# Coxeter Lecture Series

## Fields Institute

### Thematic Program on Asymptotic Geometric Analysis

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# "How very little determines a lot"

- Lecture 1: Abstract duality, the Legendre transform and a new duality transform.

- Lecture 2: Order isomorphisms and the fundamental theorem of affine geometry.

- Lecture 3: Multiplicative transforms and characterization of the Fourier transform.

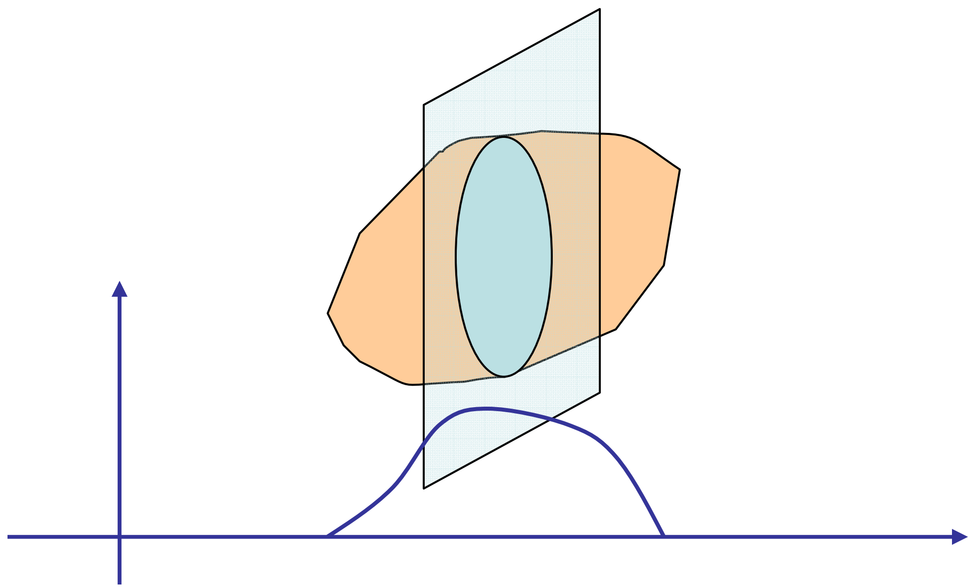
Lecture 1:

# Abstract duality, the Legendre transform and a new duality for geometric convex functions

Joint work with Vitali Milman

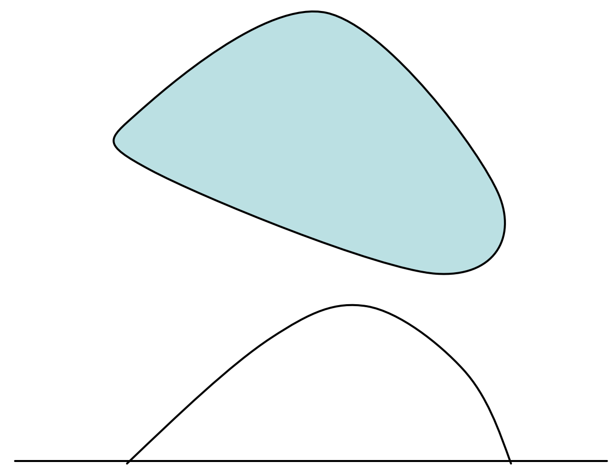
**Definition**: A function $f$ is called log-concave if $-\log(f)$ is a convex function.

These appear very naturally in geometry, for example as measure projections of convex sets:

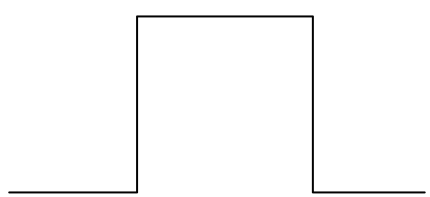


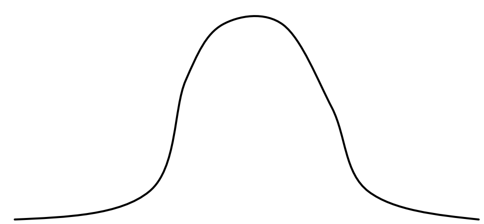
Measure projections of convex sets are dense in the family of all log concave functions [Borell]

In some sense, this is the right "completion" for the class of convex bodies.



$1_K$ is replaced by $e^{-\varphi}$, $\varphi$ convex



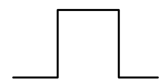


It is already common practice to use, in “Brunn Minkowski Theory”, log-concave measures instead of convex bodies.

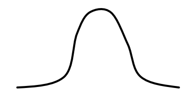
Note that for a convex body one may associate not only $1_K$ but also other log-concave functions such as $e^{-\|x\|_K^2}$ (which is occasionally more convenient).

Standard operations on convex bodies usually have a straightforward analogue for functions:

Volume = integral.



