Large Deviations 2

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- $\eta \to \eta^{u,v} \Leftrightarrow (1,0) \leftrightarrow (0,1)$.
- $$(\mathcal{A}f)(\{u_j\}) = \sum_i \sum_w p(w)(1 - \eta(u_i + w))$$ $$[f(\{u_j\}, i, u_i + w) - f(\{u_j\})]$$

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- $$\mathcal{L}_N = N^2 \mathcal{L}; \; \mathcal{A}_N = N^2 \mathcal{A}$$
- $$R = \frac{1}{N^d} \sum_i \delta_{\frac{u_i(\cdot)}{N}}$$

Initial Condition

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- Converges in probability. Deviations are possible.

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- $q:[0,T]\times\mathcal{T}^d\times Z^d\to R$,

$$(\mathcal{L}_N f)(\eta)=N^2\sum_{u,v}\Big(p(v-u)+\frac{q(t,\frac{u}{N},v-u)}{N}\Big)$$
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- $q(t,x,z) + q(t,x,-z) = 0.$

- Effect: $b(t,x) = \sum_w wq(t,x,w)$

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- Minimize over q, fixing b.
- Replace $\eta(t,u)[1-\eta(t,u+w)]$ by $\rho(t,\frac{x}{N})(1-\rho(t,\frac{x}{N}))$.

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- This does not track individual particles.

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- Can put in initial randomness. ρ_0 can be different.

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- $$I(\rho(\cdot,\cdot)) = \int_{\mathcal{S}} \int_0^T [[h(\rho)]_t + [g(\rho)]_x]^+ dx dt$$

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- Solve the equation

$$u_t + \frac{1}{2}u_{xx} + b(x,\omega)u_x = 0;\ u(T,x) = e^{\theta x}$$

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$$\lim_{t\to\infty} \frac{1}{t} \log E[e^{\theta x(t)}] = \Lambda(\theta)$$

- Then

$$I(m) = \sup_{\theta} [\theta m - \Lambda(\theta)]$$

- Solve the equation

$$u_t + \frac{1}{2} u_{xx} + b(x,\omega) u_x = 0; \; u(T,x) = e^{\theta x}$$

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- Need the limit as $T \to \infty$ of $v_T(0,0)$.

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  Can determine the limit, in an analogous manner.
- Let
$$h(x, b, \omega) = \sup_v [bv - f(x, v, \omega)]$$

$$v_\epsilon(0,0,\omega) =$$

$$\sup_{b=b(s,x,\omega)} E[\theta x(1) - \int_0^1 h(\frac{x(s)}{\epsilon}, b(s,x(s),\omega),\omega)ds]$$

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- With $T = \epsilon^{-1}$, and rescaling space and time
  $$v_T(0,0,\omega) = \sup_{b=b(s,x,\omega)} E[\theta\frac{x(T)}{T} - \frac{1}{T}\int_0^T h(x(s), b(s,x(s),\omega),\omega)ds]$$

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- $$v(0,0) = \sup_{(b, P_b)} E^{P_b}[\theta\, b(\omega) - h(b(\omega), \omega)]$$

Last Slide

THE END