Cayley Complexity of One Degree of Freedom Linkages in 2D

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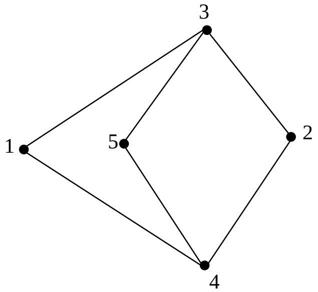
1-dof Linkages



- One degree of freedom (1-dof) linkage (mechanism) in 2D
- Linkage (G, δ): $G = (V, E)$, $\delta : E \to \mathbb{R}$

Cayley Configuration Space

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  $\Phi_f(G, \delta) := \{\delta^*(f) : \textit{linkage}\,(G \cup f, \delta, \delta^*)\ \textit{has realization}\}$

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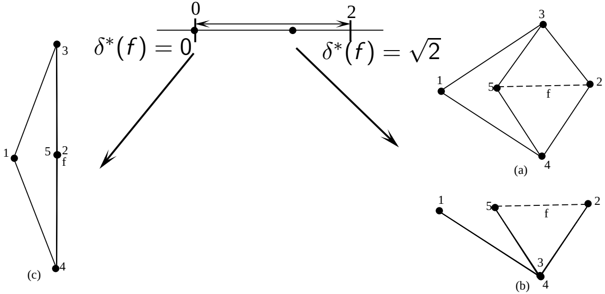
  $\Phi_f(G, \delta) := \{\delta^*(f) : \textit{linkage}\,(G \cup f, \delta, \delta^*)\ \textit{has realization}\}$
- $\Phi_f(G, \delta)$ is a set of intervals on the real line
- Each point $\delta^*(f)$ in $\Phi_f(G, \delta)$ is a Cayley configuration



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(a) Cayley complexity: algebraic complexity of interval endpoint values

Definition

Quadratically Solvable (QS) values: solutions to triangularized quadratic system with coefficient in $\mathbb{Q}$ (in extension field over $\mathbb{Q}$ by nested square-roots)

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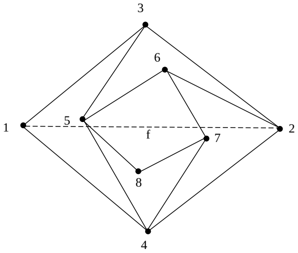
(b) Cayley size: number of intervals

(c) Cayley computational complexity: time complexity of obtaining all intervals (as function of Cayley size)

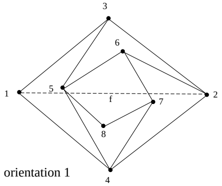
A Natural Class of Graphs

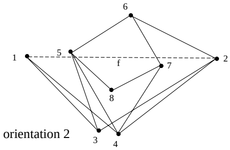
Cayley configurations $\delta^*(f)$ can be efficiently converted to Cartesian configurations provided:

- Completeness: $G \cup f$ minimally rigid (implies $(G \cup f, \delta, \delta^*(f))$ has finitely many realizations for each $\delta^*(f)$)
- Low realization complexity: linear realization complexity if local orientations are specified



Note: any $f = (i, i+2)$ guarantees both properties



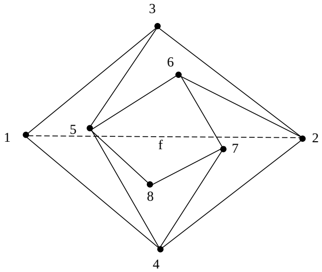


Quadratically Solvable Graphs

Definition

$G \cup f$ Quadratically Solvable (QS) from f: $\exists$ a ruler and compass realization of $(G \cup f, \delta, \delta^*(f))$ starting from f

Hence: Cayley configuration $\delta^*(f)$ $\xrightarrow{\text{efficient conversion}}$ Cartesian configuration



Note: for any $f = (i, i+2)$, $G \cup f$ *is QS starting from* f

A Class of Quadratically Solvable Graphs

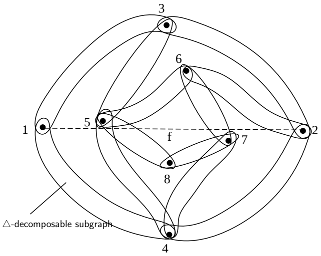
Definition

G is $\triangle$-decomposable if it is a single edge, or can be divided into 3 $\triangle$-decomposable subgraphs s.t. every two of them share a single vertex.