$1_K$



$e^{-\varphi}$

Obviously we have $$\int_{\mathbb{R}^n} 1_K = vol(K)$$

But also note that

$$\int_{\mathbb{R}^n} e^{-\|x\|_K^2} dx = \int_0^\infty vol(x : e^{-\|x\|_K^2} > t) dt = c_n vol(K)$$

## Minkowski addition = sup-convolution

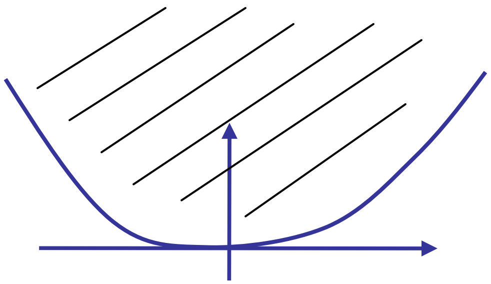
$$1_K \text{ ``}+\text{'' } 1_T = 1_{K+T}$$

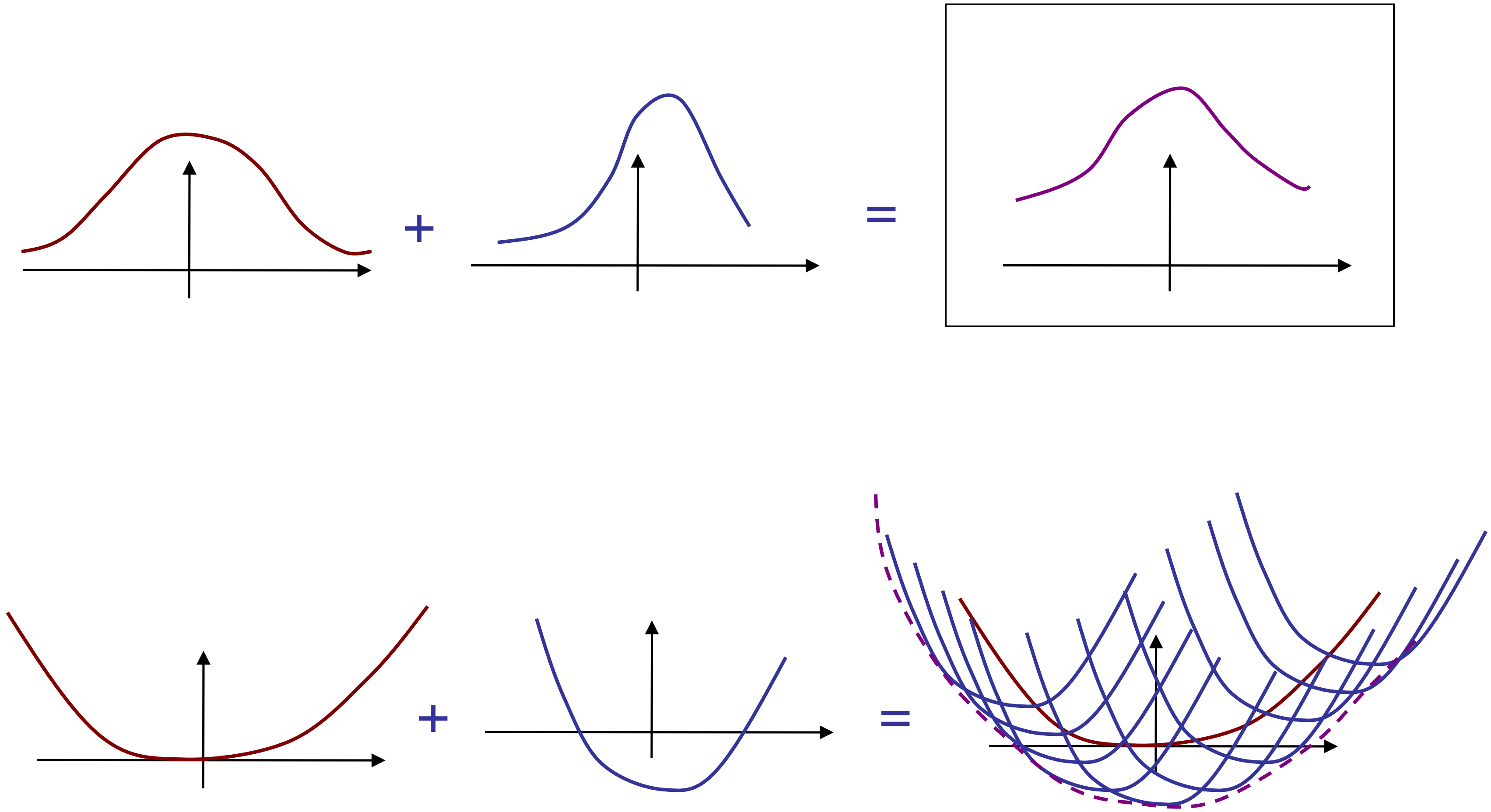
$$\left(e^{-\varphi} \text{ ``}+\text{'' } e^{-\psi}\right)(x) = \sup_y e^{-\varphi(y)} e^{-\psi(x-y)}$$

In the level of convex functions, this is
<u>Minkowski</u> addition of epi-graphs:

$$\sup_{y+z=x} e^{-\varphi(y)} e^{-\psi(z)} = e^{-(\varphi \Box \psi)}$$

$$epi(\varphi \Box \psi) = epi(\varphi) + epi(\psi)$$





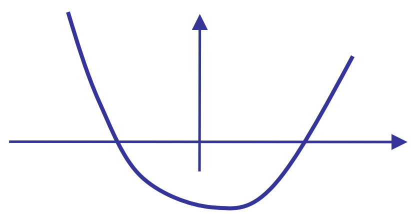

Homothety = ?

