L22/01

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Formal power series

term is redundant

We could define a formal power series over a ring R to be a function for $\mathbb{N}$ to R.

could be a field, or the ring of matrices over a field

More traditionally, it is given as a sum

$$\sum_{k\geq 0} a_k t^k$$

where $a_k \in R$; the terms in the sum are elements of the polynomial ring $R[t]$.

Any polynomial in $\mathbb{R}[t]$ is a power series, and the addition and multiplication rules are the rules we use for polynomials. By

$$[t^k, A(t)] \qquad (*)$$

we denote the coefficient of t^k in the series $A(t)$.

If

$$A(t) = \sum a_k t^k, \qquad B(t) = \sum b_k t^k,$$

then $$[t^k, A(t)+B(t)] = [t^k, A(t)] + [t^k, B(t)] = a_k + b_k$$

and

$$[t^k, A(t)B(t)] = \sum_{i=0}^{k} [t^i, A(t)][t^{k-i}, B(t)]$$

$$= \sum_{i=0}^{k} a_i b_{k-i}$$

We define the order $\operatorname{ord}(f)$ of a power series to be the least integer i such that $(t^i, f) \neq 0$.

The norm $\|f\|$ of f is $2^{-\operatorname{ord}(f)}$.

large powers of t are small

This is a norm in the usual sense. We have

(a) $\|f\| \geq 0$, & if $\|f\| = 0$ then $f = 0$.

(b) if $\alpha \in \mathbb{R} \setminus 0$, then $\|\alpha\| = 1$ and $\|\alpha f\| = \|f\|$

(c) $\|f+g\| \leq \min\{\|f\|, \|g\|\} \leq \|f+g\|$

non-Archimedian

As $\mathrm{ord}(fg) = \mathrm{ord}(f) + \mathrm{ord}(g)$, we have

$$\|fg\| \leq \|f\| \, \|g\|$$

note: R might have divisors of zero so $\leq$, not $=$

Let V be a metric space. A sequence $(v_n)_{n\geq 0}$ is **Cauchy** if, for each real $\varepsilon > 0$, there is an integer N such that $|v_m - v_n| < \varepsilon$ if $m, n \geq N$. Two Cauchy sequences are **equivalent** if their difference converges to 0. The equivalence classes of Cauchy sequences form the completion of V with respect to the metric.

Theorem The ring of formal power series $R[[t]]$ is the completion of the ring of polynomials $R[t]$.

Completions

Let R be a ring. Then

(a) $R^{\mathbb{N}}$ is a ring

(b) Given a metric on $R^{\mathbb{N}}$, the Cauchy sequences form a subring of $R^{\mathbb{N}}$; calling $\mathcal{C}(R)$

(c) The Cauchy sequences that converge to zero form an ideal E in $\mathcal{C}(R)$. The quotient $\mathcal{C}(R)/E$ is the completion.

Theorem Let $(f_r)_{r\geq 0}$ be a sequence of power series. If $\|f_r\| \to 0$ as $r \to \infty$, then $\sum_{r\geq 0} f_r$ is defined. $\square$

Lemma Let f be a power series with constant term 0. Then series $1+f+f^2+\cdots$ is defined and is the multiplicative inverse of $1-f$. $\square$

So a power series is invertible if & only if its constant term is invertible.

Generating Functions

Let Ω be a set (e.g. of strings formed from a & b)

and let $wt : \Omega \to \mathbb{N}$ be a function from Ω to $\mathbb{N}$

s.t. for all non-negative integers i,

$$|\{\alpha \in \Omega : wt(\alpha) = i\}| < \infty .$$

We call wt a weight function. Then the sum

$$\sum_{\alpha \in \Omega} t^{wt(\alpha)}$$

is the generating function for (Ω, wt).

Rules

(a) sum

(b) predict

(c) sequences

Sum: Suppose $\Omega_1 \cap \Omega_2 = \emptyset$, and the weight functions are wt_1, wt_2 respectively. Define wt on $\Omega_1 \cup \Omega_2$ by

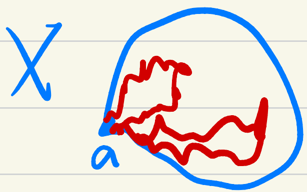
$$wt(\alpha) = \begin{cases} wt_1(\alpha), & \alpha \in \Omega_1; \\ wt_2(\alpha), & \alpha \in \Omega_2. \end{cases}$$

If the generating functions for Ω_1 & Ω_2 are respectively A_1 & A_2, the generating function for (Ω, w) is $A_1 + A_2$.

Products: We define a weight function on $\Omega_1 \times \Omega_2$ by

$$wt(\alpha_1, \alpha_2) = wt_1(\alpha_1) + wt_2(\alpha_2)$$

If A_1 & A_2 are the generating functions for Ω_1 & Ω_2, the generating function for $(\Omega_1 \times \Omega_2, wt)$ is $A_1 A_2$.



Sequences:

Assume no element of Ω has weight zero. The set

$$\bigcup_{k \geq 0} \Omega^k$$

has generating function $\frac{1}{1-A}$. We may

denote this by W^*.

Examples

(a) Assume $\Omega = \{a, b\}$ & $wt(a) = wt(b) = 1$. The generating function for Ω is $2t$. The generating function for $\Omega^* = \frac{1}{1-2t} = \sum_k 2^k t^k$

(b) Assume $\Omega = \{a, b\}$ & $wt(a) = 1$, $wt(b) = 2$.

