L08/01

Topics

Quantum walks

Quantum homomorphisms

Tools

Algebraic graph theory

- spectral decomposition
- positive semidefinite matrices
- Kronecker product
- algebras, normal matrices
- walk generating functions
- homomorphisms & automorphisms

Contents

Norms

matrix exponential $\exp(A)$ $\qquad$ A square

$$\exp(A) := \sum_{n \geq 0} \frac{1}{n!} A^n$$

The problem is to show that the limit exists.

Assume V is a vector space over $\mathbb{C}$. A norm on V is a real-valued function on V such that:

(a) if $x \in V$, then $\|x\| \geq 0$ and, if $\|x\| = 0$, then $x = 0$.

(b) if $x \in V$, $c \in \mathbb{C}$ then $\|cx\| = |c|\,\|x\|$.

(c) $\|x+y\| \leq \|x\| + \|y\|$.

Examples:

(a) Euclidean norm: If $x = x_1, \dots, x_d$ then

$$\|x\| = \sqrt{\sum_i |x_i|^2}$$

(b) Assume $p \in \mathbb{R}$ & $p \geq 1$. Then

$$\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$$

This is the ℓ^p-norm. The ℓ^2-norm is the Euclidean norm.

(for us, $p \in \{1, 2, \infty\}$)

A sequence is summable if its ℓ^1-norm is finite. It is bounded if its ℓ^∞-norm is finite.

A normed vector space V is complete if every Cauchy sequence of elements of V converges to an element of V. A Banach space is a complete normed vector space.

(c) If $\langle u,v\rangle$ is an inner product on V then $\langle u,u\rangle^{1/2}$ is a norm on V.

Remark: if V is real, and $\langle u,u\rangle := \sum u_i^2$, we recover the Euclidean norm. If V is complex then we assume $\langle u,v\rangle = \sum_i \bar{u}_i v_i$

linear in the <u>first</u> coordinate

(d) If $V = Mat_{m\times n}(\mathbb{C})$ we may define

$$\langle A, B\rangle = tr(A^* B) = sum(\bar{A} \circ B)$$

This is an inner product, and provides what is known as the Frobenius norm or the Hilbert-Schmidt norm, or the matrix norm, or the trace norm.

ambiguous

wrong

(e) Induced norms Let V be a normed vector space (with norm $\|\cdot\|$) and assume $L: V \to V$ is linear. We define $\|L\|$ by

$$\|L\| = \sup_{\|v\|=1} \{\|Lv\|\}$$

exists because $\dim(V) < \infty$

This may also be called an operator norm.

Remarks:

(a) We can extend this definition to linear maps $L: V \to W$ with V & W normed

(b) $\|\cdot\|$ is an induced norm on an algebra of operators, then $\|AB\| \leq \|A\| \|B\|$.

Exercise

If $\dim(V) < \infty$, all norms are equivalent, we work with what is most convenient.

If $\dim(V)$ is infinite, the choice of norms matters.

real polynomials

Consider norms on $\mathbb{R}[t]$

choose function $f \geq 0$

$$\langle p, q \rangle = \int p(t) q(t) f(t) dt$$

Exponential

If we have a proof that $\exp(a)$ is defined for all complex a, and we replace $|\cdot|$ by $\|\cdot\|$ (for some induced norm) and a by the square matrix A, then we have a proof that $\exp(A)$ is defined for all matrices A.

Algebras

An algebra is a vector space with an associative multiplication & a multiplicative identity. — not always assumed (by analysts)

Examples

(a) real or complex polynomials. — infinite dimensional

$\mathbb{R}[t]$

(b) $n \times n$ matrices over a field.

An algebra is a special type of ring.

An algebra $\mathcal{A}$ over $\mathbb{C}$ is *-closed if $M^* \in \mathcal{A}$ whenever $M \in \mathcal{A}$. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

An element M of a ring is nilpotent if $M^k = 0$ for some k. An ideal J is nilpotent if there is an integer k such that $J^k = 0$

the least value of k is the index

Example

The $n \times n$ upper-triangular matrices form an algebra; the strictly upper triangular matrices form a nilpotent ideal (of index n) in it.

