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Lemma For any graph Y, we have $Y \cong M(Y, 1)$.

Proof Any measurement on Y is a standard basis vector. Two standard basis vectors e_u & e_v are adjacent if & only if $u \sim v$. $\square$

If $\mathcal{P} = (P_1, \ldots, P_m)$ and $\mathcal{Q} = (Q_1, \ldots, Q_n)$ are measurements on X & Y respectively, we define $\mathcal{P} \otimes \mathcal{Q}$ to be $\{P_u \otimes Q_v\}$. We claim that $\mathcal{P} \otimes \mathcal{Q}$ is a measurement on $X \times Y$.⊛ A consequence of this is

Lemma $\mathcal{M}(X, d) \times \mathcal{M}(Y, e)$ is isomorphic to a subgraph of $\mathcal{M}(X \times Y, de)$.⊛

⊛ exercises

We note that if $Z \xrightarrow{q} X$ and $Z \xrightarrow{q} Y$, then

$Z \to \mathcal{M}(X,d)$ and $Z \to \mathcal{M}(Y,e)$ and hence

$$Z \to \mathcal{M}(X,d) \times \mathcal{M}(Y,e) \longrightarrow \mathcal{M}(X \times Y, de).$$

Therefore $Z \xrightarrow{q} X \times Y$. Accordingly the category of graphs & q-hom has products, and these correspond to products in the category of graphs & homomorphisms.

Lemma If $X \xrightarrow{q} Z$ and $Y \xrightarrow{q} Z$, then $X \cup Y \xrightarrow{q} Z$. $\square$

In the category of graphs, $X \cup Y$ is the sum of X & Y; this result shows that $X \cup Y$ is also the sum in the quantum category.

$X \xrightarrow{q} Y$ and $Y \xrightarrow{q} X$.

Lemma X and $M(d, X)$ are q-homomorphism equivalent

Proof Since $M_d(X) \to M_d(X)$, we have $M_d(X) \xrightarrow{q} X$.

Next, $X \xrightarrow{q} X$ and therefore $X \to M_d(X)$ and (as homomorphisms are quantum homomorphisms) we have $X \xrightarrow{q} M_d(X)$.

Operations on homomorphisms

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If $\mathcal{P}$ is a quantum homorphism, we define the algebra $\mathrm{alg}(\mathcal{P})$ to be the matrix algebra generated by the elements of $\mathcal{P}$. If this algebra is commutative, we say $\mathcal{P}$ is classical.

If $\mathcal{P}$ & $\mathcal{Q}$ are quantum homomorphisms from X to Y, we define their **direct sum** $\mathcal{P} \oplus \mathcal{Q}$ to be the $|V(X)| \times |V(Y)|$ matrix with

$$(\mathcal{P} \oplus \mathcal{Q})_{ij} = \mathcal{P}_{ij} \oplus \mathcal{Q}_{ij}$$

$\mathcal{P}_{ij}$
$\mathcal{Q}_{ij}$

Lemma $\mathcal{P} \oplus \mathcal{Q}$ is a quantum homomorphism. Further $\mathcal{P} \oplus \mathcal{Q}$ is classical if & only if $\mathcal{P}$ & $\mathcal{Q}$ are. $\square$

If $P : X \xrightarrow{q} Y$ & $Q : Y \xrightarrow{q} Z$, we define

$P * Q$ by setting

$$(P * Q)_{ij} = \sum_r P_{ir} \otimes Q_{rj}$$

We call it the **coproduct** of P & Q. If P & Q have

index one, the coproduct is the usual matrix product.

Lemma If $P : X \xrightarrow{q} Y$ & $Q : Y \xrightarrow{q} Z$, then

$P * Q : X \xrightarrow{q} Z$ $\square$

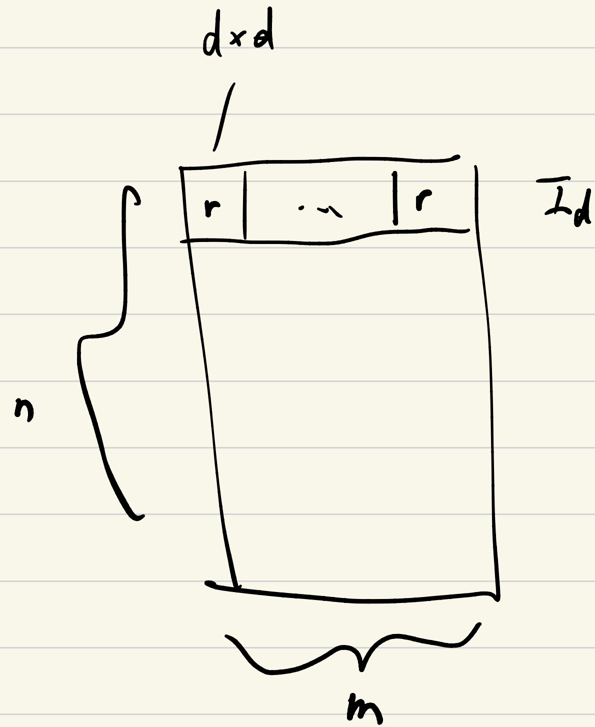
Lemma The coproduct is associative. Further

$P * Q$ is classical if & only if P & Q are.

$(A \otimes B) \otimes C = A \otimes (B \otimes C)$

We present a useful application of the direct sum operation.

