L17/1

Contents:

Some graph products

If X & Y are graphs, a product of X & Y will be a graph with vertex set $V(X) \times V(Y)$.

We consider three cases.

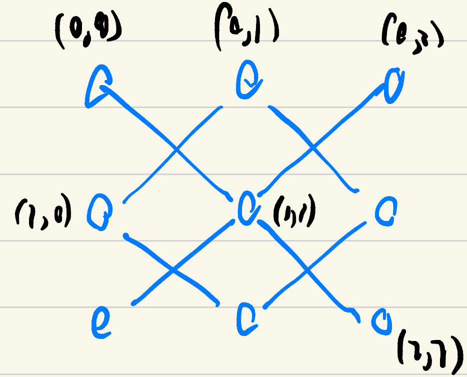
The direct product $X \times Y$ is the graph with vertex set $V(X) \times V(Y)$, where $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 \sim x_2$ & $y_1 \sim y_2$.

$\sim$ has three different meaning here

0, 2

e.g

$P_3 \times P_3$



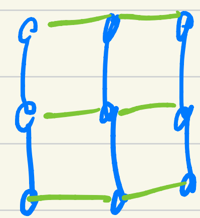
Lemma $A(X \times Y) = A(X) \otimes A(Y)$ $\square$

The Cartesian product $X \square Y$ has vertex set $V(X) \times V(Y)$, and $(x_1, y_1) \sim (x_2, y_2)$ if either

(a) $x_1 = x_2$, $y_1 \sim y_2$, or

(b) $x_1 \sim x_2$, $y_1 = y_2$.

$P_3 \square P_3$



Lemma $A(X \square Y) = A(X) \otimes I + I \otimes A(Y)$ $\square$

Corollary

$$\exp(A(X \square Y)) = \exp(A(X) \otimes I)\ \exp(I \otimes A(Y))$$

$$= \exp(A(X)) \otimes \exp((A(Y))$$

The strong product $X \boxtimes Y$ has

$$A(X \boxtimes Y) = A(X) \otimes A(Y) + A(X) \otimes I + I \otimes A(Y)$$

$$= A(X \times Y) + A(X \square Y)$$

Homomorphisms

Let X & Y be graphs. A map $\psi : V(X) \to V(Y)$ is a homomorphism if $\psi(u) \underset{Y}{\sim} \psi(v)$ whenever $u \underset{X}{\sim} v$.

Examples

(a) If X is a subgraph of Y, the inclusion map.

(b) A graph is m-colourable if & only if there is a homomorphism $\psi : X \to K_m$

preimage of a vx is a coclique

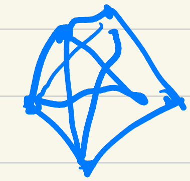
(c) An automorphism is an invertible homomorphism to itself.

(there is a better definition to come)

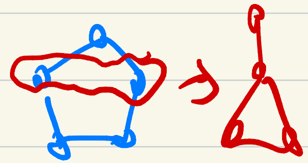
The set of homomorphisms $\psi: X \to X$ form a monoid. A graph is a core if all homomorphisms $\psi: X \to X$ are automorphisms.

Examples

(a) any complete graph is a core



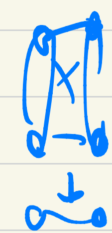
(b) any odd cycle is a core



(c) Petersen?

Lemma Suppose Y is a subgraph of X. If there is a homomorphism $\psi : X \to Y$ but there is no homomorphism from X to a subgraph of Y then Y is a core. $\square$

e.g. X is bipartite with at least one edge if & only if K_2 is a core of X.



we note (without proof) two interesting properties of cores:

(1) all cores of X are isomorphic

(2) if X is vertex-transitive, then any core $X^\bullet$ of X is vertex-transitive, and $|V(X^\bullet)| \mid |V(X)|$.

If $\psi : X \to Y$ is a homomorphism and $y \in V(Y)$, the preimage $\{x : \psi(x) = y\}$ is a fibre of ψ. The fibres of ψ partition $V(X)$ into cocliques.

We will consider homomorphisms into infinite graphs.

Example

The orthogonality graph Ω_d has

vertices: unit vectors in $\mathbb{R}^d$

edges: u & v are adjacent if $\langle u,v\rangle = 0$

A homomorphism $X \to \Omega_d$ is an orthogonal representation of X

Clearly the maximum size of a clique in Ω_d is d — the cliques are the orthonormal bases.

Theorem If $d \geq 3$, there is no clique in Ω_d that contains exactly one vertex from each d-clique. $\square$

Gleason

chromatic number

Consequently, if $d \geq 3$, then $\chi(\Omega_d) > d$

Isomorphisms & Automorphisms

An isomorphism from X to Y is a bijection ψ from $V(X)$ to $V(Y)$ such that $\psi(u) \underset{Y}{\sim} \psi(v)$ if and only if $u \underset{X}{\sim} v$.

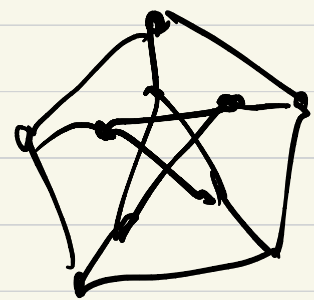
An automorphism of X is an isomorphism from X to itself. The set of all automorphisms of X is a group, denoted $\mathrm{Aut}(X)$.

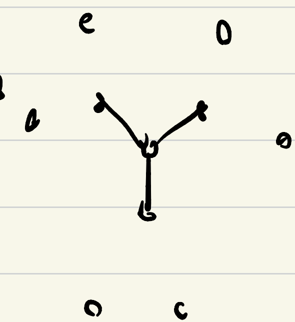
An automorphism of X is a permutation of $V(X)$.

Hence $\mathrm{Aut}(X) \leq \mathrm{Sym}(X)$, equality if & only if $X = K_n$ or $\bar{K}_n$

and therefore $\mathrm{Aut}(X)$ is a permutation group.

$$\mathrm{Aut}(\bar{X}) = \mathrm{Aut}(X)$$





A linear map $L: V \to V$ is determined by its values on a basis of V. So if $e_1, \dots, e_n$ is a basis, the values $L(e_1), \dots, L(e_n)$ determine L. If $L(e_i) \in \{e_1, \dots, e_n\}$ for all i, then L permutes the basis set, and we say L is a permutation operator. The matrix representing L relative to the given basis is a permutation matrix.

If π is a permutation of $\{1,\dots,n\}$, we use i^π to denote the image of i under π. We see that π determines a permutation operator (on $\mathbb{R}^V$) and a permutation matrix P.

Problem. If $Pe_i = e_{i^\pi}$ or $Pe_{i^\pi} = e_i$?

In any case, if P is the permutation matrix corresponding to a permutation π, then

$$\pi \in Aut(X) \iff PA = AP.$$

It is sometimes convenient to define $Aut(X)$ to be the set of permutation matrices that commute with the adjacency matrix of A.

Theorem If the eigenvalues of $A(X)$ are all simple, $\mathrm{Aut}(X)$ is abelian with exponent dividing two.