L15/01

$d(\theta^4-3\theta^2+1) = \theta^3-2\theta$

P_n $\quad 0 - 1 - 2 - 3 - 4 - \cdots$

$\quad 1 \quad \theta \quad \theta^2-1 \quad \theta^3-2\theta \quad \theta^4-3\theta^2+1 \quad \cdots$

eigenvalue θ

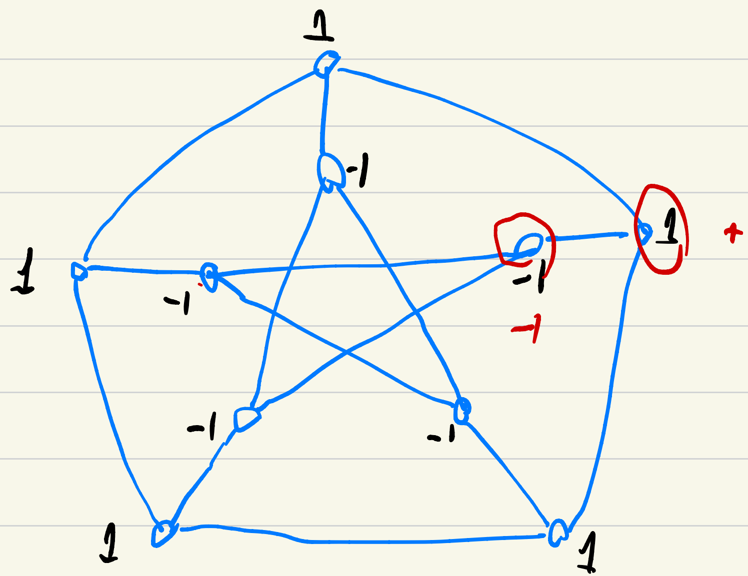
1) no eigenvector for P_n is zero on the first vertex

2) we're 1 on the first vx, we get a reccurrence for the k-th entry.

(3) looking at the last vx, we get a poly which has θ as a zero

$\theta^5-4\theta^3+3\theta = 0$

Petersen



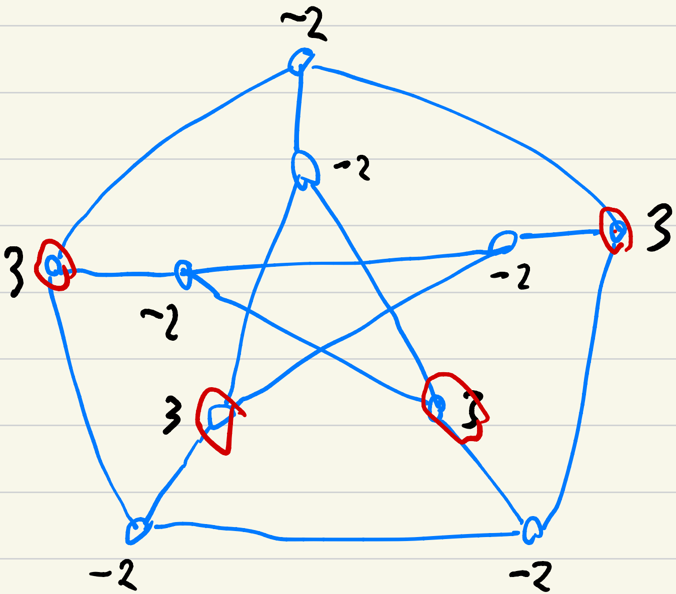
eigenvalue 3

eigenvalue 1

-2

$$\alpha \le \frac{v}{1-\frac{k}{\tau}} = \frac{10}{1-\frac{3}{-2}}$$

$$= 4$$


−2
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3
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3
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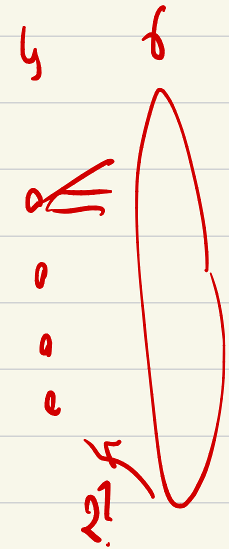
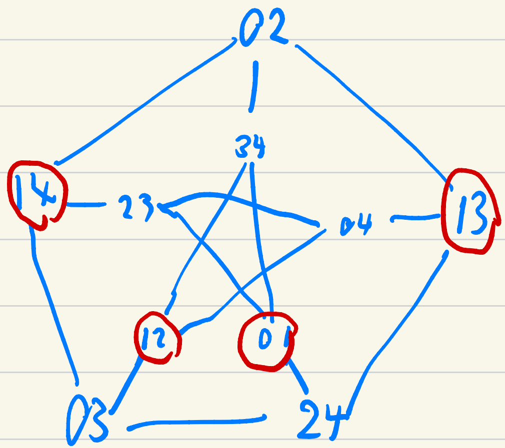
−2





$\{0,1,2,3,4\}$

Tensor & Kronecker products

If V and W are vector spaces over $\mathbb{F}$,
then $V \otimes W$ is the set of all finite $\mathbb{F}$-linear
combinations of the symbols $v \otimes w$ $(v \in V, w \in W)$
where:

$$(v \otimes (w_1 + w_2)) = v \otimes w_1 + v \otimes w_2$$

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$$

$$\alpha(v \otimes w) = \alpha v \otimes w = v \otimes \alpha w$$

The symbols $v \otimes w$ do not in general form a basis.

