INTERACTING PARTICLE SYSTEMS AND LANDAU DAMPING

3 November 2010
Fields Institute

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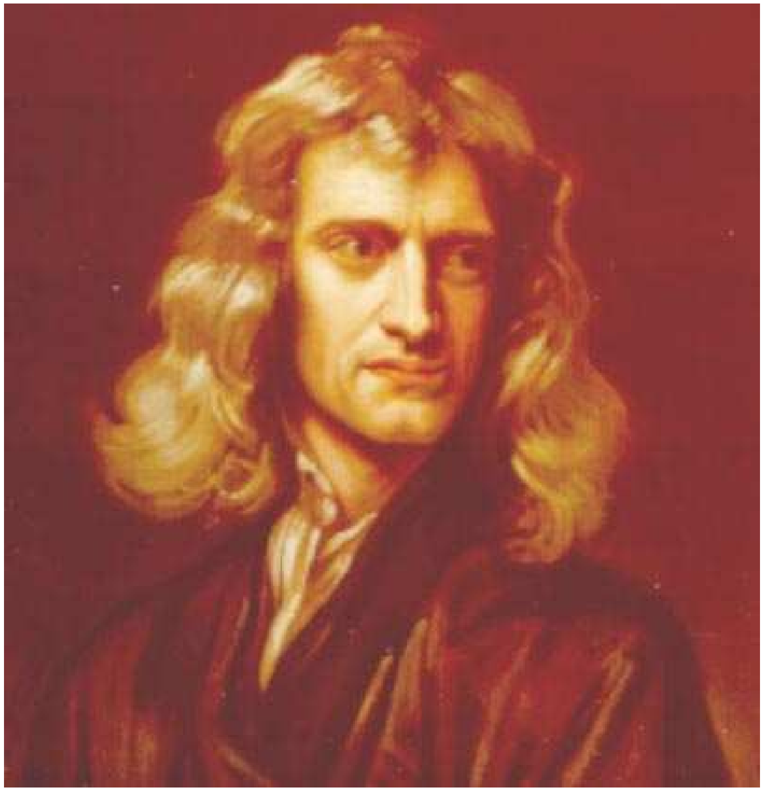
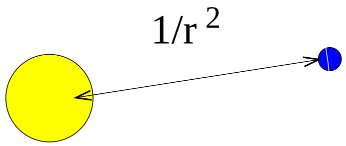
1684: Newton's law of universal attraction





The Newton equations for point masses

$$x_i = x_i(t) \in \mathbb{R}^3, \quad \text{mass } m_i, \qquad i = 1...N$$

$$\ddot{x}_i = -\sum_{j \neq i} m_j \, \nabla W(x_i - x_j)$$

$$W(x) = -\frac{\mathcal{G}}{4\pi\,|x|} \text{ Newton (gravitational) potential}$$

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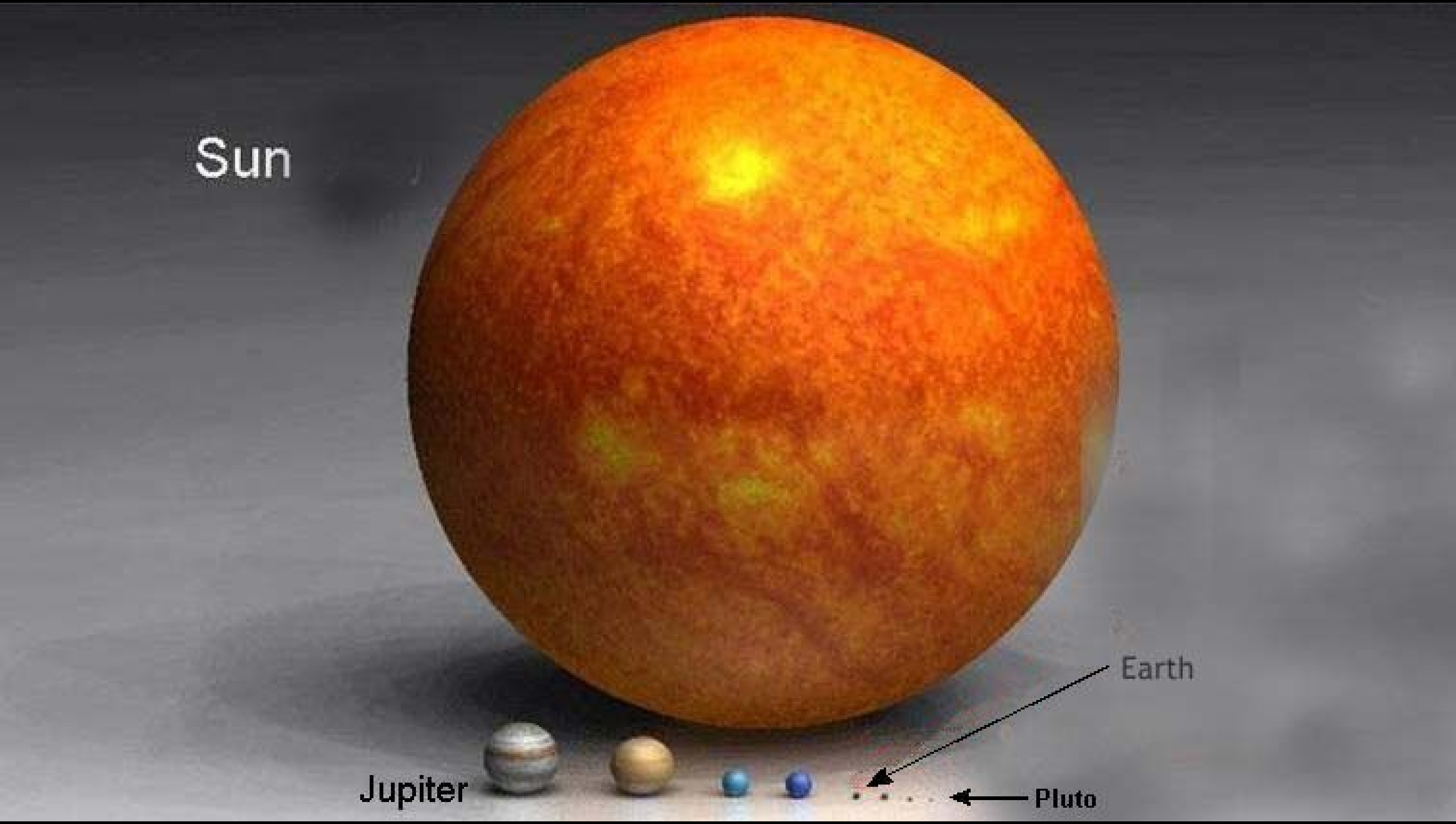
What do trajectories look like as $t \to \infty$??