Theorem Assume $\mathcal{P}$ is a quantum homomorphism from X to Y with index d. If a subgroup G of $\mathrm{Aut}(Y)$ acts transitively on $V(Y)$, there is a quantum homomorphism $X \xrightarrow{\mathcal{Q}} Y$ such that each entry has rank $d|Y|$ and index $d|G|$. $\square$



$\|v(y)\|_r = d$

Proof Assume $P: X \xrightarrow{q} Y$ and let G be a transitive subgroup of $\mathrm{Aut}(Y)$. If $\sigma \in G$, then σ gives a permutation of $V(Y)$, and hence a permutation of the columns of P. Denote this matrix by P^{σ}. Then the direct sum $\bigoplus_{\sigma \in G} P^{\sigma}$ is a quantum homomorphism from X to Y, and each entry is the direct sum of the same projections.

If ρ has index d, the direct sum of $|G|$ copies of ρ has index $d|G|$. The rank of an entry is $d|G:G_1| = d|Y|$. $\square$

Fractional Isomorphisms

Let X & Y be graphs with respective adjacency matrices A & B. A matrix S determines a fractional isomorphism from X to Y if

(a) it is doubly stochastic.

(b) $AS = SB$

A doubly stochastic matrix is necessarily square (as we will see).

Examples

(a) permutation matrices — so isomorphisms are fractional isomorphisms

(b) If X & Y are k-regular graphs on n vertices then $AJ = JA$, and $\frac{1}{n}J$ is a fractional isomorphism.

(c) $S: X \to Y$ and $T: Y \to Z$ are fractional isomorphisms then $TS: X \to Z$ is a fractional isomorphism.

We can view a doubly stochastic matrix as a weighted adjacency matrix for a directed graph. Recall that a directed graph is weakly connected if its underlying graph is connected; it is strongly connected if for each pair of vertices u & v, there is a directed path from u to v.

We translate these terms into matrix theory. An $n \times n$ matrix is reducible if it contains a submatrix of order $m \times (n-m)$ with all entries 0.

It is decomposable if it is permutation equivalent to a matrix of the form $\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$ where M & N are square.

$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$ A, C square

reducible $\Rightarrow$ not strongly connected

If a matrix is indecomposable, its digraph is weakly connected; if it is irreducible it is strongly connected.

Lemma If S is doubly stochastic, it is square. Further, each weakly connected component is strongly connected.

$$\begin{array}{cc} & m \\ m & \left.\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}\right\} n \end{array}$$

$A\mathbf{1} = \mathbf{1}$

entries of $\mathbf{1}^T A$ sum to n

Proof Assume $S = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$ with A $m \times m$. Then $\underset{\sim}{1}^T A \leq \underset{\sim}{1}$ and so $\underset{\sim}{1}^T A \underset{\sim}{1} \leq m$. On the other hand $A\underline{1} = \underset{\sim}{1}$ and so $\underset{\sim}{1}^T A \underset{\sim}{1} = m$. Hence each column of A sums to 1, and therefore A is doubly stochastic. The lemma follows. $\square$

From this proof we see that a doubly stochastic matrix must be square.

If R & S are doubly stochastic, so is RS. Hence the fractional isomorphisms of a graph form a monoid (that contains $\mathrm{Aut}(X)$).

If $AR = RB$ and $AS = SB$ and $0 \le \alpha \le 1$, then

$$A(\alpha R + (1-\alpha)S) = \alpha RB + (1-\alpha)SB$$

from which it follows that the fractional isomorphism from X to Y form a closed convex polytope.

The permutation matrices are vertices of the convex polytope formed by the fractional automorphisms of X; if these are the only vertices we say X is compact. (This is a strong condition.)

Tinhofer

Fractional isomorphism and equitable partitions

Let $\mathcal{F}(X,Y)$ denote the set of fractional isomorphisms from X to Y, and abbreviate $\mathcal{F}(X,X)$ to $\mathcal{F}(X)$.

Lemma If $R \in \mathcal{F}(X,Y)$, then $R^T \in \mathcal{F}(Y,X)$ and $RR^T \in \mathcal{F}(X)$ (and $R^T R \in \mathcal{F}(Y)$).

(Note that RR^T & $R^T R$ are doubly stochastic.)

Proof If R is doubly stochastic and $AR = RB$, then $R^T A = R^T A^T = B^T R^T = BR^T$. Hence if $R \in \mathcal{F}(X, Y)$, then $R^T \in \mathcal{F}(Y, X)$. Further $RBR^T = ARR^T$ and since RBR^T, A and RR^T are symmetric, we see that $ARR^T = RR^T A$. $\square$

Theorem Assume $R \in \mathcal{F}(X)$. Then the vertex sets of the strong components of the directed graph determined by R are the cells of an equitable partition of X.

Proof We will need to apply the Perron-Frobenius theorem, as follows:

Claim If A is a non-negative real matrix and its underlying directed graph X is strongly connected, then the spectral radius of X is a simple eigenvalue. $\square$

Proof (of theorem) We have $AR = RA$. Since R is doubly stochastic, its weak components are strong. Hence the spectral radius of each component has a simple eigenvalue.

Further, each row of R sums to 1, and so the 1-eigenspace of R is spanned by the characteristic vector of the components. The projection onto this subspace is a polynomial in A, and it has form

$$\begin{bmatrix} \frac{1}{m_1} J_1 & & \\ & \ddots & \\ & & \frac{1}{m_r} J_r \end{bmatrix}$$

where m_i is the number of vertices in the i-component.

This matrix is the normalized characteristic matrix of the "component" partition of X, and since it commutes with A, this partition is equitable. $\square$