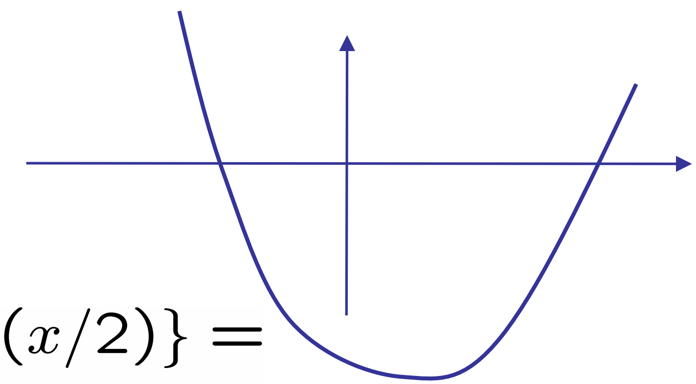
Homothety: $\quad 2 \text{ “}*\text{” } 1_K = 1_{2K}$

$$\left(2 \text{ “}*\text{” } e^{-\varphi}\right)(x) = e^{-2\varphi(x/2)}$$

$$(2 \text{ “}*\text{” } f)(x) = (f(x/2))^2$$

Again, in the level of convex functions, this is usual homothety of the epigraphs





$$epi(2\varphi(x/2)) = \{(x,y) : y \geq 2\varphi(x/2)\} =$$

$$= \{(2x, 2y) : y \geq \varphi(x)\} = 2epi(\varphi)$$

Some of the main theorems for convex bodies have been proven for log-concave functions, and the analogy has been fruitful in both directions: New functional inequalities, and applications of them back into convex bodies, with strengthened powers.

$$vol\left(2 \text{ ``}*\text{'' } e^{-\varphi}\right)(x) = \int e^{-2\varphi(x/2)} \neq 2^n vol(e^{-\varphi})$$

$$vol(e^{-\varphi(x)}) = \int_{epi(\varphi)} e^{-x_{n+1}} dx$$

## A first example (well known): Brunn-Minkowski

$$vol(\lambda K + (1-\lambda)T) \geq vol(K)^{\lambda} vol(T)^{(1-\lambda)}.$$

$$\int (\lambda \text{ ``}*\text{''} \ 1_K \text{ ``}+\text{''} \ (1-\lambda) \text{ ``}*\text{''} \ 1_T) \ \geq \ \left(\int 1_K\right)^{\lambda} \left(\int 1_T\right)^{(1-\lambda)}.$$

Its functional analogue is

$$vol(\lambda \text{ ``}*\text{''} \ f \text{ ``}+\text{''} \ (1-\lambda) \text{ ``}*\text{''} \ g) \geq vol(f)^{\lambda} vol(g)^{(1-\lambda)}.$$

Which looks more familiar as

The Prekopa-Leindler inequality.

$$\int h \ \geq \ \left(\int f\right)^{\lambda} \left(\int g\right)^{(1-\lambda)}$$

$$h(x) = \sup_{\{\lambda y + (1-\lambda) z = x\}} f(y)^{\lambda} g(z)^{(1-\lambda)}.$$

Question: what is the polar of a log-concave function?

$$K^\circ = \{x \in R^n : \langle x, y\rangle \leq 1,\ \forall y \in K\}$$

Similar question: what is the support function?

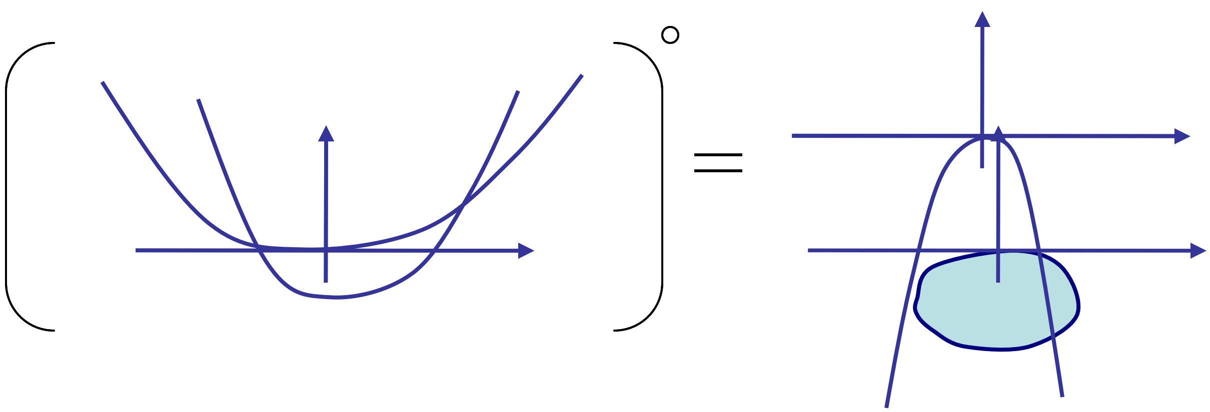
$$h_K(x) = \sup\{\langle x, y\rangle : y \in K\}$$

Related question: what is the Minkowski functional of a log-concave function?

$$m_K(x) = \|x\|_K = \inf\{r > 0 : x/r \in K\} = h_{K^\circ}(x)$$

The support should, intuitively, be defined for all functions, and the polar (and Minkwoski) only for those which "include the origin" in some sense.