Sun
Earth
Jupiter
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Kolmogorov–Arnold–Moser theorem

- Let H_0 be a completely integrable Hamiltonian

(e.g. independent periodic trajectories of planets interacting with only the Sun)

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Epistemologic paradox

The K-A-M Theorem "never" applies to real systems (planets are not small enough!)

Still has been a revolution in classical mechanics, for mathematicians **and** physicists.

Another approximation of interest

So many particles that the system looks continuous!

Let us enjoy again a numerical simulation by Dubinski.

The mean field approximation

$$\boxed{N \geq 10^{12} \text{ simple equations}}$$

for positions x_i and velocities v_i

$$\Big\downarrow{\scriptstyle N\to\infty}$$

$$\boxed{\textbf{one} \text{ (complicated) equation}}$$

for $\mu_t(dx\,dv)$

$\mu_t[A]$: fraction of mass at time t within A

$$\sum_j \qquad \longrightarrow \int_{x',v'}$$

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$$\sum_j m_j \, W\big(x_j(t) - x\big) \longrightarrow \int W(x' - x)\, \mu_t(dx'\,dv')$$

The Vlasov equation

μ_t is preserved by the flow (conservation of mass)

vol is preserved by the flow (Liouville theorem)

$$\Longrightarrow f(t,x,v) = \frac{\mu_t(dx\,dv)}{\mathrm{vol}\,(dx\,dv)} \quad \text{is also preserved:}$$

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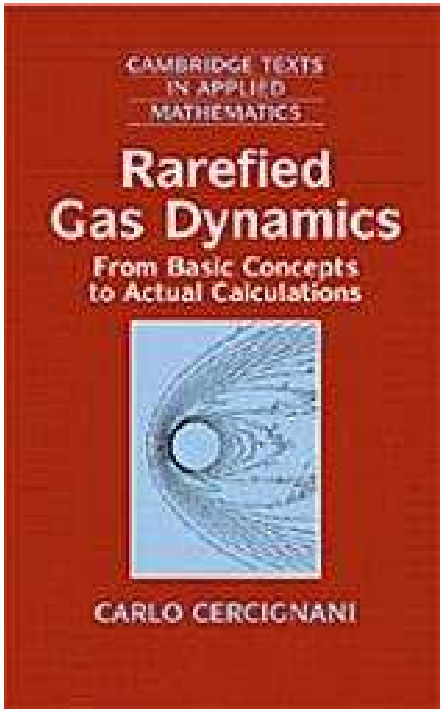
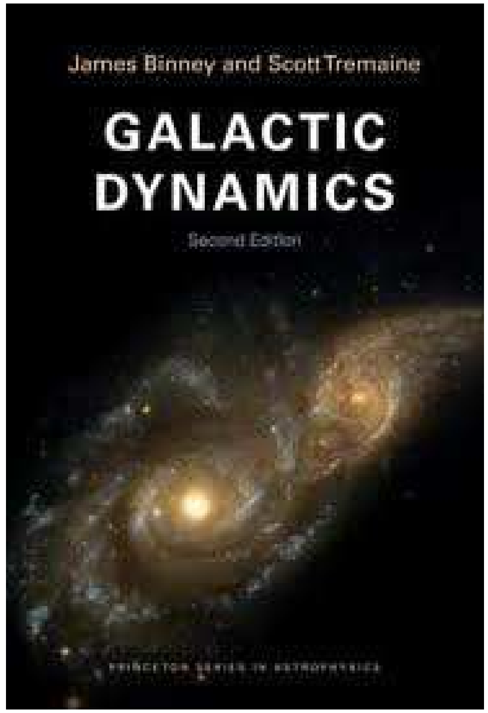
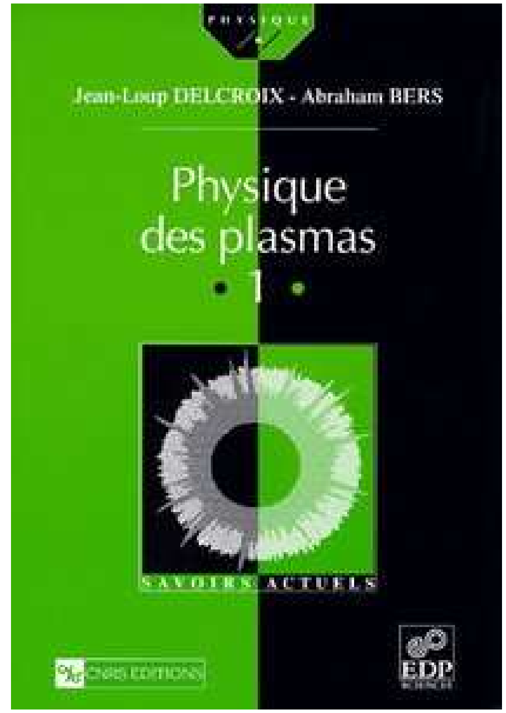
NB: Rigorous justification is still open for singular interactions (gravitation/electric: $W \sim \pm 1/r$ in $d = 3$)