But if $e_1, \dots, e_m$ & $f_1, \dots, f_n$ are bases for V & W respectively, then $\{ e_i \otimes f_j : 1 \le i \le m, 1 \le j \le n \}$ is a basis for $V \otimes W$.

We call $V \otimes W$ the tensor product of V & W.

More concretely, if A is a $k \times l$ matrix and B is $m \times n$, the Kronecker product $A \otimes B$ is the $km \times ln$ matrix

$$\begin{pmatrix} A_{1,1}B & \cdots & A_{1l}B \\ \vdots & & \\ A_{k1}B & \cdots & A_{kl}B \end{pmatrix}$$

$$(A \otimes B)_{is,jt} = A_{ij} B_{st}$$

$$\begin{pmatrix}1\\2\end{pmatrix} \otimes \begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}3\\4\\6\\8\end{pmatrix}$$

$$\begin{pmatrix}1\\2\end{pmatrix} \otimes \begin{pmatrix}3 & 4\end{pmatrix} = \begin{pmatrix}3 & 4\\6 & 8\end{pmatrix}$$ outer product

columns of A

A is $m \times n$ $\quad \mathrm{vec}(A) = \begin{pmatrix}Ae_1\\ \vdots \\ Ae_m\end{pmatrix}$

linear

vec : $m \times n$ matrices $\longrightarrow$ $mn \times 1$ matrices

We can also define $A \otimes B$ to be the linear operator such that

$$A \otimes B : u \otimes v \longmapsto Au \otimes Bv.$$

We list some important properties of the tensor product:

(a) $(A \otimes (B_1 + B_2)) = A \otimes B_1 + A \otimes B_2$

$((A_1 + A_2) \otimes B) = A_1 \otimes B + A_2 \otimes B$

distributivity

$\alpha(A \otimes B) = \alpha A \otimes B = A \otimes \alpha B$

(b) If the required products exist, then

$$(A \otimes B)(C \otimes D) = AC \otimes BD \qquad (*)$$

(e) $\operatorname{tr}(A \otimes B) = \operatorname{tr}(A)\operatorname{tr}(B)$

Note that (b) implies that if $Ay = \lambda y$ & $Bz = \mu z$, then $(A \otimes B)(y \otimes z) = \lambda\mu(y \otimes z)$ — Kronecker products of eigenvectors are eigenvectors, corresponding eigenvalues are products.

The Schur product $A \circ B$ of matrices A & B of order $m \times n$

same order is the $m \times n$ matrix s.t.

$$(A \circ B)_{i,j} = A_{i,j} B_{i,j}$$

bad student's product

Note that $A \circ B$ is a principal submatrix of $A \otimes B$?

The Schur product plays nicely with the Kronecker product; thus

$$(A \otimes B) \circ (C \otimes D) = (A \circ C) \otimes (B \circ D)$$

Also

$$\langle A, B\rangle = \operatorname{tr}(A^* B) = \operatorname{sum}(\bar{A} \circ B)$$

sum of the entries

$$\operatorname{sum}(M) := \sum_{i,j} M_{i,j}$$

Schur

Theorem If A & B are $n \times n$ matrices and A & B are positive semidefinite, so is $A \circ B$.

If $\mathcal{A}$ & $\mathcal{B}$ are algebras, the set of operators

$$\{A \otimes B : A \in \mathcal{A}, B \in \mathcal{B}\}$$

$(A \otimes B)(C \otimes D) = \sim$

$\{A \otimes I : A \in \mathcal{A}\}$

spans an algebra denoted $\mathcal{A} \otimes \mathcal{B}$.

Suppose $\mathcal{A} = Mat_{m \times m}(\mathbb{F})$. $\mathcal{B} = Mat_{n \times n}(\mathbb{F})$ and $\mathcal{W} = Mat_{m \times n}(\mathbb{F})$

If $M \in \mathcal{W}$, we have a map $M : \to AMB^*$ $(A \in \mathcal{A}, B \in \mathcal{B})$

This map is a homomorphism, in fact an isomorphism.

$A \otimes B$

(*) It follows that $\mathcal{A} \otimes \mathcal{B} \cong End(Mat_{m \times n}(\mathbb{F}))$

So the general form of an element of $\mathrm{End}(\mathrm{Mat}_{m\times n})$ is

$$M \longmapsto \sum_i A_i M B_i^*$$