1-dof $\triangle$-decomposable graph: drop an edge f from a $\triangle$-decomposable graph

Note: $\triangle$-decomposable implies minimally rigid

- Graph construction from f: each step appends a new vertex shared by 2 $\triangle$-decomposable subgraphs
- This is also a (*unique*) QS realization sequence of corresponding linkage starting from f
- Hence $\triangle$-decomposable $\implies$ QS



A Class of Quadratically Solvable Graphs

Theorem (Owen & Power, 2005)

QS $\implies$ $\triangle$*-decomposable for planar graphs*

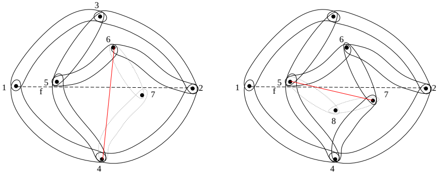
- Strong conjecture:
  $\triangle$-decomposable implies QS for general graphs
- In this talk, we only consider $\triangle$-decomposable graphs
- Will refer to them as QS graphs

QS Cayley complexity

Definition

G has QS Cayley complexity with respect to non-edge f: all interval endpoints – of $\Phi_f(G, \delta)$ – are QS

Extreme graphs: $O(n)$ of them, one per step of QS realization sequence, obtained by adding an extreme edge



Theorem

A 1-dof QS graph G has QS Cayley complexity on $f \iff$ all of its extreme graphs starting from f are QS.

This is probably folklore. For completeness, formally proven in (Gao & Sitharam, 2008).

Outline

1. Characterizing QS Cayley complexity
   - It is a Property of G Independent of Choice of Non-edge f
   - Algorithmic Characterization (4-cycle Theorem)
   - Finite Forbidden-Minor Characterization

2. Cayley Size & Cayley Computational Complexity
   - Guaranteeing Computational Complexity $O(n)$ & Cayley size $O(1)$

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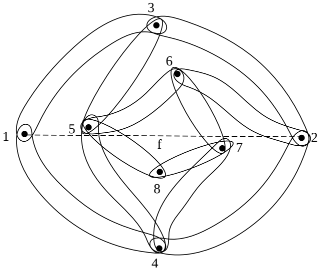
Choice of f

- Possible f: $(i, i+2)$ for any i
  By *possible* f we mean *any* non-edge f s.t. $G \cup f$ is QS.



- Does Cayley complexity depend on choice of f?

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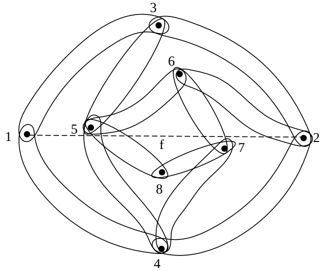


- Does Cayley complexity depend on choice of f?
- - NO.

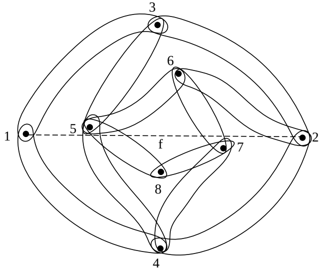
Independent of Choice of f

Theorem (Sitharam, Wang, Gao)

1-dof QS graph G either has QS Cayley complexity on all possible f or on none of them.

Proof is non-trivial.

Thus: our measure of QS Cayley complexity is robust. Characterizing G of QS Cayley complexity with a specific f is sufficient.