Best result so far: Hauray–Jabin (2007): $W \sim \log 1/r$...

Boltzmann and Vlasov equations: pillars of kinetic theory







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Entropy increases (Boltzmann's H Theorem)	Entropy is constant (from Liouville's Theorem)
Gaussian equilibria $\rho\, e^{-\lvert v\rvert^2/T}$	Infinite-dim space of equilibria Ex. any $f(v)$

1946: Landau's "amazing discovery"

Landau linearizes the Vlasov equation around $f^0(v)$: for entire (analytic) data, force damps to 0 with rate $\lambda_L =$

$$\inf_k \inf \left\{ \mathcal{R}e\, \xi;\ -4\pi^2 \, |k|^2 \, \widehat{W}(k) \int_0^\infty \int_{\mathbb{R}^d} f^0(v)\, e^{-2i\pi k t \cdot v}\, e^{2\pi \xi t}\, t\, dt\, dv = 1 \right\}$$

Ex: $f^0(v) = e^{-|v|^2}$: Coulomb interaction $\lambda_L > 0$;
Newton interaction, $\lambda_L > 0$ only for scales $< L_J$

Long-time behavior of Vlasov equation

- Landau damping: perturbations may damp away spontaneously, in an apparently irreversible way (approach to equilibrium)

- Since then the large-time behavior of Vlasov has been much much discussed. “Well-accepted” and observed e.g. in astrophysics: relaxation in a “short” time, before entropy increases. Fundamental!

- Static approaches: Lynden-Bell, Robert, Miller... But no one has any theoretical explanation based on dynamics

... except for the Landau damping perturbative effect.

But ... Is the linearization reasonable?? $f = f^0 + h$

$$\frac{\partial h}{\partial t} + v \cdot \nabla_x h + F[h] \cdot \left(\nabla_v f^0 + \nabla_v h \right) = 0 \qquad \textbf{(NLin V)}$$