Going to the world of epi-graphs of convex functions, it's not clear what should be the support?...
For polar one might try to take the polar set of the epigraph. This need not be an epigraph (could be compact, and anyhow points in the wrong direction)



# Duality

Let $K \subset R^n$ be a convex body (including the origin). The dual body is defined by

$$K^\circ = \{x \in R^n : \langle x, y\rangle \leq 1, \ \forall y \in K\}$$

$$(K^\circ)^\circ = K$$

Denote by $B_2^n$ the Euclidean unit ball.

$$(B_2^n)^\circ = B_2^n$$

**Question**: what is the dual of a log-concave function?

## The Legendre transform

$$(\mathcal{L}f)(x) = \sup_{y \in R^n} \left( \langle x, y \rangle - f(y) \right)$$

The resulting function is always convex and l.s.c., and for such functions,

$$\mathcal{L}\mathcal{L}f = f.$$

A.M. Legendre, 1752-1833
W. Fenchel, 1905-1988

## The dual of a log-concave function

Let $f = e^{-\varphi}$ be a log-concave function.

In [Artstein-Klartag-Milman] we suggested:

$$f^\circ = e^{-\mathcal{L}\varphi}$$

The function which is self dual is

$$g = g^\circ = e^{-\frac{|x|^2}{2}}$$

$$f = e^{-\varphi} \qquad\qquad f^\circ = e^{-\mathcal{L}\varphi}$$

Unpleasant: $1_K^\circ = e^{-\|\cdot\|_K^*} \neq 1_{K^\circ}$

Although: $(e^{-\|x\|_K^2/2})^\circ = e^{-\|x\|_{K^\circ}^2/2}$

More generally: $(e^{-\|x\|_K^p/p})^\circ = e^{-\|x\|_{K^\circ}^q/q}$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$ .

## Blaschke-Santalo Inequality

L.A. Santalo, 1911-2001

Let $K \subset R^n$ be a centrally symmetric convex body.

$$Vol(K)Vol(K^\circ) \le Vol(B_2^n)^2$$

With equality only for ellipsoids.

For general convex bodies:

$$\inf_{x_0}\{Vol(K)Vol((K-x_0)^\circ)\} \le Vol(B_2^n)^2$$

## Santalo's Inequality – functional version

Let $f : R^n \to R$ be an even convex function.
Denote $g(x) = e^{-|x|^2/2}$. Then

$$\int e^{-f} \int e^{-\mathcal{L}f} \le (\int g)^2$$

(Ball ['86]) with equality only for gaussians. For general convex functions:

$$\inf_{x_0}\{\int e^{-f} \int e^{-\mathcal{L}f_{x_0}}\} \le (\int g)^2$$

where $f_{x_0}(x) = f(x - x_0)$. (A-Klartag-Milman ['04])

In the notation given above, the functional Santalo's inequality reads:

For even functions: $$\int f \int f^\circ \leq \left(\int g\right)^2$$

For general functions: $$\inf_{x_0}\left\{\int f \int (f_{x_0})^\circ\right\} \leq \left(\int g\right)^2$$

In particular, one re-captures usual Santalo by applying this to $e^{-\|x\|_K^2/2}$ and using that

$$(e^{-\|x\|_K^2/2})^\circ = e^{-\|x\|_{K^\circ}^2/2}$$

and that the volume is proportional to that of the original body.

Doubts whether this definition is “pollution” arose from theorems in [Fradelizi-Meyer] and also the following conjecture of Cordero

$$Vol(K \cap T)Vol(K^\circ \cap T) \leq (Vol(B_2^n \cap T))^2 \text{ ?}$$

(much is known in the complex case)

But these doubts were lifted after we proved:

**Theorem** [A-Milman] :
Up to linear changes, $f = e^{-\varphi} \mapsto f^\circ = e^{-\mathcal{L}\varphi}$ is the unique order reversing involution on log-concave functions.

**Notation**: The class of lower-semi-continuous convex functions on $R^n$ will be denoted by $Cvx(R^n)$

**Theorem** [A-Milman] :

Let $\mathcal{T} : Cvx(R^n) \to Cvx(R^n)$ be an order reversing involution.

Then there exists a vector $v_0 \in R^n$, a constant $C_0$, and a symmetric $B \in GL_n$ such that

$$(\mathcal{T}f)(x) = C_0 + \mathcal{L}f(Bx - v_0) + \langle x, v_0 \rangle$$

In fact: any bijective order reversing transform:

$$(\mathcal{T}f)(x) = C_0 + C_1\mathcal{L}f(Bx - v_0) + \langle x, v_1 \rangle$$

Consider a partially ordered set $(S, \leq)$.