G has QS Cayley complexity

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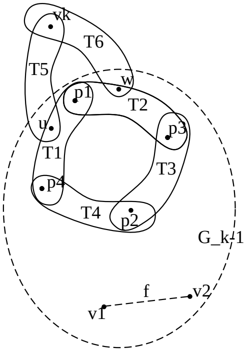
Algorithmic Characterization (4-cycle Theorem)

Theorem (Sitharam, Wang)

1-dof QS graph G has QS Cayley complexity $\iff \exists$ non-edge f ($\forall f$) each construction step from f is based on a pair of vertices taken from two adjacent QS subgraphs, from a 4-cycle of QS subgraphs

- Gives $O(n)$ time algorithm to recognize QS Cayley complexity graphs



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Finite Forbidden-Minor Characterization

- Can there exist finite forbidden-minor characterization for general 1-dof QS graphs?

Finite Forbidden-Minor Characterization

- Can there exist finite forbidden-minor characterization for general 1-dof QS graphs?
- - NO.

Will show counterexamples later.

1-Path & △-Free

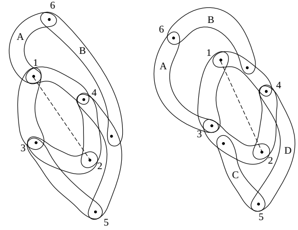
Need to look at natural subclasses: 1-Path & △-Free

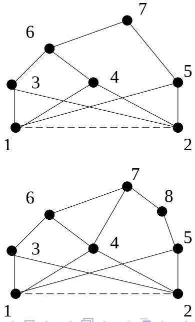
Definition

1-Path: ∃ only one "last vertex" v, that is, v is shared by exactly 2 QS subgraphs, each of them share only one vertex with the rest of the graph.

Definition

△-Free: no subgraph of G is a triangle

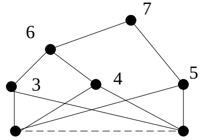




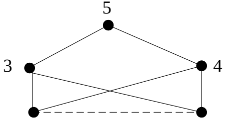
Equivalence to Planarity

Theorem (Sitharam, Wang)

A 1-path, $\triangle$-free, 1-dof QS graph G has QS Cayley complexity $\iff$ G is planar



(a)



(b)

Ex. (b) has QS Cayley complexity, (a) doesn't

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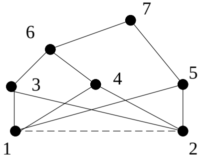

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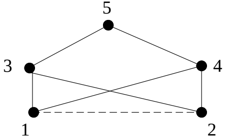


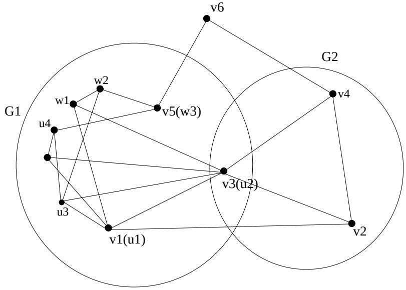
(b)

Ex. (b) has QS Cayley complexity, (a) doesn't

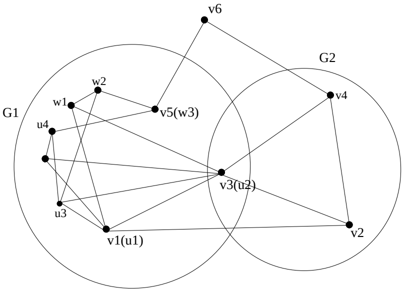
- 1-path & $\triangle$-free are necessary. Otherwise no finite forbidden-minor characterization exists