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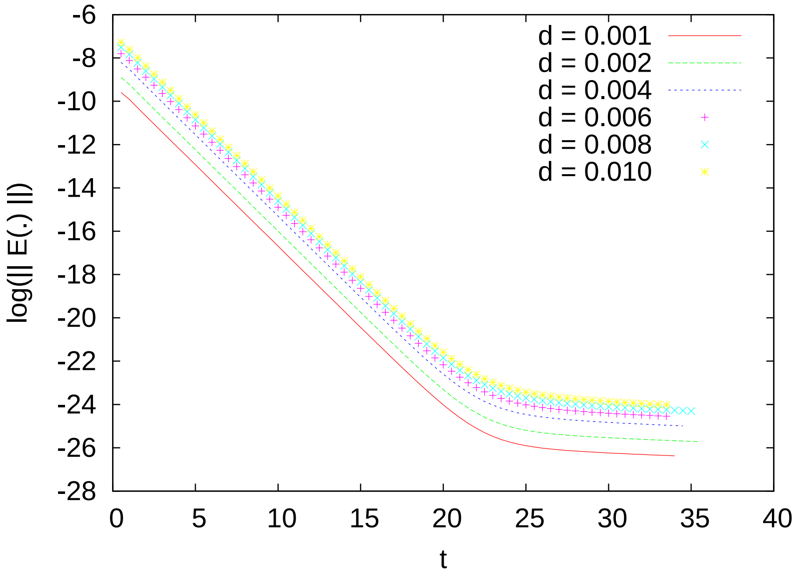
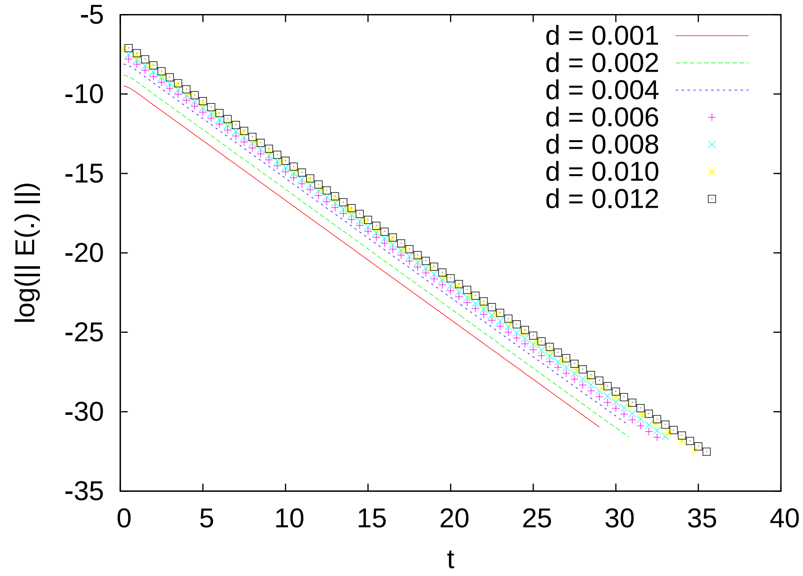
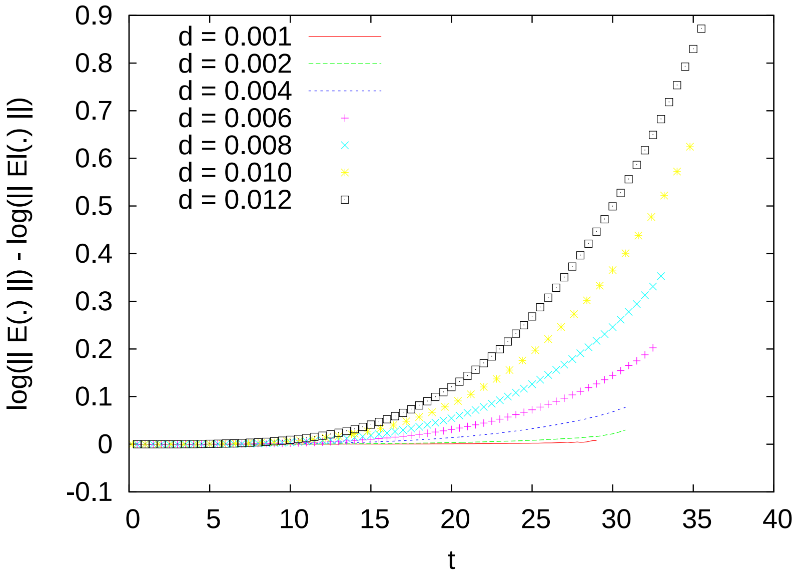
- Caglioti–Maffei (1998): at least some nontrivial

solutions decay exponentially fast







Filbet 2010

$$e^{-\frac{v^2}{2}}\Big(1+\varepsilon\cos(2\pi kx)\Big)$$

What theorem??

Confinement crucial; comes from container or dynamics

To simplify take $x \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ $(d \geq 1)$

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- Let $f = f(t,x,v)$ be the solution of the Vlasov eq. with interaction W and $f(0,\cdot) = f_i$, **then**

$F[f](t,x) = O(e^{-2\pi\lambda|t|})$, $\quad \forall \lambda < \min(\lambda_0, \lambda_i, \lambda_L)$

Mathematical comments

- One also proves: $f(t,\cdot) \xrightarrow[weak]{} f_\infty = f_\infty(v)$ as $t \to \infty$
- Quantitative estimate.
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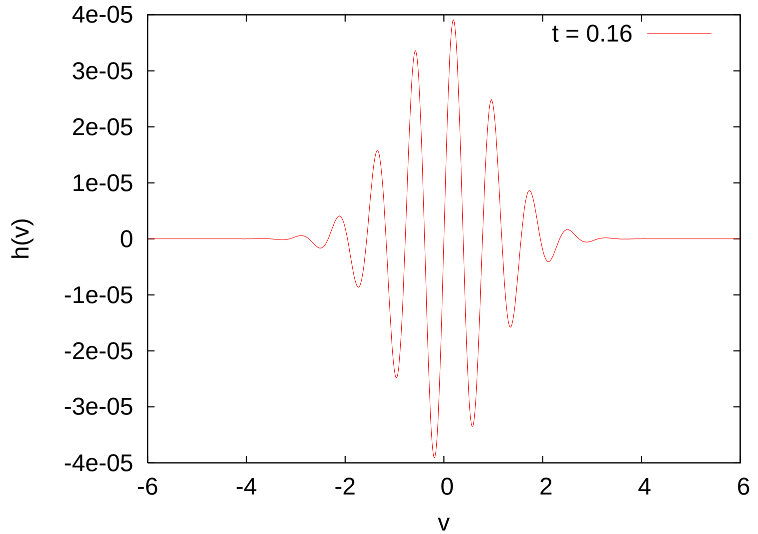
Physical comments