Normal matrices

A square matrix A is normal if $AA^* = A^*A$.

(In the real case, this means $AA^T = A^TA$)

Examples:

(a) Hermitian matrices, real symmetric matrices.

(b) Unitary matrices, orthogonal matrices

(c) If M is normal, any complex polynomial in M is normal.

(d) If L is unitary & D is diagonal, then LDL^{-1} is normal, i.e. if M is unitarily diagonalizable, it is normal. converse?

Note that if N is normal then the algebra $\langle N, N^* \rangle$ is $*$-closed and commutative.

Lemma A complex matrix M is normal if & only if

$$\langle Mu, Mu \rangle = \langle M^* u, M^* u \rangle$$

for all u.

Exercise

$\square$

Theorem If $\mathcal{A}$ is $*$-closed commutative algebra of $n \times n$ matrices, then $\mathbb{C}^n$ has an orthogonal basis consisting of eigenvectors for $\mathcal{A}$.

Definition: A vector is an eigenvector for $\mathcal{A}$ if & only if it is an eigenvector for each matrix in $\mathcal{A}$.

$A = A^* \Rightarrow \langle Ax, y \rangle = \langle x, Ay \rangle \ \forall x, y$

A fixes $\mathcal{U} \Rightarrow A$ fixes $\mathcal{U}^\perp$

Proof Three parts:

(a) If $\mathcal{A}$ fixes a subspace U of $\mathbb{C}^n$, it fixes $U^\perp$.

(b) If $\mathcal{A}$ fixes U & $L \in \mathcal{A}$, let L_0 be the "restriction" of L to U. Then $\langle L_0 : L \in \mathcal{A} \rangle$

domain = codomain

is commutative and $*$-closed.

(c) If $L \in \mathcal{A}$ and $Lz = \lambda z$, then $\ker(L - \lambda I)$ is $\mathcal{A}$-inv. $\square$

Corollary If H is normal, acting on $\mathbb{C}^n$, then $\mathbb{C}^n$ has an orthonormal basis consisting of eigenvectors for H. $\square$

Equivalently, a normal matrix is unitarily diagonalizable.

Theorem (Schur) A square complex matrix is unitarily similar to an upper triangular matrix. $\square$

Recall that the eigenvalues of a triangular matrix are its diagonal entries.

Projections

If V is an inner product space and $L : V \to V$, the adjoint L^* of L is defined by requiring that

$$\langle u, Lv \rangle = \langle L^* u, v \rangle \quad \forall u, v$$

If $L = L^*$ then L is self-adjoint.

eg L symmetric ; L Hermitian

If $L : V \to V$, then L fixes the subspace U of V if & only if L^* fixes $U^\perp$.

Projections A linear map $P: V \to V$ is a projection if

(a) $L = L^*$ self-adjoint

(b) $L^2 = L$ idempotent

If $U \leq V$ and $u_1, \ldots, u_d$ is an orthonormal basis for U, then $\sum_i u_i u_i^*$ is a projection, with image U & kernel $U^\perp$.

outer product

a) If P is a projection, so is $I-P$.

$$(I-P)^2 = I-2P+P^2 = I-P$$

b) If P, Q are projections and $PQ=0$, then

$P+Q$ is a projection.

exercise

c) If P and Q are commuting projections, PQ is a projection and $\operatorname{im}(PQ) \subseteq \operatorname{im}(P) \cap \operatorname{im}(Q)$.

$P(I-P)=0$

Spectral decomposition

Assume $M : V \to V$ is normal with distinct eigenvalues $\theta_1, \dots, \theta_d$. Let E_r be the matrix representing orthogonal projection onto the θ_r-eigenspace. Then

(a) $E_r = E_r^* = E_r^2$. $\qquad \sum_i u_i u_i^* \qquad uu^*$

(b) If $r \neq s$ then $E_r E_s = 0$.

