L29/01

Quantum Walks

Continuous walks

initial state: pure, spanned by z, density zz^*

state at time t: $U(t)zz^*U(-t)$

Since $\operatorname{tr}(U(t)zz^*U(-t)) = 1$, we see that $U(t)z$ is a unit vector.

If we measure with respect to the standard basis, the outcome is a with probability

$$\langle U(t) z z^* U(-t),\ e_a e_a^T \rangle$$

$$= \operatorname{tr}\left(U(t) z z^* U(-t)\, e_a e_a^T \right)$$

$$= e_a^T U(t) z \cdot z^* U(-t) e_a$$

$$= |e_a^T U(t) z|^2$$

POVM

$\sum Q_r = I$, $Q_r \succeq 0$

$Q_r = e_r e_r^T$

If our initial states are vertex states, and we start at e_b, the probability we observe a is

$$|e_a^T U(t) e_b|^2 = |U(t)_{a,b}|^2 = U(t) \circ \overline{U(t)}$$

and $\bar{U}(t) = U(-t)$.

$u(t) = \exp(itA)$

The mixing matrix of a walk is

$$M(t) = U(t) \circ \overline{U(t)} = U(t) \circ U(-t)$$

It is row & column stochastic. Its ab-entry is the probability that a walk starting at b is "on" a at time t.

e.g. $X = K_2$ $\quad H = A(X) = A$

initial state $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\1\end{pmatrix}$

$$U(t) = \exp(itA) = \sum_m \frac{(it)^m}{m!}\begin{pmatrix}0&1\\1&0\end{pmatrix}^m$$

$$= \cos(t)\begin{pmatrix}1&0\\0&1\end{pmatrix} + i\sin(t)\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$= \begin{pmatrix}c & is\\ is & c\end{pmatrix}$$

$|\alpha|=1$

Here $M(t) = \begin{pmatrix}c^2 & s^2\\ s^2 & c^2\end{pmatrix}$.

$$D \longrightarrow D \otimes D$$

$$\alpha D \longmapsto \alpha^2 (D \otimes D) \quad \text{not linear}$$

Interesting cases:

(a) $M(t)e_b = e_a$, $a \neq b$ perfect state transfer

(b) $M(t)e_b = e_{\underset{\sim}{b}}$, vertex b is periodic

(c) $M(t)e_b = \frac{1}{n}\underset{\sim}{1}$, local uniform mixing

For the continuous walk on K_2, we have

(a) local uniform mixing from each vertex at time $\frac{\pi}{4}$.

$$M(t) = \frac{1}{2} J, \qquad U(t) = \frac{1}{\sqrt{2}} J$$

(b) perfect state transfer at time $\frac{\pi}{2}$

$$M(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad U(t) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

(c) periodicity at time π

$$M(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad U(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

In terms of density matrices, if the initial state is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,

then at time t it is

$$\begin{pmatrix} c & is \\ is & c \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c & -is \\ -is & c \end{pmatrix} = \begin{pmatrix} c & is \\ is & c \end{pmatrix}\begin{pmatrix} c & -is \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} c^2 & -isc \\ isc & s^2 \end{pmatrix}$$

t	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π
	?	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

P_3:

$$\phi(P_3, t) = t^3 - 2t$$

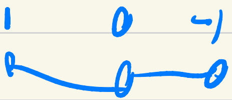
eigenvalues $\quad -\sqrt{2} \qquad 0 \qquad \sqrt{2}$

$z_r, \ \|z_r\| = 1$

$E_r = z_r z_r^*$

eigenvectors

$$\left\{ \begin{matrix} 1 & 1 & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -1 & 1 \end{matrix} \right.$$



idempotents

$$\frac{1}{4}\begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{bmatrix} \qquad \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad \frac{1}{4}\begin{bmatrix} 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}$$

$$E_1 \qquad\qquad E_2 \qquad\qquad E_3$$

$$U(t) = e^{it\sqrt{2}} E_1 + E_2 + e^{it\sqrt{2}} E_3$$

$$U\left(\tfrac{\pi}{\sqrt{2}}\right) = -E_1 + E_2 - E_3 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

gives an automorphism of P_3

We have perfect state transfer from vx 1 to vx 3 at time $\pi/\sqrt{2}$. Further $U(\pi\sqrt{2}) = E_1 + E_2 + E_3 = I$

Theorem If X admits perfect state transfer from a to b at time τ, it admits perfect state transfer from b to a at time τ.