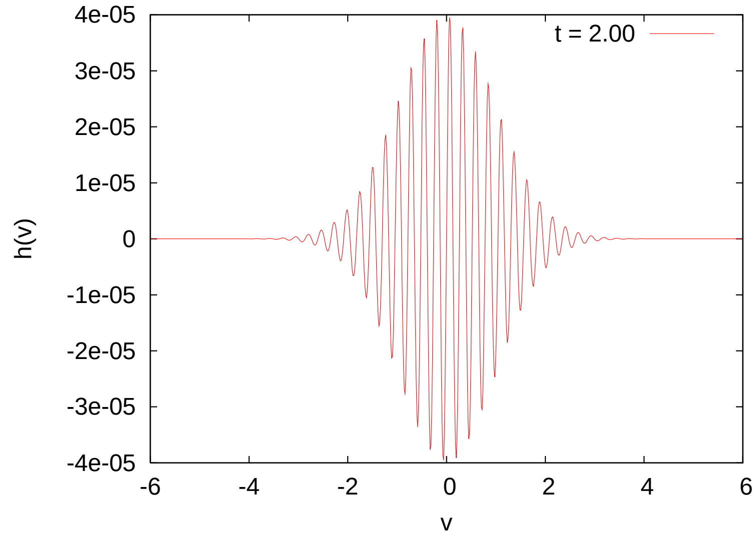
Information goes to small velocity scales (invisible!)

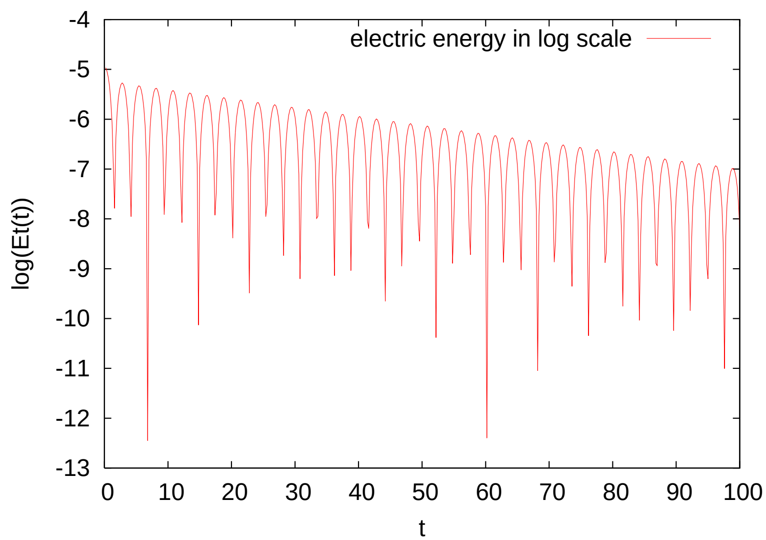
.... vanishes into thin air (≠ radiation!)

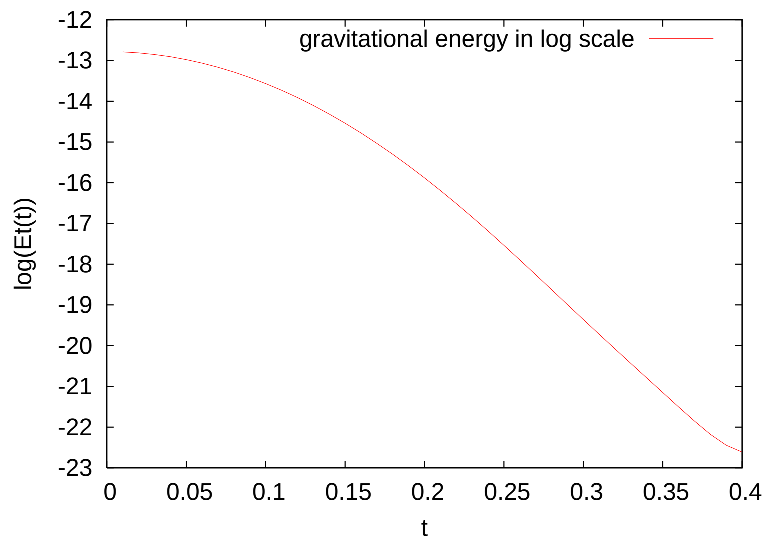
Lynden-Bell: *"A [galactic] system whose density has achieved a steady state will have information about its birth still stored in the peculiar velocities of its stars"*

Numerical illustration









Kinetic Fourier analysis

$$\widetilde{f}(k,\eta) = \iint e^{-2i\pi k\cdot x}\, e^{-2i\pi \eta\cdot v}\, f(x,v)\, dx\, dv$$

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Functional setting: wishlist

- Quantify analytic regularity
- Good behavior wrt composition (by trajectories)
- Uniform bounds in spite of the fast oscillations

Naive analytic norm:

$\|f\| = \sup_{k,\eta} |\widetilde{f}(k,\eta)|\, e^{2\pi\lambda|\eta|}\, e^{2\pi\mu|k|}$ is bad: unstable by composition or large-time limit