(c) $\sum_r E_r = I$.

(d) $\operatorname{tr}(E_r)$ is the dimension of the θ_r-eigenspace.

(e) $ME_r = \theta_r E_r$.

Further

eigenvalue of M

spectral idempotent

$$M = MI = \sum_r ME_r = \sum_r \theta_r E_r$$

and if f is a function defined on the eigenvalue of M, then

$$f(M) = \sum_r f(\theta_r) E_r$$

"functional calculus"

In particular

$$M^* = \sum_r \bar{\theta}_r E_r .$$

$$M = \sum_r \theta_r E_r$$

$$M^2 = \sum_{r,s} \theta_r \theta_s E_r E_s = \sum_r \theta_r^2 E_r$$

$$M^k = \sum_r \theta_r^k E_r$$

$p(t)$ is a polynomial

$$p(M) = \sum_r p(\theta_r) E_r$$

If f is defined on $\{\theta_1, \dots, \theta_d\}$ then

$$f(M) := \sum f(\theta_r) E_r .$$

e.g. $\theta_i \geq 0$

$$M^{1/2} = \sum_r \sqrt{\theta_r}\, E_r$$

$$\sum \theta_r^k E_r$$

$$\downarrow$$

$$\sum_{k \geq 0} \frac{M^k}{k!}$$

e.g.

$$\exp(M) = \sum_r e^{\theta_r} E_r$$

Lemma The spectral idempotents of A are polynomials in A.

Lagrange interpolating polynomials

Proof. Assume $\theta_1, \dots, \theta_d$ are the distinct eigenvalues of A and $l_i(t) := \prod_{j : j \neq i} \frac{(t-\theta_j)}{(\theta_i - \theta_j)}$. Then $l_i(\theta_j) = \delta_{i,j}$

and $l_i(A) = E_i$ by spectral decomposition. $\square$

If U is unitary with spectral decomposition

$$U = \sum \varphi_r E_r$$

then $|\varphi_r| = 1$ for all r, so

$$U = \sum_r e^{i\theta_r} E_r \qquad (\theta_r \text{ real})$$

and therefore, if $H := \sum_r \theta_r E_r$,

$$\boxed{U = \exp(iH), \quad H = H^*}$$

iH is skew Hermitian

Hamiltonian

Any unitary matrix is an exponential.

If H is Hermitian and

$$U(t) := \exp(itH) \qquad (t \in \mathbb{R})$$

then the matrices $U(t)$ for $t \in \mathbb{R}$ determine a continuous quantum walk.

Example $H = A(K_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Then $H^{2n} = I$, $H^{2n+1} = H$ and

$$U(t) = \sum_{n \geq 0} \frac{(it)^n}{n!} H^n = \sum_{n \geq 0} \frac{(it)^{2n}}{(2n)!} I + \sum \frac{(it)^{2n+1}}{(2n+1)!} H$$

$$= \begin{bmatrix} \cos(t) & i\sin(t) \\ i\sin(t) & \cos(t) \end{bmatrix}$$

Example $M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ adjacency matrix of P_3

Char. polynomial $t^3 - 2t$, eigenvalues $-\sqrt{2}, 0, \sqrt{2}$

$$tI - M = \begin{bmatrix} t & -1 & 0 \\ -1 & t & -1 \\ 0 & -1 & t \end{bmatrix}$$

$$\operatorname{adj}(tI - M) = \begin{bmatrix} t^2-1 & t & 1 \\ t & t^2 & t \\ 1 & t & t^2-1 \end{bmatrix}$$

E_0, $\sqrt{2}$: $\frac{1}{4}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}$

E_1, 0: $-\frac{1}{2}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$

E_2, $-\sqrt{2}$: $\frac{1}{4}\begin{pmatrix} 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$

$$(tI - M)\operatorname{adj}(tI - M) = (t^3 - 2t)I$$