Then the generating function for Ω is $t + t^2$ and the generating function for $\Omega^* = \frac{1}{1-t-t^2} =: A(t)$

Note that $[t^k, A(t)]$ is the number of ways we can write k as a sum of a sequence of 1's & 2's.

If $A(t) = \sum_{r\geq 0} a_r t^r$ then $1 = (1-t-t^2)A(t)$. We have

$$[t^k, (1-t-t^2)A(t)] = a_k - a_{k-1} - a_{k-2} \quad (k \geq 2)$$

and

$a_0 = 1 \quad a_1 = ?$

$$[t^k, 1] = \begin{cases} 1, & k=0; \\ 0, & k>0; \end{cases}$$

$\Rightarrow a_k = a_{k-1} + a_{k-2} \quad k \geq 2$

It follows (after some work) that a_k is the k-th Fibonacci number.

(c) Suppose $a_{k+1} = a_k + a_{k-1}$ $(k \geq 1)$. Then

$$\begin{pmatrix} a_{k+1} \\ a_k \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \overbrace{\begin{pmatrix} a_k \\ a_{k-1} \end{pmatrix}}^{F}$$

and hence

$$\begin{pmatrix} a_{k+1} \\ a_k \end{pmatrix} = A^k \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \qquad k \geq 0$$

So

$$\sum_{k\geq 0} t^k \begin{pmatrix} a_{k+1} \\ a_k \end{pmatrix} = \sum_{k\geq 0} t^k F^k \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = (I - tF)^{-1} \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}$$

and

$$\sum_{k\geq 0} t^k a_k = e_2^T \left[(I - tF)^{-1} \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \right]$$

$$\left(I - t\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\right)^{-1} = \begin{pmatrix} 1-t & -t \\ -t & 1 \end{pmatrix}^{-1} = \frac{1}{1-t-t^2} \begin{pmatrix} 1 & t \\ t & 1-t \end{pmatrix}$$

Walk generating functions

The ij-entry of A^k is the number of walks from i to j in X with length k. If we fix i & j, the generating function for the sequence $((A^k)_{i,j})_{k\geq 0}$ is $((I-tA)^{-1})_{i,j}$. So we define

$$W(X,t) = (I-tA)^{-1}$$

and call it the walk generating function of X

The matrix $(tI-A)^{-1}$ is the resolvent of A and plays a key role in spectral theory. If the eigenvalues of A are $\theta_1, \ldots, \theta_d$ with corresponding spectral idempotents $E_1, \ldots, E_d$, the spectral decomposition of $(tI-A)^{-1}$ is

$$(tI-A)^{-1} = \sum_{r=1}^{d} \frac{1}{t-\theta_r} E_r$$

Hence

$$\boxed{((tI-A)^{-1})_{i,j} = \sum_{r} \frac{(E_r)_{i,j}}{t-\theta_r}.}$$

A walk is closed if its first & last vertices are equal. The generating function for closed walks in X starting at u is $W(X,t)_{u,u}$; we denote it by $C_u(X,t)$ and use $C(X,t)$ to denote $\sum_{u \in V(X)} C_u(X,t)$, the generating function for all closed walks in X. Finally $C_u^{(1)}(X,t)$ is the generating function for closed walks on u that return just once.

Lemma $$C_u(X,t) = \frac{1}{1 - C_u^{(1)}(X,t)}$$

Proof A closed walk starting at u is a sequence of closed walks that start at u and return exactly once.

The lemma follows. $\square$

Remarks:

(a) If u has valency k then $C_u^{(1)}(X,t) = kt^2 + \dots$

(b) $$C_u^{(1)}(X,t) = 1 - \frac{1}{C_u(X,t)}$$

We have $C_u(X,t) = (I-tA)^{-1}_{u,u}$ and so

$$t^{-1}C_u(X,t^{-1}) = ((tI-A)^{-1})_{u,u}$$

By Cramer's rule

$$(tI-A)^{-1}_{u,u} = \frac{\det(tI-A(X\setminus u))}{\det(tI-A(X))} = \frac{\varphi(X\setminus u,t)}{\varphi(X,t)}$$

Lemma $t^{-1}C_u(X,t^{-1}) = \dfrac{\varphi(X\setminus u,t)}{\varphi(X,t)}$

If A has eigenvalues $\theta_1, \ldots, \theta_d$ with respective multiplicities $m_1, \ldots, m_d$, then

$$\operatorname{tr}(A^k) = \sum_{r=1}^{d} m_r \theta_r^k$$

and hence

$$\sum_{k \geq 0} t^k \operatorname{tr}(A^k) = \sum_{k,r} m_r \theta_r^k t^k = \sum_r \frac{m_r}{1 - t\theta_r}.$$

But LHS is

$$\operatorname{tr}(I - tA)^{-1} = \sum_{u \in V(X)} C_u(X,t) = \sum_u \frac{t^{-1} \phi(X \setminus u, t^{-1})}{\phi(X, t^{-1})}$$

Now if $p(t)$ is a polynomial with zeros $\theta_1, \dots, \theta_d$ and respective multiplicities $m_1, \dots, m_d$ then

$$\frac{p'(t)}{p(t)} = \sum_r \frac{m_r}{t-\theta_r}$$ (exercise)

So

$$\sum_r \frac{m_r}{1-t\theta_r} = t^{-1} \sum_r \frac{m_r}{t^{-1}-\theta_r} = t^{-1} \frac{\phi'(X, t^{-1})}{\phi(X, t^{-1})}$$

and therefore $\sum_{u \in V} \phi(X-u, t) = \phi'(X, t)$.