Call a bijection $T : S \to S$ an "order reversing isomorphism" if

(1) $x \leq y$ if and only if $Tx \geq Ty$

Call it an "abstract duality" if also

(2) $T \circ T = Id_S$

We became interested in characterizing such transforms for various classes connected with convexity

Note that if you know <u>one</u> order-reversing isomorphism on the class, the question becomes the same as that of characterizing "order preserving isomorphisms":

$$x \leq y \quad \text{if and only if} \quad Tx \leq Ty$$

**Theorem** [Boroczky-Schneider]: On the class of convex bodies (compact, with the origin in the interior) there is essentially only one abstract duality:

$$\mathcal{T}K = K^\circ$$

$$K^\circ = \{x \in R^n : \langle x, y\rangle \le 1,\ \forall y \in K\}$$

**Theorem** [Slomka, A-M]: Same with other classes of closed convex sets including 0.

The involution condition in the above theorem can be replaced it with the condition that

$$\mathcal{T}(K_1 \cap K_2) = conv(\mathcal{T}K_1 \cup \mathcal{T}K_2)$$

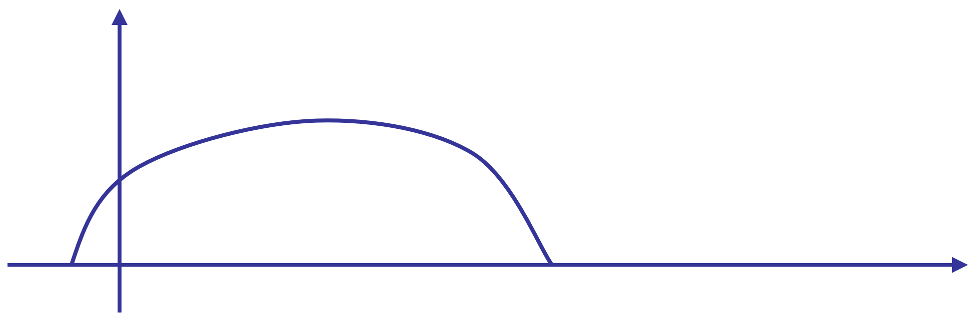
and

$$\mathcal{T}(conv(K_1 \cup K_2)) = \mathcal{T}K_1 \cap \mathcal{T}K_2 \quad .$$

(omit the constant map)

Many similar theorems..

Denote by $Conc(R^n)$ the class of functions concave on their support and with $f(0) > 0$.



**Theorem** [A-Milman]: On $Conc(R^n)$ there is essentially only one abstract duality:

$$\mathcal{T}f(x) = \inf_{\{y: f(y)>0\}} \frac{(1-\langle x, y\rangle)_+}{f(y)}.$$

Question: what is the polar of a log-concave function?

$$f = e^{-\varphi} \qquad\qquad f^\circ = e^{-\mathcal{L}\varphi}$$

What is the support function?

$$h_K(x) = \sup\{\langle x, y\rangle : y \in K\}$$

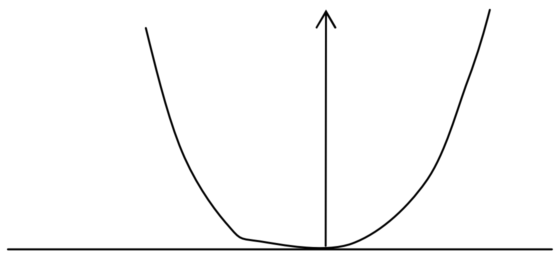
What is the Minkowski functional?

$$m_K(x) = \|x\|_K = \inf\{r > 0 : x/r \in K\} = h_{K^\circ}(x)$$

How come duality is defined for all log-concave functions? How come it does not coincide with usual Polarity under the standard embedding?

**Notation**: The subclass of $Cvx(R^n)$ consisting of non-negative functions satisfying $f(0) = 0$ will be denoted by $Cvx_0(R^n)$

(later referred to as "geometric convex functions")



**Remark**: This class qualifies for geometric applications. In fact, in all cases where log-concave functions were used in AGA, it sufficed to use ones which have maximum $1$, at the origin.

**Question**: How many abstract dualities are there on the class $Cvx_0(R^n)$?

**Easy to check**: The class $Cvx_0(R^n)$ is invariant under the Legendre transform

$$(\mathcal{L}f)(x) = \sup_{y \in R^n} \left( \langle x, y \rangle - f(y) \right)$$

Turns out there's another one!

**Theorem** [A-Milman]: There are exactly two abstract dualities on the class of geometric convex functions.

**Theorem** [A-Milman]: There is a non-trivial gauge transform on the class $Cvx_0(R^n)$ (the so called "Minkwoski functional")

What is the other abstract duality transform?
What is the gauge transform?

