k, then

(a) $c_0 = n$ (b) $c_1 = 0$ (c) $c_2 = 2|E(X)|$

(d) $c_3 = 6 \times \#$ triangles (e) c_4 - depends on number of 4-cycles and $2K_2$s.

From the characteristic polynomial, we can determine the length of the shortest odd cycle.

what about even cycles?

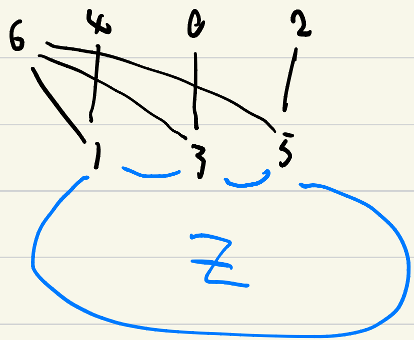
Constructions

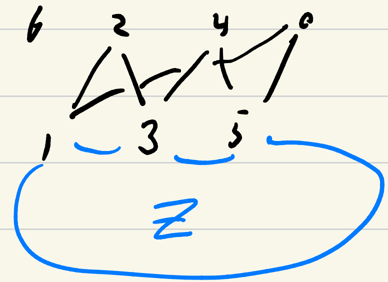
1) local switching

2) generalized tensor product

3) 1-sums

Local switching





$$\begin{array}{c|cccc|ccc|c}
 & 6 & 4 & 0 & 2 & 1 & 3 & 5 & \\ \hline
6 & & & & & 1 & 1 & 1 & \\
4 & & 0 & & & 1 & 0 & 0 & 0 \\
0 & & & & & 0 & 1 & 0 & \\
2 & & & & & 0 & 0 & 1 & \\ \hline
1 & 1 & 1 & 0 & 0 & & & & \\
3 & 1 & 0 & 1 & 0 & & & & \\
5 & 1 & 0 & 0 & 1 & & & & \\ \hline
 & & 0 & & & & & &
\end{array}$$

$$\begin{array}{c|cccc|ccc|c}
 & 6 & 4 & 0 & 2 & 1 & 3 & 5 & \\ \hline
6 & & & & & 0 & 0 & 0 & \\
4 & & 0 & & & 0 & 1 & 1 & 0 \\
0 & & & & & 1 & 0 & 1 & \\
2 & & & & & 1 & 1 & 0 & \\ \hline
1 & 0 & 0 & 1 & 1 & & & & \\
3 & 0 & 1 & 0 & 1 & & & & \\
5 & 0 & 1 & 1 & 0 & & & &
\end{array}$$

So pattern is

$$A_1 = \begin{bmatrix} C & B_1 \\ B_1^T & C \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & B_2 \\ B_2^T & C \end{bmatrix}$$

Assume Q is 4×4 & $Q^TQ = I$, Then

$$\begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & B_1 \\ B_1^T & C \end{pmatrix}\begin{pmatrix} Q^T & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & QB_1 \\ B_1^TQ^T & C \end{pmatrix}$$

Hence A_1 & A_2 are similar if $QB_1 = B_2$

Set $Q = \frac{1}{2} J_4 - I$. Then

(a) $Q^2 = \frac{1}{4} J^2 - J + I = I$

(b)
$$\frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

(c)
$$\begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} J = J \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix}$$

Extension:

Assume B is 01-matrix of order $2k \times l$. Assume further that each column consists of 0's, or of 1's, or exactly half its entries are 1.

Set $Q = \frac{1}{k} J_{2k} - I$. Then $Qz = \frac{\mathbf{1}^T z}{k} \underset{\sim}{1} - z$

and so if $\mathbf{1}^T z = k$ then $Qz = 1 - z$

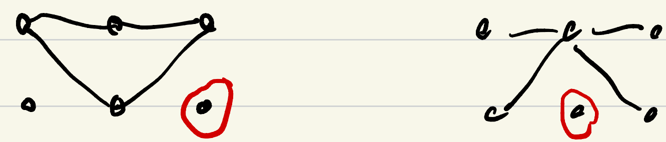
Also $Q^2 = \frac{1}{k^2} J_{2k}^2 - \frac{2}{k} J_{2k} + I = I$.

Then QB is 01 &

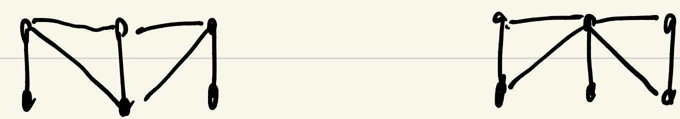
$$\begin{pmatrix} 0 & B \\ B^T & A \end{pmatrix}, \quad \begin{pmatrix} 0 & QB \\ B^TQ & A \end{pmatrix}$$

are cospectral with cospectral complements.

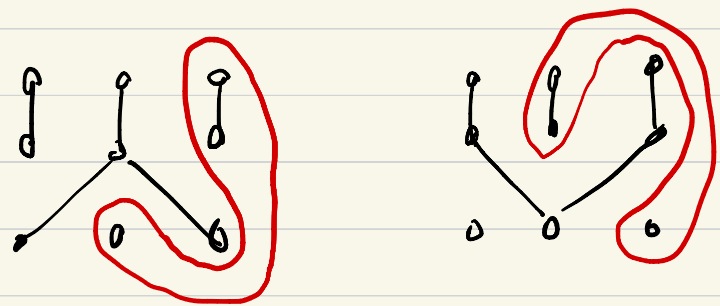
Partitioned tensor product



smallest pair of cospectral graphs



smallest pair of connected cospectral graph



smallest pair of cospectral forests

More general pattern:

$$G \; Z \; H \; S \; G \qquad\qquad H \; S \; G \; Z \; H$$

$$H \qquad\qquad\qquad\qquad G$$

$$\begin{bmatrix} A_1 & 0 & L \\ 0 & A_1 & L \\ L^T & L^T & A_2 \end{bmatrix} \qquad \begin{bmatrix} A_2 & 0 & L^T \\ 0 & A_2 & L^T \\ L & L & A_1 \end{bmatrix}$$

$$\begin{pmatrix} 0 & B_1 \\ B_1^T & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & B_2 \\ B_2^T & 0 \end{pmatrix} = \left[\begin{array}{c|c} 0 & \begin{matrix} 0 & B_1 \otimes B_2 \\ B_1 \otimes B_2^T & 0 \end{matrix} \\ \hline \begin{matrix} 0 & B_1^T \otimes B_2 \\ B_1^T \otimes B_2^T & 0 \end{matrix} & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{c|c} \begin{matrix} 0 & B_1 \otimes B_2 \\ B_1^T \otimes B_2^T & 0 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 0 & B_1 \otimes B_2^T \\ B_1^T \otimes B_2 & 0 \end{matrix} \end{array}\right]$$