Two remarkable families of analytic norms:

$$\|f\|_\lambda = \sum_{n\in\mathbb{N}} \frac{\lambda^n \ \|f^{(n)}\|_\infty}{n!} \qquad \|f\|_\lambda = \sum_{k\in\mathbb{Z}} e^{2\pi\lambda|k|} \ |\widehat{f}(k)|$$

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$\Rightarrow$ **Basic norm for nonlinear problem**

$$\|f\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma);p}} = \sum_{k\in\mathbb{Z}^d} \sum_{n\in\mathbb{N}^d} \frac{\lambda^n}{n!} e^{2\pi\mu|k|} (1+|k|)^\gamma \left\| \left(\nabla_v + 2i\pi\tau k\right)^n \widehat{f}(k,v) \right\|_{L^p(dv)}$$

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Procedure { **hybridize** these two spaces; add a Sobolev correction; and a time-shift (gliding regularity)

$\Rightarrow$ **Basic norm for nonlinear problem**

$$\|f\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma);p}} = \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}^d} \frac{\lambda^n}{n!}\, e^{2\pi\mu|k|}\,(1+|k|)^\gamma \left\|\left(\nabla_v + 2i\pi\tau k\right)^n \widehat{f}(k,v)\right\|_{L^p(dv)}$$

NB: Not in final statement! (compare with naive norms)

Crucial twist: regularity interpretation

Instead of

$$\boxed{F(t,\cdot) \xrightarrow[t\to\infty]{} 0}$$

prove

$$\boxed{\sup_{t\geq 0} \|f(t,\cdot)\|_{\mathcal{Z}_t^{\lambda,\mu;1}} < +\infty}$$

Regularity exists!

It drives Landau damping, Cf. Riemann–Lebesgue lemma

$\Longrightarrow$ It can be measured (in a way...)

A key step in the proof

Analyze the linear equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F[f] \cdot \nabla_v \overline{f} = 0$$

where $\overline{f} = \overline{f}(t, x, v)$ is given, **not stationary**,

still $\quad \sup_{t \geq 0} \|\overline{f}(t)\|_{\mathcal{Z}_t} \leq C.$

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Equation on $\|\rho(t)\|$? looks like

$$\|\rho(t)\| \leq S(t) + \int_0^t K(t, \tau) \, \|\rho(\tau)\| \, d\tau$$

$$K(t, \tau) = O(\tau), \qquad \int_0^t K \, d\tau = O(t)....$$

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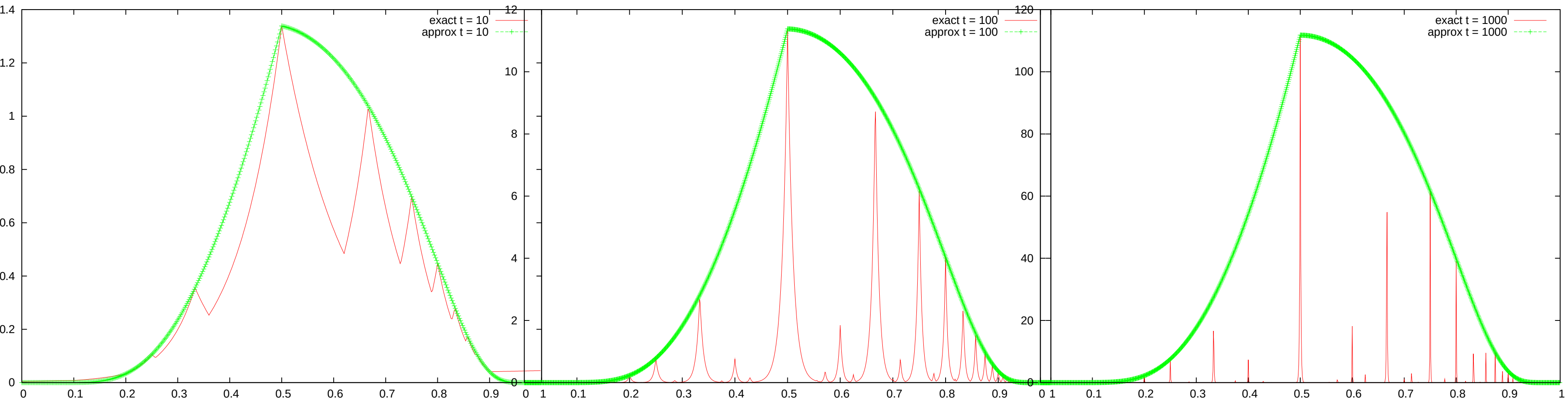
... So $\|\rho(t)\| = O(\exp c\, t^2)$...... very bad estimate!!

Refined analysis of the time-response kernel

For a correct choice of parameters,

$$K(t,\tau) \simeq (1+\tau) \sup_{\ell \neq k \neq 0} |\widehat{W}(k-\ell)|\, e^{-\alpha\,|k(t-\tau)+\ell\tau|}\, e^{-\alpha\,|\ell|}$$

Coupling of (k,ℓ) stronger if W more singular!



