Coxeter Lecture Series

Fields Institute

Thematic Program on Asymptotic Geometric Analysis

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Lecture 3:

Multiplicative transforms and characterization of the Fourier transform.

Based on joint work with Semyon Alesker, Dmitry Faifman, Herman Konig and Vitali Milman.

This talk will revolve around the main ingredient in the proofs of the two following theorems:

Theorem [Alesker-A-Faifman-Milman]
The Fourier transform is characterized by mapping products to convolutions.

Theorem [A-Konig-Milman]
The derivative transform is characterized by the satisfying the chain rule.

To formally state the result regarding the Fourier transform, we need a few standard definitions and results.

The Schwartz space of rapidly decreasing complex valued functions on R^n is denoted by $S(n)$ and consists of functions satisfying

$$\sup_{x \in R^n} \left|\frac{\partial^\alpha f(x)}{\partial x^\alpha}(1 + |x|^l)\right| < \infty$$

where $\frac{\partial^\alpha f}{\partial x^\alpha} = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$.

The Fourier transform $F : S(n) \to S(n)$, defined by

$$(Ff)(\xi) = \int_{R^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx$$

is a linear topological isomorphism.

The space $S(n)$ has two structures of algebra given by the point-wise product and by convolution:

$$\cdot \quad : S(n) \times S(n) \to S(n),$$

$$* \quad : S(n) \times S(n) \to S(n).$$

Both operations are continuous with respect to both arguments simultaneously.

We denote by $S'(n)$ the topological dual of $S(n)$, also referred to as "distributions of tempered growth", and equip it with the weak topology.
(we mainly need it to contain delta-functions).

The canonical mapping $S(n) \to S'(n)$, is defined by

$$\langle \phi, f \rangle = \int_{R^n} f(x)\phi(x)dx.$$

has dense image and is injective. Identify $S(n) \subset S'(n)$

The following claim is standard:

The point-wise product on $S(n)$ extends to a separately continuous map $S(n) \times S'(n) \to S'(n)$ which is given explicitly by

$$\langle \phi, \psi \cdot f \rangle = \langle \phi \cdot \psi, f \rangle$$

and $S'(n)$ is thus a module over $S(n)$.

Also convolution extends to a separately continuous map $S(n) \times S'(n) \to S'(n)$ given by

$$\langle \phi, \psi * f \rangle = \langle \phi * (-Id^*)\psi, f \rangle$$

and $S'(n)$ is again a module over $S(n)$.
Here $((-Id^*)\psi)(x) = \psi(-x)$.

The Fourier transform extends to $S'(n)$ and maps product to convolution and vice-versa.

Theorem [Alesker-A-Milman]: Fourier transform is characterized by exchanging product and convolution.

More precisely: let $F : S \to S$, bijective, with bijective extension $F' : S' \to S'$ such that

$$F'(f * g) = (Ff) \cdot (F'g), \quad F'(f \cdot g) = (Ff) * (F'g)$$

Then it is the Fourier transform up to a linear term and possibly conjugation.

Theorem [A-A-Faifman-M]: Enough to assume just one of the equations, and then an extra diffeomeorphism of the base term appears.

Remark: [Jaming, '10] Has results regarding Zp, though with continuity etc.

Actually, without even explaining the setting, one may compose the unknown transform (exchanging product and convolution) with the Fourier transform, to get one which satisfies $T(f \cdot g) = (Tf) \cdot (Tg)$

So, the question becomes one about multiplicative transforms.

Remarks:

1. Difference between real valued and complex valued.
2. Non-degeneracy conditions can be important.
3. Assuming also that $T(f * g) = (Tf) * (Tg)$ simplifies things but is NOT needed.
4. <u>No continuity or linearity assumed.</u>

Theorem [A-A-Faifman-M]: For complex valued functions. Assume we are given a bijective map $F : S \to S$ which admits a bijective extension $F' : S' \to S'$ such that for every $f \in S$ and $g \in S'$ we have

$$F'(f \cdot g) = (Ff) \cdot (F'g)$$

Then there exists a C^∞-diffeomorphism $u : R^n \to R^n$ such that either $Ff = f \circ u$ for all $f \in S$
or $Ff = \overline{f \circ u}$ l . $f \in S$

Theorem [Alesker-A-Milman]: For real valued function, the only bijective $F : S \to S$, with bijective extension $F' : S' \to S'$ such that $F'(f \cdot g) = (Ff) \cdot (F'g)$ is composition with a diffeomorphism.

$$T(f \cdot g) = (Tf) \cdot (Tg)$$

If a certain function f is zero on an interval, then for all functions g supported on the interval,

$$f \cdot g = 0$$

Thus, $$(Tf) \cdot (Tg) = 0$$

And this for all g supported on the interval.

