L26/02: Discrete Walks

Contents:

Invariant subspaces

Let P & Q be two projections on $\mathbb{C}^m$.

We want spectral information for $U = (2P-I)(2Q-I)$. We work in the algebra $\mathcal{A} = \langle P, Q \rangle = \langle R, C \rangle$.

Lemma If P & Q are projections on $\mathbb{C}^m$, then $\mathbb{C}^m$ is the direct sum of 1- & 2-dimensional $\mathcal{A}$-invariant subspaces.

Proof. A 1-dimensional $\mathcal{A}$-invariant subspace is spanned by a common eigenvector for P & Q. Let W be the orthogonal complement to the span of the common eigenvectors, we decompose it into 2-dimensional subspaces.

Since P & Q are Hermitian, W is $\mathcal{A}$-invariant. (why?)

Note that QPQ is Hermitian and so W is spanned by eigenvectors for QPQ.

There are two cases according as QPQ is zero on W or not.

Assume first that QPQ is not zero on W. Then there is is a non-zero vector z such that $QPQz = \mu z$ & $\mu \neq 0$

So

$$\mu Qz = Q \cdot QPQz = QPQz = \mu z$$

and hence $Qz = z$ and

$$QPz = QPQz = \mu z$$

Therefore the subspace spanned $\{z, Pz\}$ is $\mathcal{A}$-invariant.

We treat the case where QPQ is zero on W. If Q is zero on W, then $\langle P, Q \rangle = \langle P \rangle$ on W and therefore W has a basis of eigenvectors for $\langle P, Q \rangle$. So Q is not zero on W and hence there is z in W such that $Qz = z$. (the eigenvalues of Q are 0 & 1.)

As $QPz = QPQz = 0$, we see that the span of $\{z, Pz\}$ is $\langle P, Q \rangle$-invariant.

Eigenvalues

Our next step is to work out the eigenvalues of U.

Because U is unitary, its eigenvalues are complex with norm one, so the real eigenvalues of U can only be 1 or -1.

There are matrices L & M with orthonormal columns s.t. $P = LL^*$, $Q = MM^*$. Let y be an eigenvector for M^*PM and set $z = Ly$.

$QPQ = M(M^*PM)M^*$

$P = LL^*$ $\qquad Q = MM^*$

y - eigenvector for $M^* P M$

$z := My$

Claim The following matrices are positive semidefinite, and have the same nonzero eigenvalues with the same multiplicities.

$$QP \sim PQ = \begin{cases} P^2Q \sim PQP \\ PQ^2 \sim QPQ \\ LL^*Q \sim L^*QL \\ PMM^* \sim M^*PM \end{cases}$$

these two are "small"

If M is a square matrix, $\|M\|_2 := \sup_{\|x\|=1} \{x^* M x\}$.

We have $\|MN\|_2 \leq \|M\|_2 \|N\|_2$. If M is Hermitian, then $\|M\|_2$ equals the spectral radius of M

spectral radius

Claim If P & Q are projections, then $\rho(PQ) \leq 1$.

Proof Let $\rho(M)$ be the spectral radius of the matrix M.

Define $\|M\|_2$ to be $\sup\{\|Mx\|_2 : \|x\|_2 = 1\}$. Then $\|MN\|_2 \leq \|M\|_2 \|N\|_2$

If P is a projection on a Hilbert space, then

$$\langle Px, (I-P)x \rangle = 0$$

and so $\langle x,x \rangle = \langle Px + (I-P)x, Px + (I-P)x \rangle = \langle Px, Px \rangle + \underbrace{\langle (I-P)x, (I-P)x \rangle}_{\geq 0}$

This implies that $\|Px\|_2 \leq 1$ and so $\|P\|_2 \leq 1$.

If Q is a projection as well, then

$$\|PQ\|_2 \leq \|P\|_2 \|Q\|_2 \leq 1.$$

Assume y is an eigenvector of M^*PM and $z := My$.

Claim! If $M^*PMy = y$, then $z \in \mathrm{im}(P) \cap \mathrm{im}(Q)$.

Proof If $M^*PMy = y$, then

$$y^*M^*(I-P)My = y^*M^*My - y^*M^*PMy = 0$$

and, as $I-P \succeq 0$, it follows that $(I-P)My = 0$

and $My \in \mathrm{im}(P)$. Since $QMy = MM^*My = My$

we have $My \in \mathrm{im}(Q)$.

Claim 2. If $M^*PMy = 0$, then $My \in \ker(P) \cap \operatorname{im}(Q)$.

Proof $0 = y^*M^*PMy$ and $PMy = 0$.

So $My \in \ker(P)$. As before, $QMy = My \in \operatorname{im}(Q)$.