... As $t \to \infty$, K concentrates on discrete times τ

(compensation by oscillations — except if “resonance”)

As in the plasma echo experiment (Malmberg 1967)

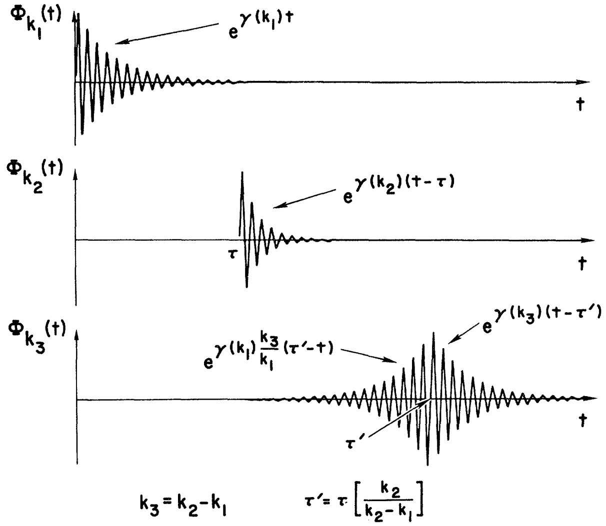


FIG. 1. Approximate variation of the principal Fourier coefficients of the self-consistent field for the case $k_3 \cong k_1 \cong \frac{1}{2}k_2$. Upper line: response to the first pulse; middle line: response to the second pulse; lower line: echo.

where

$$\tan\delta = \gamma(k_1)(k_3 - k_1)/\omega_p(k_3 + k_1)$$

and

$$\tan\delta' = \gamma(k_3)(k_1 - k_3)/\omega_p(k_1 + k_3).$$

It is interesting to note that the echo is not symmetric in that it grows up at the rate $\exp[\gamma(k_1)k_3/k_1(\tau'-t)]$ and damps away at the rate $\exp[\gamma(k_3)\times(t-\tau')]$.

The results of both the first- and second-order calculations are summarized in Fig. 1. The exponentials written in this figure indicate the general dependence of the envelopes of the oscillating curves, which have actually been drawn for the case where $k_1 \simeq k_3$.

The above calculation was based on the collisonless Boltzmann equation and is invalidated if collisions are strong enough to destroy the phase information before the echo can appear. Small angle Coulomb collisions are particularly effective in this regard, since the Fokker-Planck operator representing these collisions enhances the collision rate by a factor $(k\bar{v}\tau)^2 \simeq (\omega_p\tau)^2$ when operating on a perturbation of the form $e^{ikv\tau}$. By working in a marginal range, one might be able to use this effect as a tool to measure the Coulomb collision rate, even though the neutral collision rate is somewhat higher.

We have considered several variations on the above calculation. Although in this paper we have discussed explicitly only second-order echoes, higher order echoes are also possible. For example, a third-order echo is produced when the velocity space perturbation from the first pulse is modulated by a spatial harmonic of the electric field from the second pulse. The echo then occurs at $t = \tau 2k_2/(2k_2 - k_1)$ or $t = 2\tau$ when $k_2 = k_1$. This result is more closely related to echoes of other types[3] which are also third order for small amplitudes.

It is possible also to have spatial echoes, and these will probably be easier to observe experimentally than the temporal echoes described above. If an electric field of frequency ω_1 is continuously excited at one point in a plasma and an electric field of frequency $\omega_2 > \omega_1$ is continuously excited at a distance l from this point, then a spatial echo of frequency $\omega_2 - \omega_1$ will appear at a distance $l\omega_1/(\omega_2 - \omega_1)$ from the point where the second field is excited.

Finally, although our discussion has been entirely in terms of electron wave echoes, it is clear that the above treatment can be extended in a straightforward manner to include ion dynamics, and this leads to temporal as well as spatial ion wave echoes.

An observation of plasma echoes would be of fundamental interest, since it would experimentally verify the reversible nature of collisionless damping. The analogy with spin echoes[3] strongly suggests the possible use of the ehco technique as a means for studying collisional relaxation phenoma in plasmas.

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Stabilizing effect due to delay: baby models

- $\varphi(t) \leq \int_0^t \tau\, \varphi(\tau)\, d\tau \Longrightarrow \varphi(t) = O(e^{t^2})$

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Baby model for gravitation interaction

$$\varphi_k(t) \leq a(kt) + \frac{c\,t}{k^2}\,\varphi_{k+1}\left(\frac{kt}{k+1}\right) \qquad k \in \mathbb{N}$$

$$\Longrightarrow \varphi_k(t) \lesssim a(kt)\exp((ckt)^{1/3})$$

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Loss of time-decay, superpolynomial but sub-exponential

$\Longrightarrow$ can be compensated by the linear exponential decay

Similar estimates established on the true model, via technical exponential moment bounds

Overcome the loss

Loss of regularity in a perturbative regime is often curable by the Newton scheme (Kolmogorov, Nash ...)

