L05/02

Contents:

Eccentricity & eigenvalue support

covering radius

If $u \in V(X)$, the eccentricity $ecc(u)$ of u is the maximum distance from u of a vertex in X. (Hence the maximum of $ecc(u)$ as u runs over $V(X)$ is the diameter of X.)

If $ecc(u) = k$, the vectors $(A+I)^r e_u$ for $r = 0, \dots, k$ are linearly independent. But these $k+1$ vectors lie in the span of the vectors $E_r e_u$, and so $k+1$ is bounded above by the size of eval support of u.

Thus we have:

Lemma If $u \in V(X)$, then

$$\text{size of } e(u) \text{ supp} = |\{\theta_r : E_r e_u \neq 0\}| \geq ecc(u) + 1. \quad \square$$

We will make significant use of this before long.

Remark If X is vertex-transitive, the eigenvalue support of a vertex is the spectrum of X. Exercise

Cospectral & strongly cospectral vertices

Lemma If X admits perfect state transfer from vertex a to vertex b, then a & b are cospectral.

Proof If $U(t)e_a e_a^T U(-t) = e_b e_b^T$ then since $U(t)E_r = e^{it\theta_r}E_r$,

$$E_r e_b e_b^T E_r = E_r U(t) e_a e_a^T U(-t) E_r = e^{it\theta_r} E_r e_a e_a^T E_r e^{-it\theta_r} = E_r e_a e_a^T E_r$$

Taking trace of the first & last terms yields $e_b^T E_r e_b = e_a^T E_r e_a$,

since this holds for all r, we have that a & b are cospectral. $\square$

$e_a^T E_r^T E_r \underline{e_a}$ — $e_a^T E_r e_a$ $e_b^T E_r e_b$ $e_b^T E_r^T E_r \underline{e_b}$

Lemma Any two vertices in a strongly regular graph are cospectral.

Proof We saw that if X is strongly regular, we have

$$A^2 = kI + aA + c(J-I-A) = (k-c)I + (a-c)A + cJ.$$

It follows that if $m \geq 0$, then $A^m \in \operatorname{span}\{I, J, A\}$ and therefore all polynomials in A have constant diagonal. $\square$

A graph is **walk-regular** if all vertices are cospectral. We have the following examples:

(a) vertex-transitive graphs

(b) strongly regular graphs

(c) complements of walk-regular graphs

(d) direct & Cartesian products of walk-regular graphs

exercises

Theorem If X is walk-regular & $|V(X)| \geq 3$, then X has at most $\frac{n}{2}$ simple eigenvalue.

Before starting the proof, I sketch the strategy.

(a) If z is an eigenvector with simple eigenvalue θ and k is the valency of X, then we may assume z is a ± 1-vector and $k-\theta$ is even.

(b) the simple eigenvalues of X lie in $\{-k, \ldots, k-2, k\}$, whence the number s of simple eigenvalues is at most k

(c) the simple eigenvalues of $\bar{X}$ are congruent to $n-1-k \pmod 2$ and so $s \le n-k$. Hence $2s \le n+1$.

Since is even, the result follows.

We will make of the relation between the eigenvalues of X and those of $\bar{X}$, when X is regular:

(a) If X has valency k, then $\bar{X}$ has valency $n-1-k$

(b) If θ is an eigenvalue of X and there is an eigenvector z of θ such that $\mathbf{1}^T z = 0$, then z is an eigenvector for $\bar{X}$ with eigenvalue $-\theta-1$.

Proof Assume $|V(X)| = n$. If X is not connected, each component is walk-regular and X has no simple eigenvalues. So assume X is connected with valency k and note that k is simple. why?

Let λ be a simple eigenvalue, not equal to k and with eigenvector z. Then the idempotent E_λ is a non-zero scalar multiple of zz^T. Since E_λ has constant diagonal, so does zz^T and hence we can assume z is a ± 1-vector. As $zz^T \mathbf{1} = 0$ we have $z^T \mathbf{1} = 0$. Therefore exactly half the entries of z are equal to 1, the rest to -1.

It follows that n is even. Since $\lambda z_a = \sum_{b \sim a} z_b$ is congruent to $k \pmod 2$, we see that $k - \lambda$ is eve

Assume X has exactly s simple eigenvalues.

If the eigenvalues of X are

$$k = \theta_1 \geq \cdots \geq \theta_n.$$

then the eigenvalues of $\bar{X}$ are

$$n-1-k \geq -\theta_n - 1 \geq \cdots \geq -\theta_2 - 1$$

Since $\bar{X}$ is walk-regular, if it is not connected, it has no simple eigenvalues and $\bar{X}$ has at most two.