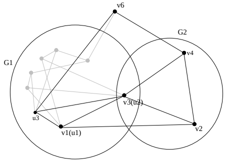
1-Path & △-Free are Necessary

Counter example 1: not △-free



- Has QS Cayley complexity since the sole extreme graph is QS.
- Can extend the graph to make G_1 have an arbitrary clique as minor


v6
G2
G1
w2
w1
v5(w3)
v4
u4
v3(u2)
u3
v2
v1(u1)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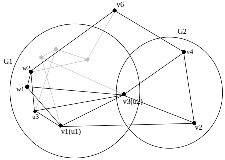
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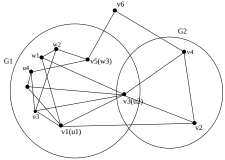
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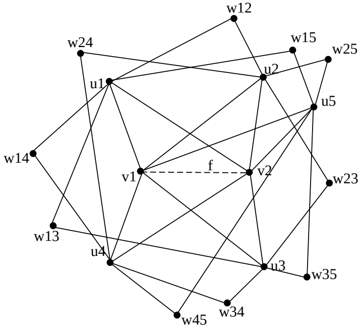
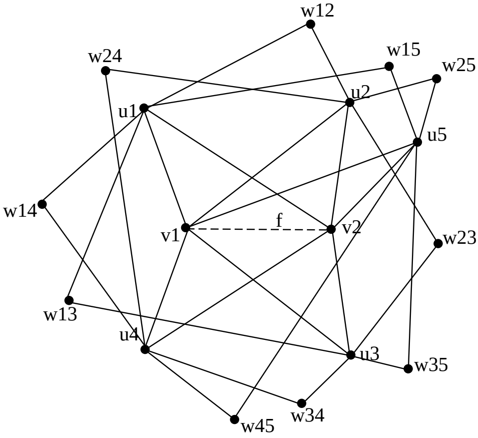
1-Path & △-Free are Necessary

Counter example 2: not 1-path



- Has QS Cayley complexity (can be checked using the 4-cycle theorem).
- Can be made to have an arbitrary clique as minor.


w12
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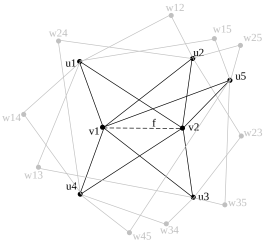
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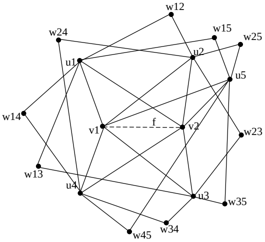
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Cayley size & Cayley computational complexity

Recall the three aspects of complexity of Cayley Configuration Spaces

(a) Cayley complexity

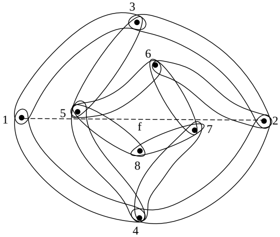
(b) Cayley size: number of intervals

(c) Cayley computational complexity: complexity of obtaining all intervals

- Have characterization of (a)
- Let's consider (b) and (c)

Cayley size & Cayley computational complexity

- Suppose G has QS Cayley complexity



- Are we guaranteed to have small Cayley size & low Cayley computational complexity?

Cayley size & Cayley computational complexity

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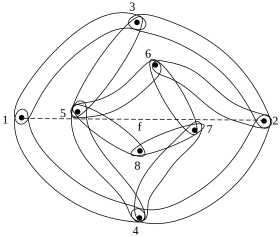


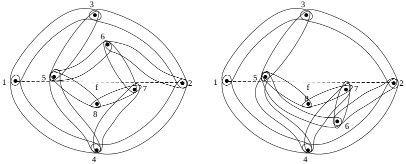
- Are we guaranteed to have small Cayley size & low Cayley computational complexity?

- Only if we specify necessary orientations of the realizations

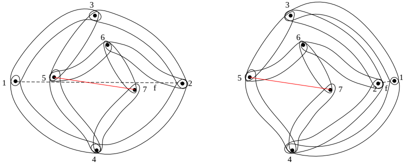
Necessary Orientations

A natural, minimal set of local orientations for both forward & backward QS realization sequences

- forward orientations from f



- backward orientations for all extreme linkages



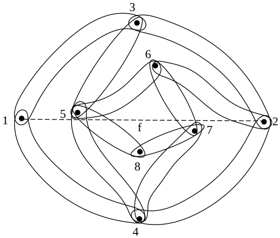
Blow-up of Cayley Size & Computational Complexity without Orientations

Is either type of orientations sufficient without the other?