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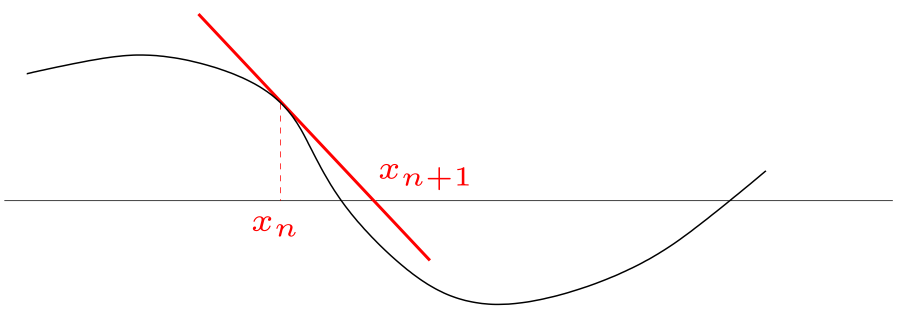
The Newton scheme to solve $\Phi(x) = 0$

$$x = \lim x_n, \qquad \Phi(x_n) + D\Phi(x_n) \cdot (x_{n+1} - x_n) = 0$$



Ex: $\Phi(x) = x^2 - \alpha$, get $x = \sqrt{\alpha}$, "Babylonian algorithm"

The Newton scheme converges tremendously fast: $O(\varepsilon^{2^n})$

Newton scheme for the Vlasov equation

$f^0 = f^0(v) \quad$ (homogeneous stationary state)

$$f^n = f^0 + h^1 + \ldots + h^n$$

$$\begin{cases} \partial_t h^1 + v \cdot \nabla_x h^1 + F[h^1] \cdot \nabla_v f^0 = 0 \\ h^1(0,\,\cdot\,) = f_i - f^0 \end{cases}$$

$$\begin{cases} \partial_t h^{n+1} + v \cdot \nabla_x h^{n+1} + F[f^n] \cdot \nabla_v h^{n+1} + F[h^{n+1}] \cdot \nabla_v f^n \\ \qquad\qquad = -F[h^n] \cdot \nabla_v h^n \\ h^{n+1}(0,\,\cdot\,) = 0. \end{cases}$$

Long-time estimates along the Newton scheme

- $f^n = f^0 + h^1 + \ldots + h^n$
- Control simultaneously density function + trajectories

$S^n_{t,\tau} = (X,V)_t \longmapsto (X,V)_\tau$ in the force field $F[f^n]$

$\Omega^n_{t,\tau} = S^n_{t,\tau} \circ (S^0_{t,\tau})^{-1}$ ("scattering")

- Propagate a bunch of controls including

$$\boxed{\sup_{\tau \geq 0} \left\| \int h^k_\tau \, dv \right\|_{\mathcal{Z}^{\lambda_k,\mu_k}_\tau} \leq \delta_k}$$

$$\boxed{\sup_{t \geq \tau \geq 0} \left\| h^k_\tau \circ \Omega^{k-1}_{t,\tau} \right\|_{\mathcal{Z}^{\lambda_k(1+b),\mu_k;1}_{\tau - \frac{bt}{1+b}}} \leq \delta_k, \qquad b(t) = \frac{B}{1+t}}$$

At each stage, in $\mathcal{Z}$ norms...

1[†]) Estimate $\Omega^n - \mathrm{Id}$ (uniformly in n) and $\nabla\Omega^n - I$

2[†]) Estimate $\Omega^n - \Omega^k$ ($k \leq n-1$; small when $k \to \infty$)

3[†]) Estimate $(\Omega^k)^{-1} \circ \Omega^n$

4) Estimate h^k_τ, ∇h^k_τ, $\nabla^2 h^k_\tau$ ($k \leq n$) along $\Omega^n_{t,\tau}$

5[*]) Estimate $\int h^{n+1}\, dv$

6) Deduce an estimate on $F[h^{n+1}]$

7[*]) Estimate $h^{n+1} \circ \Omega^n$

8) Estimate $\nabla h^{n+1} \circ \Omega^n$

9) Show $(\nabla h^{n+1}) \circ \Omega^n \simeq \nabla(h^{n+1} \circ \Omega^n)$

---

† by classical fixed point

* Using the equation

Use the amazingly fast convergence of the Newton scheme $(O(\varepsilon^{2^n}))$ to absorb the very large constants

Mathematical conclusions

- Get the desired regularity: $\sup_{t\geq 0} \|f(t)\|_{\mathcal{Z}_t^{\lambda,\mu;1}} < +\infty$

- It all works well in first place because linearized Vlasov is a completely integrable system!

$\Rightarrow$ Unexpected analogy with KAM: Use a Newton scheme to overcome a loss of "regularity" in the perturbation of a completely integrable Hamiltonian system

- Note: We use more than the super-geometric convergence of Newton: here it is really useful to know that error $\leq \exp(-n^s)$

- Accordingly, there is no C^k version to be seen so far ("**KA** rather than **M**"): new open problem among many other (extension to nonhomogeneous equilibria, etc.)

Physical conclusions

Landau meets Kolmogorov

Three of the most famous paradoxes of modern classical physics are related:

$$\begin{cases} \text{KAM Theorem} \\ \text{plasma echoes} \\ \text{Landau damping} \end{cases}$$

.... but only in the nonlinear regime!

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Nature of Landau damping

Relaxation from regularity, driven by confined mixing.

First steps in a virgin territory? Universal problem of constant-entropy relaxation in particle systems