L 12/02

Contents

Group theory

If $U(t) = \exp(itA)$ then

$$\{U(t) : t \in \mathbb{R}\}$$

is a group. The map $t \longmapsto U(t)$ is a homomorphism from $\mathbb{R}$ to $U(d)$. $U(d, \mathbb{F})$

If z is an eigenvector for A with eigenvalue θ then $U(t)z = e^{it\theta} z$ and we have a homomorphism from $U(t)$ to the unit circle in the complex plane.

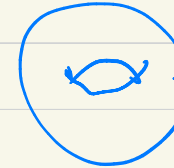
We define $\mathbb{R}/\mathbb{Z}$ to be the circle group.

If $r \in \mathbb{R} \setminus 0$, then $\mathbb{R}/(r\mathbb{Z})$ is isomorphic to
to the circle group.

Theorem A subgroup of $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$ is either
discrete or dense. $\square$

The set

$$\mathcal{U} = \{U(t) : t \in \mathbb{R}\}$$



is a subgroup of $U(d)$, a so-called 1-parameter subgroup. It need not be closed.

As $U(t) = \sum_r e^{it\theta_r} E_r$ we have a homomorphism from $((\theta_1, \theta_2, \dots, \theta_m))$ to $U(t)$. The group

$$\{(e^{it\theta_1}, \dots, e^{it\theta_m}) : t \in \mathbb{R}\}$$

is a torus, i.e., a direct product of circle groups

If D is a density

$$\{U(t)\,D\,U(-t) : t \geq 0\}$$

is a forward orbit of $\mathcal{U}$ acting on the set of density matrices. Thus we have perfect state transfer from D_1 to D_2 if & only if D_2 lies on the forward orbit of D_1.

Recurrence

Theorem Assume we have a continuous quantum walk on X with transition matrix $U(t)$. Assume $\tau > 0$. Then for each $\varepsilon > 0$, there is an integer k such that $\|U(k\tau) - I\| < \varepsilon$.

Proof Consider the sequence

$$I, U(\tau), U(2\tau) \ldots$$

If this sequence is periodic, the theorem holds. Assume it is not. Then we have an infinite sequence in the compact set $U(d)$, and this sequence has a limit point. Call this limit point T. Note that T lies in the closure of this sequence.

(T need not lie in the sequence.) Any ball of radius ν about T contains two distinct points of our sequence, say $U(k\tau)$ & $U(l\tau)$. By the triangle inequality $\|U(l\tau) - U(k\tau)\| < 2\nu$ and since $U(t)$ is unitary $\|U((l-k)\tau) - I\| < 2\nu$. We choose $\nu < \frac{1}{2}\varepsilon$ to get the theorem. $\square$

Poincaré recurrence

In fact a stronger result holds: given $\varepsilon > 0$ there is an integer L such that each sequence

$$U(k), \dots, U(k+L)$$

almost periodic

contains a point within ε of I.

We note that I is not that special — we see that if the quantum visits a state it returns infinitely often to an ε-neighbourhood of that state

The minimum period

Suppose a is a periodic vertex in a continuous quantum walk on X. Define $Per(a)$ to be the set of times t such that $U(t) e_a e_a^T U(-t) = e_a e_a^T$, i.e., the set of periods of a.

Lemma If a is a periodic vertex of X with valency >0, then there is a number τ such that $Per(a) = \{\tau m : m \in \mathbb{Z}\}$.

Proof Per(a) is an additive subgroup of $\mathbb{R}$, hence it is discrete or dense.

If Per(a) is dense, then there is a sequence $(t_k)_{k\geq 0}$ of elements of Per(a) with limit 0. Since $U(t)$ is differentiable,

$$\lim_{k\to\infty} \frac{1}{t_k}(U(t_k)-I) = U'(0) = iA$$

$\frac{d}{dt}\exp(tA) = A\exp(tA)$

If $U(t)e_ae_a^TU(-t) = e_ae_a^T$, then e_a must be an eigenvector for $U(t)$. By the preceding limit calculation it is also eigenvector for A, implying that a is an isolated vertex. $\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$

We conclude that $\mathrm{Per}(a)$ is discrete, and the lemma follows. $\square$

The previous lemma implies that the minimum period exists. We now derive a lower bound for it. (In two steps.)