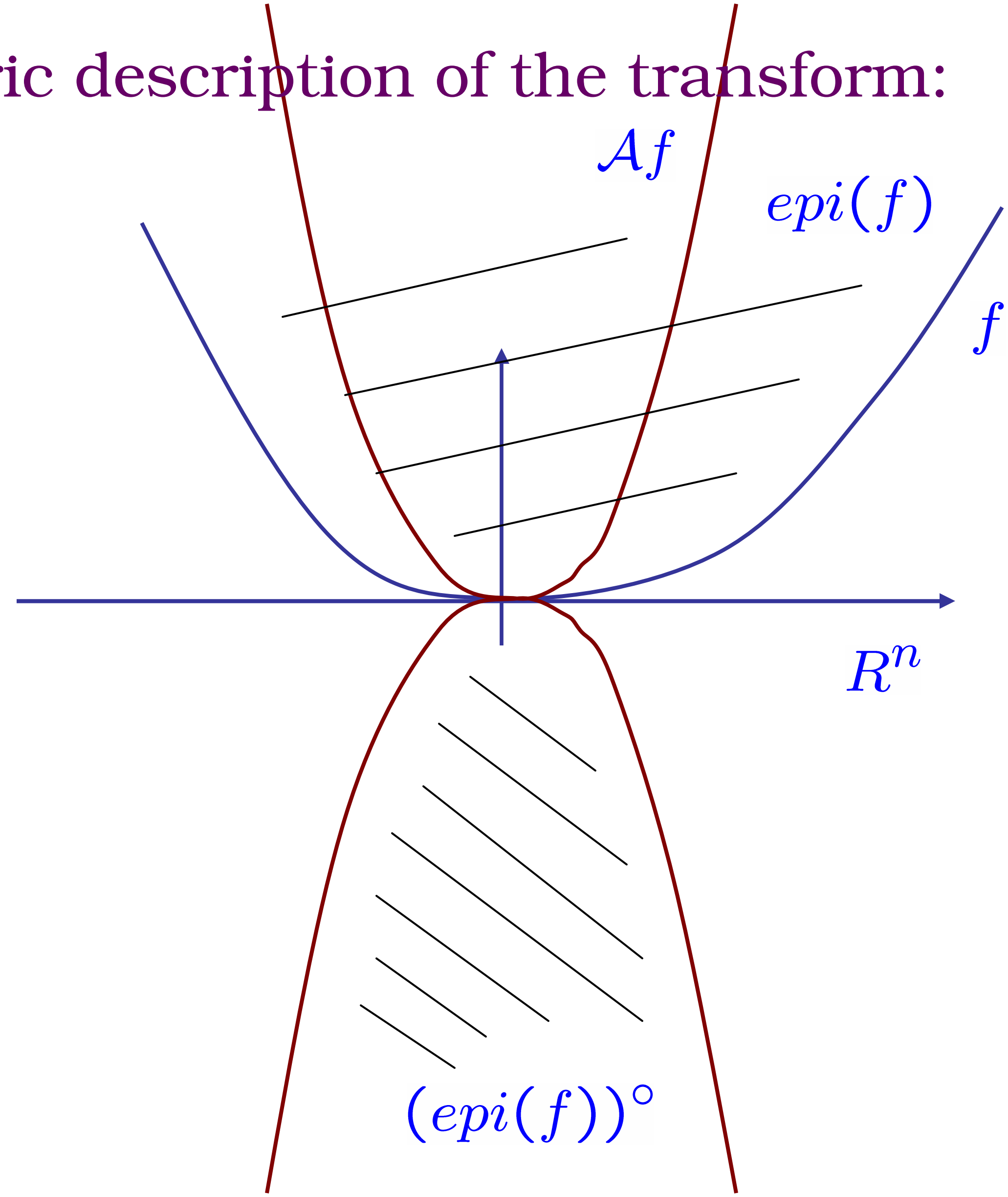
## The second duality transform

**Definition**: For a function $f \in Cvx_0(R^n)$ , let

$$(\mathcal{A}f)(x) = \begin{cases} \sup\limits_{\{y \in R^n : f(y) > 0\}} \dfrac{\langle x, y\rangle - 1}{f(y)} & x \in \{f(y) = 0\}^\circ \\ +\infty, & \text{otherwise} \end{cases}$$

**Geometric definition**: For a function $f \in Cvx_0(R^n)$ let $\mathcal{A}f$ be defined so that it epi-graph is the reflection of the dual of the epi-graph of $f$ .

## Geometric description of the transform:

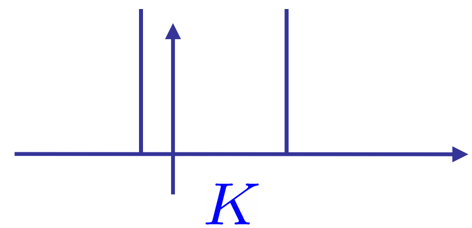


## Examples:

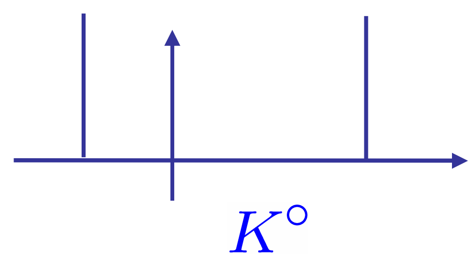
$$f(x) = \|x\|_K \qquad (\mathcal{A}f)(x) = \|x\|_{K^\circ}$$

$$f(x) = \|x\|_K^p \qquad (\mathcal{A}f)(x) = c_p\|x\|_{K^\circ}^p$$

$$f(x) = \begin{cases} 0, & \text{if } \; x \in K \\ +\infty, & \text{otherwise} \end{cases}$$




$$(\mathcal{A}f)(x) = \begin{cases} 0, & \text{if } \; x \in K^\circ \\ +\infty, & \text{otherwise} \end{cases}$$



**Definition**: For a function $f \in Cvx_0(R^n)$ , let

$$(\mathcal{A}f)(x) = \sup_{\{y \in R^n : f(y) > 0\}} \frac{\langle x, y \rangle - 1}{f(y)}$$

**Theorem** [A-M]: Every duality on the class $Cvx_0(R^n)$ is up to linear terms either $\mathcal{L}$ or $\mathcal{A}$ .

$$f = e^{-\varphi}$$

So one can also define $f^{\odot} = e^{-\mathcal{A}\varphi}$

**Examples:** $1_K^{\odot} = 1_{K^\circ}$

$$f(x) = e^{-\|x\|_K} \qquad f^{\odot}(x) = e^{-\|x\|_{K^\circ}}$$

$$f(x) = e^{-\|x\|_K^p} \qquad f^{\odot}(x) = e^{-c_p\|x\|_{K^\circ}^p}$$

In particular, there are many self-dual functions.