If $\bar{X}$ is connected, its valency $n-1-k$ is simple and so it has exactly s simple eigenvalues.

The simple eigenvalues of X lie in

$$\{-k, \ldots, k-2, k\}$$

which implies $s \leq k+1$. Considering the complement, we also have that $s \leq n-k$ and thus $2s \leq n+1$.

Since n is even, the result follows $\square$

We say two vertices a & b in a graph are **strongly cospectral** if $E_r e_b = \pm E_r e_a \ \forall r$.

Note that this implies $\|E_r e_b\| = \|E_r e_a\|$ for all r, and therefore a & b are cospectral.

Lemma If the eigenvalues of X are simple, then cospectral vertices are strongly cospectral. $\square$

(Exercise)

Other constructions of strongly cospectral vertices are given in the notes.

Theorem If X admits perfect state transfer from a to b, then a & b are strongly cospectral.

Proof If

$$e_b e_b^T = U(\tau) e_a e_a^T U(-\tau)$$

then $e_b = \gamma U(\tau) e_a$ and $E_r e_b = \gamma e^{i\theta_r \tau} E_r e_a$. Since e_b & $U(\tau) e_a$ both have norm one, $|\gamma| = 1$ & $|\gamma e^{i\theta_r \tau}| = 1$. Since $E_r e_b$ & $E_r e_a$ are real, $\gamma e^{i\tau\theta_r} = \pm 1$ $\square$

Theorem If any two vertices in X are strongly cospectral, $|V(X)| \leq 2$.

Proof If all vertices of X are strongly cospectral to the vertex a, then $E_r e_a$ spans the θ_r-eigenspace. Then all eigenvalues are simple and, as X is walk-regular, $|V(X)| \leq 2$. $\square$

We offer a characterization of strongly cospectral vertices.

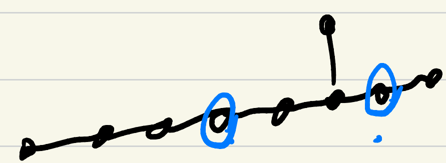
Theorem Vertices a, b in $V(X)$ are strongly cospectral if & only if there is an orthogonal matrix Q such that:

(a) Q is a polynomial in A, and is rational,

(b) $Qe_a = e_b$,

(c) $Q^2 = I$.

$$\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}$$



Proof We show that if a & b are strongly cospectral φ exists as stated. (The converse is an easy exercise.) Since a & b are strongly cospectral, there are scalars $\sigma_r = \pm 1$ such that $E_r e_b = \sigma_r E_r e_a$. Let $p(t)$ be a polynomial such that $p(\theta_r) = \sigma_r$. Then

$$p(A)e_a = \sum_r p(\theta_r) E_r e_a = \sum_r \sigma_r E_r e_a = \sum_r E_r e_b = e_b .$$

Further $\rho(A)^2 = \left(\sum \rho(\theta_r) E_r\right)^2 = \left(\sum_r \underline{\sigma_r}\, E_r\right)^2 = \sum E_r = I$

and so we define $Q = \rho(A)$.

It remains to show that Q is rational, which requires some field theory.

Let $\mathbb{E}$ be the extension field of $\mathbb{Q}$ generated by the eigenvalues of X. If $\alpha \in \mathrm{Aut}(\mathbb{E})$ and θ is an eigenvalue of X, the coefficients of the minimal polynomial $\psi(t)$ of θ are integers. So α fixes $\psi(t)$ and consequently it permutes the eigenvalues of X.

Let $(E_r)^\alpha$ be the matrix we get by applying α

to each entry of E_r. Then

$$A E_r^{\alpha} = (A E_r)^{\alpha} = (\theta_r E_r)^{\alpha} = \theta_r^{\alpha} E_r^{\alpha}$$

and hence E_r^{α} is a spectral idempotent of A with eigenvalue θ_r^{α}. Therefore $((E_r)_{a,a})^{\alpha} > 0$. As $(E_r)_{ab} = \sigma_r (E_r)_{a,a}$ this implies $(E_r)_{ab}$ and $((E_r)_{a,b})^{\alpha}$ have the same sign.

Accordingly $Q = \sum_r \sigma_r E_r$ is fixed by α, and so fixed by each automorphism of $\mathbb{E}$.

Since $\mathbb{E}$ is the splitting field of $\mathbb{O}(X,t)$, it follows that $\mathbb{Q}$ is rational. $\square$

My very strong recommendation for a text on Galois theory is David A. Cox, "Galois Theory", Wiley (2004)