Lemma Assume X is a graph with eigenvalues $\theta_1, \ldots, \theta_m$ in decreasing order. Assume C & D are density matrices and $\operatorname{tr}(PQ) = 0$. If we have transfer from C to D at time t, then $t \geq \dfrac{\pi}{\theta_1 - \theta_m}$.

$e_a e_a^T e_b e_b^T = 0$ if $a \neq b$

Proof Suppose $C \succ 0$, it has a unique positive definite square root $C^{1/2}$. We have

$$\langle C, U(t) C U(-t) \rangle = \mathrm{tr}(C U(t) C U(-t))$$

$$= \mathrm{tr}(C^{\frac{1}{2}} U(t) C^{\frac{1}{2}} C^{\frac{1}{2}} U(-t) C^{\frac{1}{2}})$$

$$= \mathrm{tr}((C^{\frac{1}{2}} U(t) C^{\frac{1}{2}})(C^{\frac{1}{2}} U(-t) C^{\frac{1}{2}}))$$

and therefore $\langle C, U(t) C U(-t) \rangle = 0$ if & only if $C^{\frac{1}{2}} U(t) C^{\frac{1}{2}} = 0$

Now

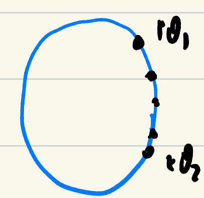
$$\operatorname{tr}(C^{\frac{1}{2}} U(t) C^{\frac{1}{2}}) = \sum_r e^{it\theta_r} \operatorname{tr}(C^{\frac{1}{2}} E_r C^{\frac{1}{2}})$$

why?

The matrices $C^{\frac{1}{2}} E_r C^{\frac{1}{2}}$ are positive semidefinite and

$$\sum_r \operatorname{tr}(C^{\frac{1}{2}} E_r C^{\frac{1}{2}}) = \sum \operatorname{tr}(C^{\frac{1}{2}} I C^{\frac{1}{2}}) = \operatorname{tr}(C) = 1$$

Consequently $\operatorname{tr}(C^{\frac{1}{2}} U(t) C^{\frac{1}{2}})$ is a convex combination of the eigenvalues $e^{i\theta_r t}$ of $U(t)$. The lemma follows.



When t is small, the eigenvalues of $U(t)$ lie on a small arc of the unit circle containing 1. If they lie on an arc of length less than π, then there convex hull does not contain 0. Therefore if $t(\theta_1 - \theta_m) < \pi$, then $tr(PQ) \neq 0$. $\square$

Remark If $D_a = e_a e_a^T$ and $D_b = e_b e_b^T$ $(b \neq a)$, then $D_a D_b = 0$. So the above bound applies to perfect state transfer between vertex states.

Lemma If X is a graph and the eigenvalue support of the vertex a is $\theta_1 > \dots > \theta_m$, then the minimum period of a is at least $\frac{2\pi}{\theta_1 - \theta_m}$.

Proof If a is periodic at time τ, then, as we saw above,

$U(\tau) e_a = \lambda e_a$ with $|\lambda| = 1$. So

$$\lambda E_r e_a = E_r U(\tau) e_a = e^{i\tau\theta_r} E_r e_a$$

and therefore $e^{i\tau\theta_r} = \lambda$ and $e^{i\tau(\theta_r - \theta_s)} = 1$

Hence

$$\tau(\theta_r - \theta_s) = 2m_{r,s}\pi$$

and therefore

$$\tau \geq \frac{2\pi}{\theta_1 - \theta_m} \qquad \square$$

We have that θ_1 is the spectral radius of X and therefore $-\theta_m \leq \theta_1$. So $\tau \geq \frac{\pi}{\theta_1}$.