Known: The maximizer in “Santalo” must be radial.

Our new understanding is thus that the Legendre transform should be viewed as the extension of the Support function, which is also order reversing:

$$f = e^{-\varphi} \qquad\qquad h_f = \mathcal{L}\varphi$$

$$h_{1_K} = \|\cdot\|_{K^\circ} = h_K$$

Whereas duality should be defined only for geometric convex functions, and is the extension of polarity, given by (the order reversing)

$$f = e^{-\varphi} \qquad\qquad f^{\odot} = e^{-\mathcal{A}\varphi}$$

$$1_K^{\odot} = 1_{K^\circ}$$

Note how well this works out with homothety:

$$f = e^{-\varphi} \qquad\qquad 2 * f = e^{-2\varphi(\cdot/2)}$$

$$h_f = \mathcal{L}\varphi \qquad\qquad h_{2*f} = \mathcal{L}(2\varphi(\cdot/2)) = 2h_f$$

And of course with Minkowski addition

$$h_{f+g} = h_f + h_g$$

(recall that $f + g$ was defined by sup-convolution)

Question: what is the polar of a log-concave function?

$$f = e^{-\varphi} \qquad\qquad f^{\odot} = e^{-\mathcal{A}\varphi}$$

What is the support function?

$$f = e^{-\varphi} \qquad\qquad h_f(x) = (\mathcal{L}\varphi)(x)$$

What is the Minkowski functional?

$$m_K(x) = \|x\|_K = \inf\{r > 0 : x/r \in K\} = h_{K^\circ}(x)$$

Thus we see it is natural to define the gauge function of a geometric log-concave function by

$$f = e^{-\varphi} \qquad\qquad m_f(x) = (\mathcal{L}\mathcal{A}\varphi)(x)$$

## The $\mathcal{J}$-transform: A Gauge transform

**Properties**: $\mathcal{J} = \mathcal{L}\mathcal{A} = \mathcal{A}\mathcal{L}$

An order preserving involution

Unique (with identity)

Acts on rays

Unique extension of the Minkowski-map

**Explicit definition**: For a function $f \in Cvx_0(R^n)$ , let

$$(\mathcal{J}f)(x) = \inf\{r > 0 : rf(\frac{x}{r}) \leq 1\}$$

**Examples:** $m_{1_K}(x) = \|x\|_K = m_K(x)$

$f(x) = e^{-\|x\|_K^2/2}$ $m_f = \mathcal{L}(\|\cdot\|_{K^\circ}^2/2) = \|\cdot\|_K^2/2$

$f(x) = e^{-\|x\|_K^p}$ $m_f(x) = c_p\|x\|_K^q$

It is the <u>only</u> extension of the Minkwoski functional to log-concave functions which is order reversing.

$$(\mathcal{J} f)(x) = \inf\{r > 0 : r f(\frac{x}{r}) \leq 1\}$$

Turns out: $\mathcal{J}$ is induced by a point map on $R^n \times R_+$ :

$$F(x, y) = (\frac{x}{y}, \frac{1}{y})$$

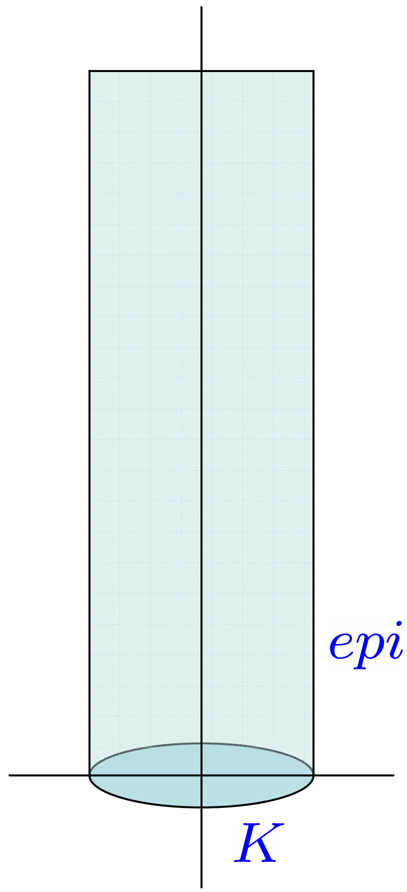
$$F(epi(f)) = epi(\mathcal{J} f)$$

(In particular, Legendre transform is polarity + this point map – for geometric convex functions)