Proof 1: If $U(\tau) e_a e_a^T U(-\tau) = e_b e_b^T$ then, taking complex conjugates, we get

$$e_b e_b^T = \overline{U(\tau)} e_a e_a^T \overline{U(-\tau)} = U(-\tau) e_a e_a^T U(\tau)$$

and so $U(\tau) e_b e_b^T U(-\tau) = e_a e_a^T$.

Proof 2: $M(t)$ is symmetric. $\square$

Products

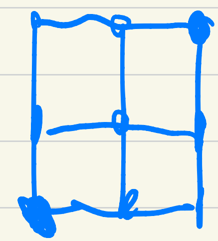
Theorem $U_{X \square Y}(t) = U_X(t) \otimes U_Y(t)$ $\square$

$\Rightarrow$ If X has pst from u to v at time τ and Y has pst from w to x at time τ, then $X \square Y$ has pst from (u, w) to (v, x) at time τ.

Corollary The d-cube has perfect state transfer from $\underset{\sim}{0}$ to $\underset{\sim}{1}$ at time $\pi/2$.

The Cartesian power $P_3^{\square d}$ has diameter $2d$.

Corollary $P_3^{\square d}$ has perfect state transfer between pairs of vertices at distance $2d$. $\square$



A walk has uniform mixing at time τ if there is local uniform mixing at time τ from each vertex; equivalently if $U(\tau)$ is flat.

flat = $\left\{ \begin{array}{l} \text{all entries have the} \\ \text{same absolute} \\ \text{value} \end{array} \right.$

Theorem If X & Y have uniform mixing at time τ, so does $X \square Y$. $\square$

Complete graphs

multiplicities

As $X(K_n) = J - I$ has eigenvalues $(n-1)^{(1)}$ & $(-1)^{(n-1)}$,

$E_0 = \frac{1}{n} J$ and $E_1 = (I - \frac{1}{n} J)$. Hence we have the spectral decomposition

$$A = (n-1)\frac{1}{n}J + (-1)\left(I - \frac{1}{n}J\right)$$

and

$$U(t) = e^{(n-1)it}\frac{1}{n}J + e^{-it}\left(I - \frac{1}{n}J\right).$$

Therefore

$$U(t) = e^{-it}\left[\frac{e^{nit}}{n} J + I - \frac{1}{n} J \right]$$

$$= e^{-it}\left[I + \frac{e^{nit}-1}{n} J \right]$$

and so $M_{K_n}(t)$ converges to I as $n \to \infty$.

Bipartite graphs

If X is bipartite, there is a partition of its vertices into two classes, each of which is a coclique. So we can write $A(X)$ in partitioned form

$$A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad A(P_3)$$

Lemma If X is bipartite, A and $-A$ are similar.

Proof

$$\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -B \\ -B^T & 0 \end{pmatrix}. \qquad \square$$

Corollary If X is bipartite & θ is an eigenvalue of X, then $-\theta$ is an eigenvalue with the same multiplicity. $\square$

If $\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix}$ then

$$\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} By \\ B^T x \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -By \\ B^T x \end{pmatrix} = \begin{pmatrix} -\theta x \\ \theta y \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix}.$$

This provides another proof of the lemma, and relates the eigenvectors of $-\theta$ to those of θ.

Further, if

$$E = \begin{pmatrix} E_{00} & E_{01} \\ E_{01}^{T} & E_{11} \end{pmatrix}$$

is a spectral idempotent of the bipartite graph X corresponding to the eigenvalue θ, then

$$\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} E \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} -E_{00} & E_{01} \\ -E_{01}^{T} & E_{11} \end{pmatrix} = \begin{pmatrix} E_{00} & -E_{01} \\ -E_{01}^{T} & E_{11} \end{pmatrix}$$

is the idempotent corresponding to $-\theta$.

X bipartite, $A = A(X)$

$$\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \exp(itA) \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} = \exp(-itA)$$

$$\begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix} \qquad \begin{pmatrix} U_{00} & -U_{01} \\ -U_{10} & U_{11} \end{pmatrix} = \begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix}^{-1} = \overline{\begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix}}.$$

So U_{00} is real & $\overline{U_{01}} = -U_{01}$ i.e. $U_{01} = iS$, S real

$$\exp(itA) = \begin{pmatrix} C_1 & iS \\ iS^T & C_2 \end{pmatrix},$$ C_1, C_2 & S real; C_1, C_2 symmetric