So, for any x such that there is some g supported on the interval with $(Tg)(x) \neq 0$, we have that $(Tf)(x) = 0$.

Things thus depend already on some kind of "non-degeneracy" condition.

For example: a simple case.
Assume that for every interval J and every $x \in J$ there exists a function g supported on J with $(Tg)(x) \neq 0$.

One gets that functions supported on a given set are mapped to functions supported on the same set.

Moreover, if two functions f_1 and f_2 coincide on some interval, then for any function g supported on it

$$f_1 \cdot g = f_2 \cdot g$$

So that

$$(Tf_1) \cdot (Tg) = (Tf_2) \cdot (Tg)$$

and by plugging in x for which $(Tg)(x) \neq 0$ we get

$$(Tf_1)(x) = (Tf_2)(x)$$

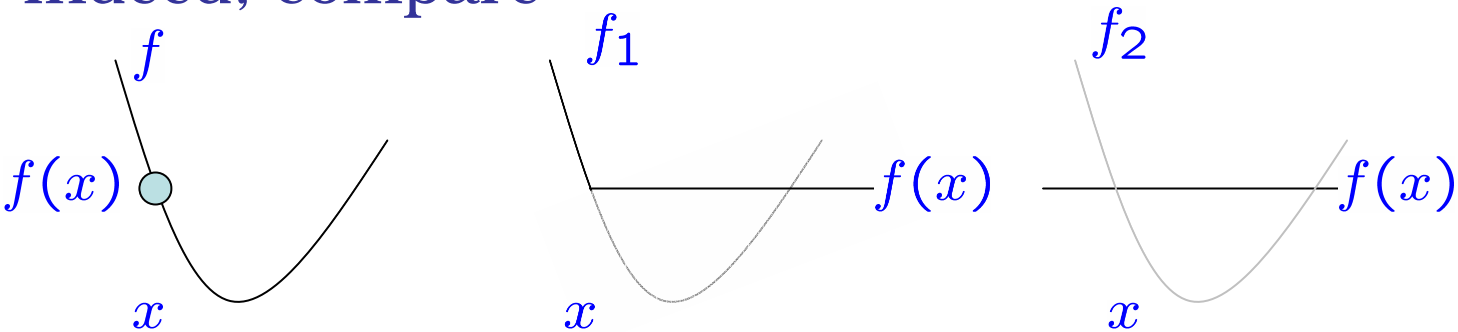
In other words, we have so called “localization”: two functions which agree on an interval, do so also in the image.

As we always deal with subclasses of continuous functions, these functions (both in the range and in the image) agree also on the endpoints.

From this we can conclude in the continuous case:
there exists some function $F : R \times R \to R$ such that

$$(Tf)(x) = F(x, f(x))$$

Indeed, compare



$$(Tf)(x) = (Tf_1)(x) = (Tf_2)(x)$$

Moreover, the function $F : R \times R \to R$ is no other than

$$F(x, f(x)) = T(f(x))(x)$$

It is thus multiplicative in the second argument

$$F(x, a \cdot b) = F(x, a)F(x, b)$$

and together with continuity this implies

$$F(x, f(x)) = f(x)^{p(x)}$$

where $p(x) : R \to R^+$ is also continuous.

Thus, the only transform from continuous functions to themselves which is non-degenerate in the way indicated above, and which satisfies

$$T(f \cdot g) = (Tf) \cdot (Tg)$$

is given by

$$(Tf)(x) = f(x)^{p(x)}$$

where $p(x) : R \to R^{+}$ is continuous.

Note that the non-degeneracy conditions are pretty strong, as they disallow even the simple (multiplicative)

$$(Tf)(x) = f(u(x))$$

for some good-natured $u(x) : R \to R$.