$f = -\log 1_K$

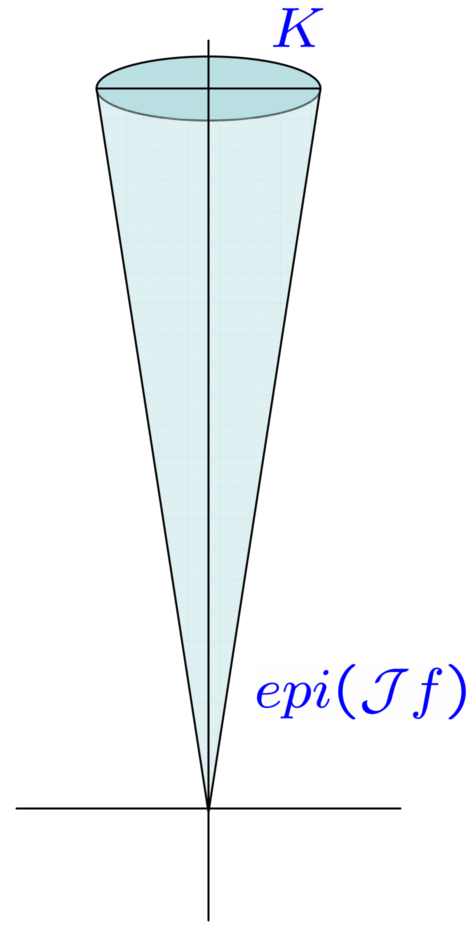
$\mathcal{J} f = \|x\|_K$

$$F(epi(f)) = epi(\mathcal{J} f)$$




$epi(f)$

$$(x, y) \mapsto (\frac{x}{y}, \frac{1}{y})$$



Fractional linear maps turn out very naturally in convexity when one considers The polar of a translation of a convex body

Letting $$F(x) = \frac{x}{1 + \langle x_0, x \rangle}$$

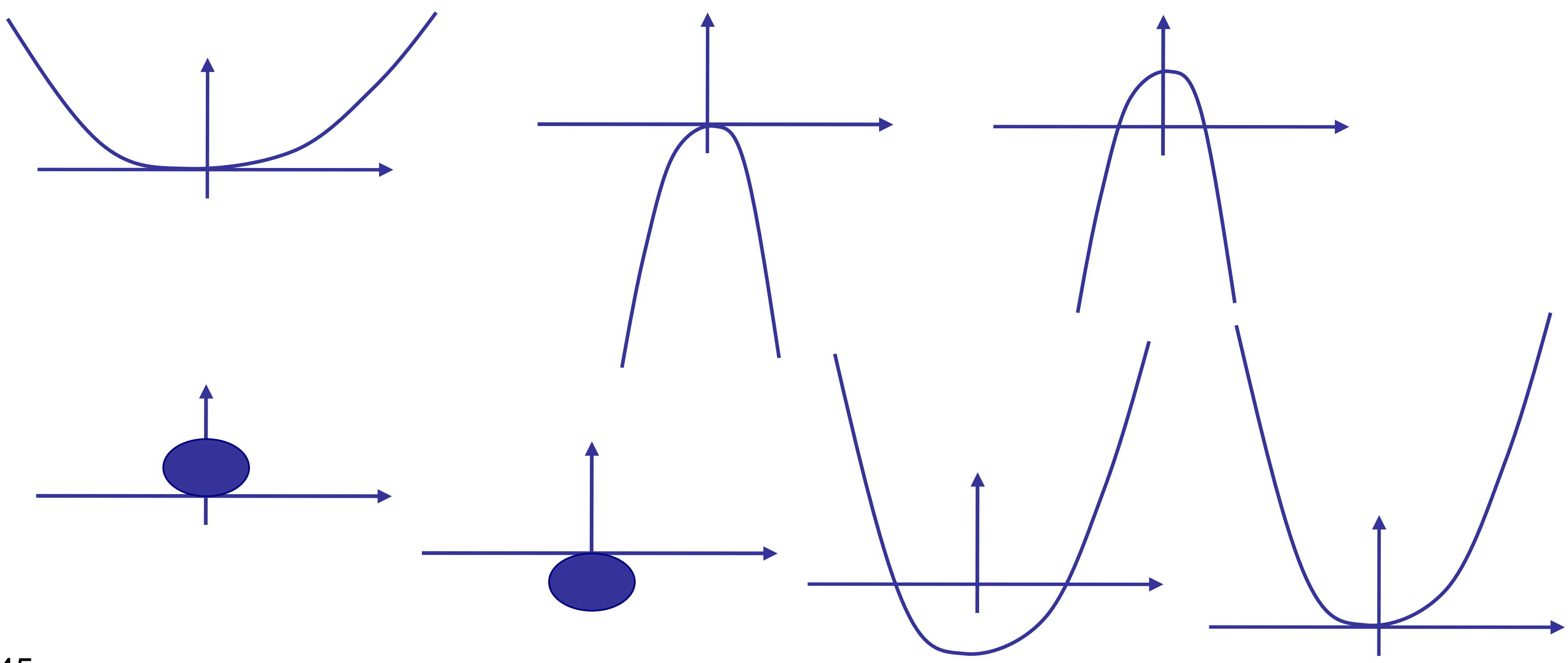
one has $$(x_0 + K)^\circ = F(K^\circ)$$

and so $$(x_0 + K^\circ)^\circ = F(K)$$

## The Legendre transform revisited

For geometric convex functions

$$epi(\mathcal{L}f) = (((epi(f))^\circ + e)^\circ - e)^\circ + e$$



# More about fractional linear maps in the next lecture..

- Lecture 1: Abstract duality, the Legendre transform and a new duality transform.

- Lecture 2: Order isomorphisms and the fundamental theorem of affine geometry.

- Lecture 3: Multiplicative transforms and characterization of the Fourier transform.

THANK YOU