Note that in case (1), $Uz = z$

and, in case (2), $Uz = -z$.

why? (see claim)

Claim 3. Assume $M^* P M y = \mu y$ with $0 < \mu < 1$ and $\cos(\theta) = 2\mu - 1$. Then

$$(\cos(\theta)+1) z - (e^{i\theta}+1) P z$$

is an eigenvector for U with eigenvalue $e^{i\theta}$

and

$$(\cos(\theta)+1) z - (e^{-i\theta}+1) P z$$

is an eigenvector for U with eigenvalue $e^{-i\theta}$.

Proof Suppose $M^*PMy = \mu y$ $\quad (0<\mu<1)$.

Then

$$Qz = MM^*.My = My = z$$

$$QPz = MM^*PMy = \mu My = \mu z$$

and therefore $\operatorname{span}\{z, Pz\}$ is $\langle P, Q\rangle$-invariant.

From this, one deduces that

C

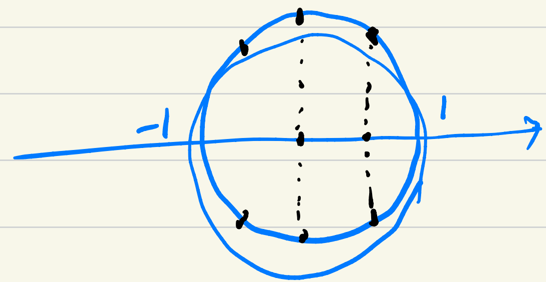
$$U[z\ Pz] = [z\ Pz]\begin{pmatrix}-1 & -2\mu \\ 2 & 4\mu-1\end{pmatrix}$$

$t^2-(4\mu-2)t+1$

With $2\mu - 1 = \cos(\theta)$, we find that the eigenvalues of C are $e^{\pm i\theta}$. With more work, one sees that the eigenvectors of U on $\operatorname{span}\{\xi, P\xi\}$ are as given. $\square$

Corollary The eigenvalues of U are determined by the eigenvalues of M^*PM (where $MM^* = Q$). $\square$

It is also possible to compute the eigenvalue multiplicities, see H. Zhan's thesis for details.



(Too many) Incidence matrices

Let X be a k-regular graph on n vertices. Then we have various incidence matrices describing relations between vertices, edges & arcs.

D_t	vxs $\times$ arcs starting on u	$n \times nk$
D_h	vxs $\times$ arcs ending on u	$n \times nk$
B	vxs $\times$ edges	$n \times nk/2$
M	arcs $\times$ edges	$nk \times nk/2$

We see that D_t^T is the characteristic matrix of the partition of arcs by initial vertex, ...

Lemma

$D_t^T D_t = I_n \otimes J$

$D_t R = D_h$

(a) $R = MM^T - I$

(b) $C = I \otimes G = \frac{2}{d} D_t^T D_t - I \otimes I$ $\square$

Define

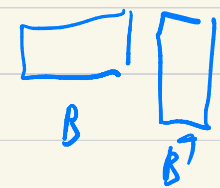
$K := \frac{1}{\sqrt{2}} M$, $L := \frac{1}{\sqrt{d}} D_t^T$, $S = L^T K$.

arcs × edges

arcs × verts

verts × edges

Lemma $SS^T = \frac{1}{2d} BB^T = \frac{1}{2d}(A + dI)$. $\square$



$= A + dI$

Theorem Assume X is d-regular and U gives the arc-reversal walk on X. Then the multiplicities of the non-real eigenvalues of U sum to $2n-4$ if X is bipartite and to $2n-2$ otherwise. If λ is an eigenvalue of X, not $\pm d$ and $\cos(\theta) = \lambda/d$, and y is an eigenvector for λ, then

$$(D_t^T - e^{\pm i\theta} D_h^T)\, y$$

is an eigenvector for U with eigenvalue $e^{\pm i\theta}$. $\square$

Line digraphs

If X is a directed graph, the **line digraph** $LD(X)$ is the directed graph with the arcs of X as vertices and with an arc from (a,b) to (c,d) if & only if $b=c$. (We could allow loops, but nothing much changes so we don't.)

Note that we have arcs $(a,b) \to (b,a)$. If we disallow these, we have a **strict line digraph**.

As for undirected graphs we have incidence matrices D_t and D_h with the same definition.

Lemma We have $D_h^T D_t = A(LD(X))$ and $D_t D_h^T = A(X)$. $\square$

Hence X & $LD(X)$ have the same non-zero eigenvalues with the same multiplicities.

The transition matrix for an arc-reversal walk is a weighted adjacency matrix for $LD(X)$.

Non-backtracking walks A walk in a graph is non-backtracking if it has no subsequences of the for aba. We can view a non-backtracking walk as a sequence of arcs such that if (ab) & (cd) are consecutive then $b=c$ and $d \neq a$. Hence it may be viewed as a walk on the strict line digraph of X, which has adjacency matrix $A(LD(X))-R$