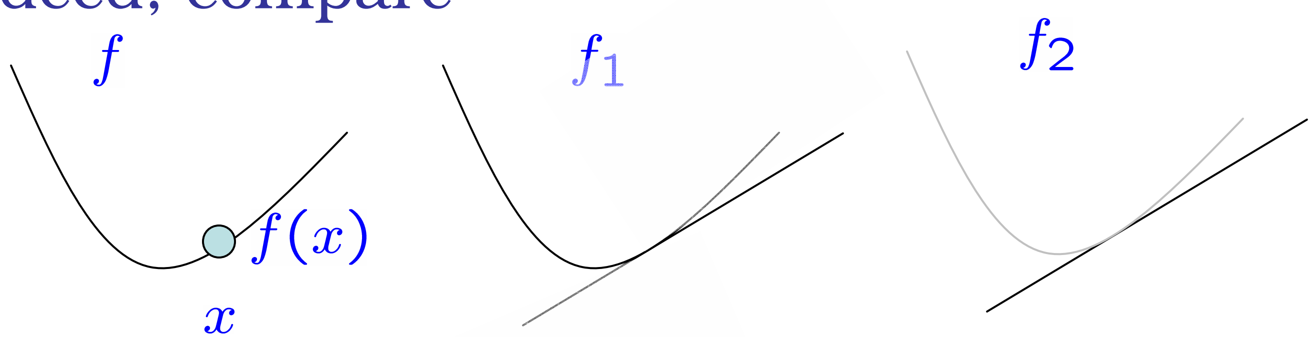
They were:
Assume that for every interval J and every $x \in J$ there exists a function g supported on J with $(Tg)(x) \neq 0$.

Similarly, in the C^1 case:

there exists some function $F : R^3 \to R$ such that

$$(Tf)(x) = F(x, f(x), f'(x))$$

Indeed, compare



$$(Tf)(x) = (Tf_1)(x) = (Tf_2)(x)$$

Now the function $F : R^3 \to R$ is given by

$$F(x, f(x), f'(x)) = T(f(x) + f'(x)(\cdot - x))(x)$$

So that it satisfies a more complicated equation

$$F(x, a \cdot b, ad + bc) = F(x, a, c)F(x, b, d)$$

and one can show after manipulations that actually

$$F(x, f(x)) = f(x)$$

that is, only the identity transform applies.

The non-degeneracy conditions
(which disallowed $(Tf)(x) = f(u(x))$)

were:
Assume that for every interval J and every $x \in J$ there exists a function g supported on J with $(Tg)(x) \neq 0$.

Without this condition, one needs to discover the point map $u(x) : R \to R$ which determines how the support of a function is changed.

This was some intuition on multiplicative maps. Similar, though different, considerations enabled us to prove the theorem on Fourier transform (there delta functions appear, which make things easier, but other difficulties come up, especially in the complex valued case.

Next let us describe the result regarding derivative, the proof of which has a similar flavor.

$$T(f \circ g) = ((Tf) \circ g)\, Tg$$

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Some well known solutions to the equation:

(a) the derivative $Tf = f'$

(b) composition $Tf = \frac{H \circ f}{H}$

(b) product of solutions, powers of solutions.

$$Tf = \frac{H \circ f}{H} \cdot |f'|^p \mathsf{sign}(f')$$

Theorem [A-Konig-M] Assume that $T : C^1(R) \to C(R)$ satisfies the chain rule

$$T(f \circ g) = ((Tf) \circ g)\, Tg$$

is locally surjective, and non-degenerate. Then there exist $p > 0$ and $0 < H \in C(R)$ such that

$$Tf = \frac{H \circ f}{H} \cdot |f'|^p \mathrm{sign}(f')$$

[Here non-degenerate means: there exists x and a bounded F in C^1 with existing limits at infinity such that (TF)(x) in not 0.]

Remarks:

Theorem [A-Konig-M] There does not exist a map $T : C(R) \to C(R)$ which satisfies the chain rule

$$T(f \circ g) = ((Tf) \circ g)\, Tg$$

co-cycle

and is non-degenerate.

$$Tf = \frac{H(f(x))}{H(x)} \cdot |f'|^p \mathrm{sign}(f')$$

co-boundary cohomology

Cohomological interpretation:

Consider the semigroup $G = (C^1(R), \circ)$ (op: composition)

It acts on $M = (C(R), \cdot)$ (op: point-wise multiplication)
by composition from the right $fH = H \circ f$

Functions from G^n to M will be denoted $F^n(G, M)$ us: n=1

The co-boundary operator $d^n : F^n(G, M) \to F^{n+1}(G, M)$
is given by (changing notation to additive)

$$\begin{aligned} d^n\varphi(g_1, \dots g_{n+1}) &= g_1\varphi(g_2, \dots, g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i \varphi(g_1, \dots, g_{i-1}, g_i g_{i+1}, \dots, g_{n+1}) \\ &+ (-1)^{n+1}\varphi(g_1, \dots, g_n) \end{aligned}$$

It
m
C
th $f = \frac{\;}{H}$

So one can say we identify $H^1(G, M) = Ker(d^1)/Im(d^0)$
as powers of the derivative

$$T(f \circ g) = ((Tf) \circ g)\, Tg$$

Theorem [A-Konig-M] Assume that $T : C^1(R) \to C(R)$ satisfies the chain rule

$$T(f \circ g) = ((Tf) \circ g)\, Ag$$

For some non-degenerate $A : C^1(R) \to C(R)$

Assume it is locally surjective, and non-degenerate.