Blow-up of Cayley Size & Computational Complexity without Orientations

Is either type of orientations sufficient without the other?
- NO.

- Can adapt existing examples of Borcea & Streinu to show exponential blow-up



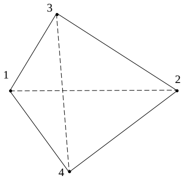
- *Already so for our standard example.*

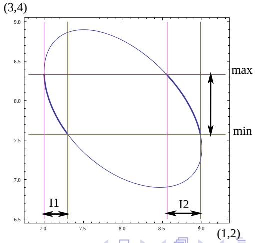
Efficient Cayley Configuration Space

Theorem (Sitharam, Wang)

For 1-dof QS graph G with QS Cayley complexity, given both forward and backward orientations, the Cayley size is $O(1)$ and the Cayley computational complexity is $O(|V|)$

- Proof non-trivial & based on the 4-cycle theorem
- Yields straightforward algorithm using quadrilateral interval mapping via 4-cycles.





Summary

- Cayley configuration space & measure of complexity
- Choice of base non-edge does not affect QS Cayley complexity.
- Algorithmic characterization (4-cycle Theorem)
- For 1-path, $\triangle$-free, 1-dof QS graphs: QS Cayley complexity $\Longleftrightarrow$ planarity
- Low Cayley size & computational complexity in the presence of necessary orientations

Proof of Planarity Theorem

Theorem

A 1-path, $\triangle$-free, 1-dof QS graph G has QS Cayley complexity $\iff$ G is planar

Proof of Planarity Theorem

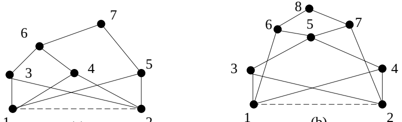
Lemma (1)

Given a 1-path, $\triangle$-free, 1-dof QS graph G with non-edge $f = (v_1, v_2)$. If

- *3 or more vertices are directly constructed on f OR*
- *exactly 2 vertices are directly constructed on f & $deg(v_1) \geq 3, deg(v_2) \geq 3$*

We have

1. *G has a $K_{3,3}$ minor*
2. *G does not have QS Cayley complexity on f*



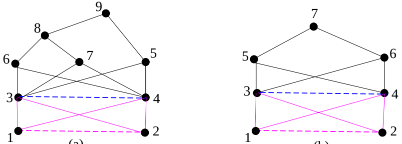
(a) (b)

Proof of Planarity Theorem

Lemma (2)

Given a 1-path 1-dof QS graph G with non-edge $f = (v_1, v_2)$ s.t. u_1, u_2 are the only 2 vertices directly constructed on f, if v_1 is a "last vertex" & v_2 is not (resp. both v_1 and v_2 are "last vertices"), then

1. $G' = G \setminus \{v_1\}$ *(resp.* $G' = G \setminus \{v_1, v_2\}$*) is 1-path 1-dof QS graph on* $f' = (u_1, u_2)$.
2. $G' = G \setminus \{v_1\}$ *(resp.* $G' = G \setminus \{v_1, v_2\}$*) has QS Cayley complexity on* f' $\iff$ G *has QS Cayley complexity on* f



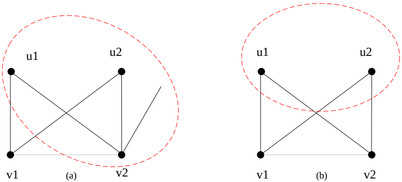
(a) (b)

Proof of Planarity Theorem.

By the two lemmas, the only interesting case is where G has exactly 2 vertices u_1, u_2 directly constructed on $f = (v_1, v_2)$, and either (a) $deg(v_1) = 2$, $deg(v_2) > 2$, or (b) $deg(v_1) = 2$, $deg(v_2) = 2$. Define G' as in Lemma (2).