Then up to power $p > 0$

$$Tf = \frac{G_1 \circ f}{G_2} \cdot f'$$

$$Af = \frac{G_2 \circ f}{G_2} \cdot f'$$

One difference between this talk and the previous is that here we investigate an equation

$$T(f \cdot g) = (Tf) \cdot (Tg)$$

whereas before it was a (double) inequality.

$$f \leq g \Leftrightarrow Tf \leq Tg$$

This is misleading as the latter may be written as

$$T(\mathsf{max}(f, g)) = \mathsf{max}(Tf, Tg)$$

The point is rather the difference between the operation of product and that of taking the maximum.

Let us briefly look at the operation of addition (say, of convex functions)

$$T(f+g) = (Tf) + (Tg)$$

Note that, for example in Cvx_0 , it is a bit similar to

$$f \leq g \Leftrightarrow Tf \leq Tg$$

but with the order relation given by

$$f \leq g \Leftrightarrow g - f \in Cvx_0$$

Operation of point-wise addition

Theorem [A-Milman] :

On the class $Cvx(R^n)$ the only bijections satisfying

$$T(f+g) = Tf + Tg$$

are induced by linear maps: $(Tf)(x) = Cf(Ax+b)$

$$epi(\mathcal{L}(f+g)) = epi(\mathcal{L}f) + epi(\mathcal{L}g)$$

Theorem [A-Milman] :

The Legendre transform is the unique Minkowski-graph-additive transform on the class $Cvx(R^n)$.

$$\|\cdot\|^*_{K_1+K_2} = \|x\|_{K_1} + \|x\|_{K_2}$$

Theorem [A-M]: The support map is the unique additive map on convex bodies $\mathcal{T}K = \|\cdot\|^*_K$

So, back to the conclusion that the Legendre transform is the right analogue of the support.

What happens when you change the condition of equality (say, as above) to "near equality"?

Usually one of to things: Stability, or Super-stability

Theorem [A-Milman] Let $n \geq 2$. Assume a bijection $T : Cvx(R^n) \to Cvx(R^n)$ is given and that for some fixed $0 < c \leq 1$ and $C \geq 1$, for every $f, g \in Cvx(R^n)$ we have that

$$c(Tf + Tg) \leq T(f + g) \leq C(Tf + Tg)$$

then there exists an affine $u : R^n \to R^n$ and a constant a such that for all $f \in Cvx(R^n)$ we have

$$(Tf)(x) = af(u(x))$$

In particular, $T(f+g) = Tf + Tg$.

Actually, it suffices to assume

This is **super**-stability

$$c_{f,g}(Tf + Tg) \leq T(f + g) \leq C_{f,g}(Tf + Tg)$$

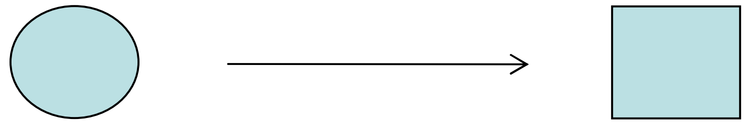
Theorem [A-Milman] Let $n \geq 2$. Assume a bijection $T : \mathcal{K}_0^n \to \mathcal{K}_0^n$ is given and that for some fixed $0 < c \leq 1$ and $C \geq 1$, for every $K_1, K_2 \subset \mathcal{K}_0^n$ we have that

$$c(TK_1 + TK_2) \subset T(K_1 + K_2) \subset C(TK_1 + TK_2)$$

then there exists $A \in GL_n$ such that

$$AK \subset T(K) \subset \left(\frac{C}{c}\right)^3 (AK).$$

So, one cannot extend the mapping



in an addition preserving manner to all $\mathcal{K}_0^n$, and not even "approximately". Same is true for order.

THANK YOU FOR YOUR (LONG LASTING) ATTENTION