1. G is planar $\Longrightarrow$ G has QS Cayley complexity on f:
   Prove by contradiction. Assume G is the minimum QS graph s.t. G does not have QS Cayley complexity on f and is planar. Clearly G' contradicts the assumption of minimality of G.

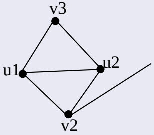
□



Proof of Planarity Theorem (cont.)

2 G has QS Cayley complexity on $f \implies G$ has no $K_{3,3}$:
Prove by contradiction. Assume G is the minimum QS graph s.t. G has QS Cayley complexity on f and G has a $K_{3,3}$ minor.

In case (a), either (v_1, u_1) or (v_1, u_2) must be contracted. Either case we obtain the graph on right. (v_3 is the first vertex constructed after u_1 and u_2)



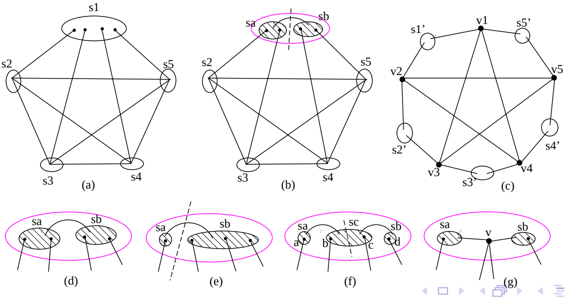
$K_{3,3}$ contains no triangles. Every way to eliminate the two triangles will produce a subgraph of G'.
In case (b), similar argument applies.

□

Proof of Planarity Theorem (cont.)

3 1-path, △-free, 1-dof QS graph G has $K_5 \implies G$ has $K_{3,3}$: To keep G △-free, some vertices of K_5 must be contracted from more than one vertices from G. To keep G 1-path we will get a $K_{3,3}$.

Therefore we have: G has QS Cayley complexity on $f \implies G$ has no $K_{3,3}$ or K_5. Thus completes the proof. □



Proof of $O(1)$ Cayley Size Theorem

Theorem

For 1-dof QS graph G with QS Cayley complexity, given both forward and backward orientations, the Cayley size is $O(1)$ and the Cayley computational complexity is $O(|V|)$

Proof of $O(1)$ Cayley Size Theorem

Levels

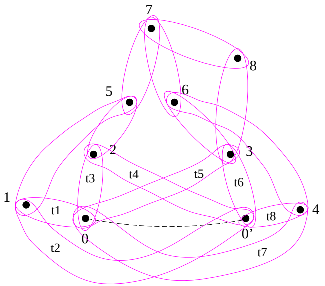
Definition

Levels: construction partial order of a 1-dof QS graph from non-edge f

L_0: endpoints of f.

L_1: can directly construct on f.

$L_i (i \geq 2)$: can directly construct given $L_0 \sim L_{i-1}$, cannot construct without L_{i-1}.



Proof of $O(1)$ Cayley Size Theorem

Lemma

For a 1-path G with QS Cayley complexity,

1. *Each level has one or two construction steps.*
2. *If L_k has two construction steps, they are based the same pair of vertices.*
3. *From L_{k+1} on, each construction step must be based on QS subgraphs in L_k or higher levels.*

Proof of $O(1)$ Cayley Size Theorem.

1. For the 1-path case, the chain of quadrilateral is obvious. The algorithm maps the attainable interval of one diagonal of a quadrilateral to the attainable interval of the other diagonal. By Lemma (2) this mapping process can be repeated, so we can finally get the interval of f.
   Since both forward and backward orientations are fixed, each mapping step is projection on a monotonic function.
   Therefore the Cayley size is $O(1)$.

□

Proof of $O(1)$ Cayley Size Theorem (cont.)

2. For graphs with two "last vertices", we can find a base 4-cycle at the common "root" of both paths. Each path maps to a single interval of a diagonal of the root 4-cycle. Considering the constraints from both paths together, the result is the intersection of two intervals.
3. For graphs with more than 2 paths, we can prove by induction on number of paths.

□

Thank you!