To hold a convex body by a frame

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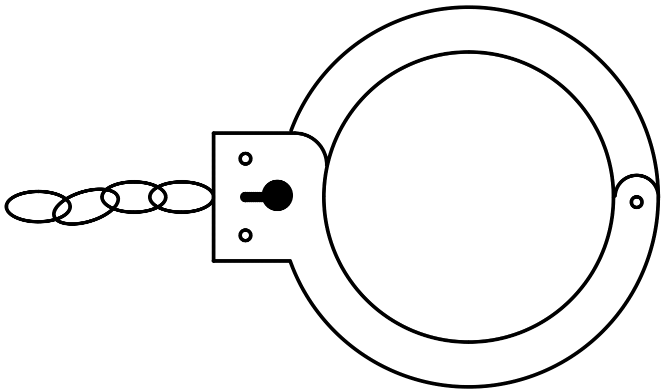
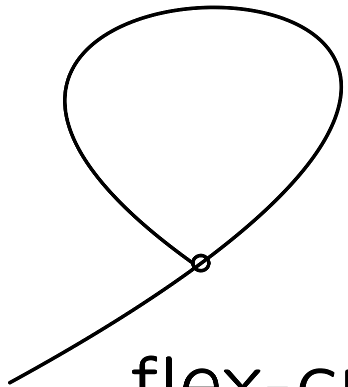
- Is it possible to hold a cube by a circular handcuff?
- How about for other convex bodies?
- By a flex-cuff or handcuffs of other shapes?





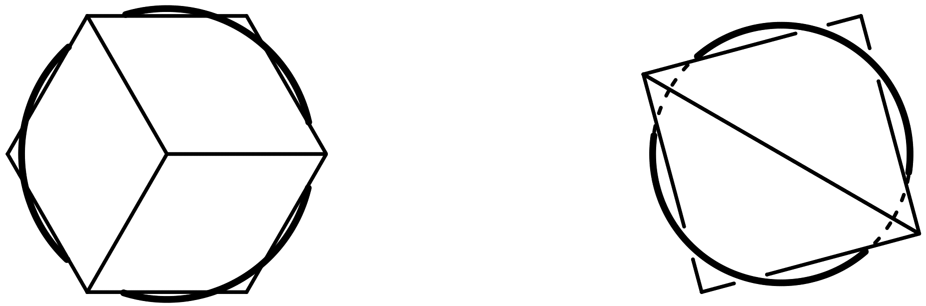
flex-cuff

- A **frame** is the (rigid) boundary curve of a convex disk in a plane.
- A frame F is said to **hold** a convex body K if
  (1) F is **attached** to K, that is,
  $F \cap int(K) = \emptyset$, $conv(F) \cap int(K) \neq \emptyset$, and
  (2) F cannot **slip out** of K by a rigid motion with keeping $F \cap int(K) = \emptyset$.

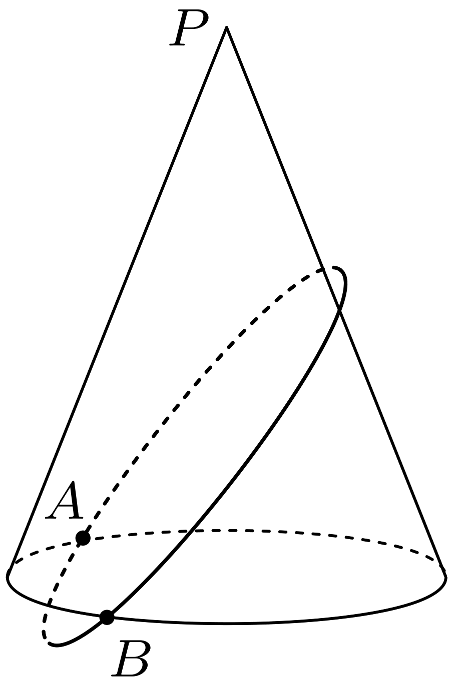
1. Circular frame

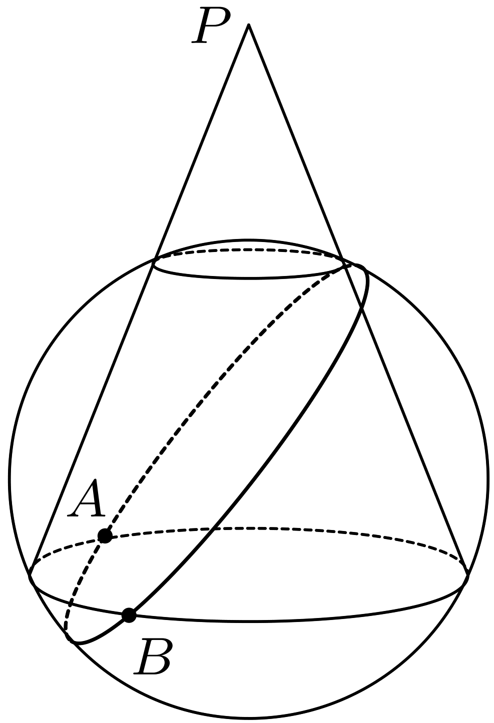
A convex body is called **circle–free** if no **circle** can hold it.

- Every ball is circle–free.
- A cube and a regular tetrahedron are not circle–free.



Example. Every circular cone is circle–free.



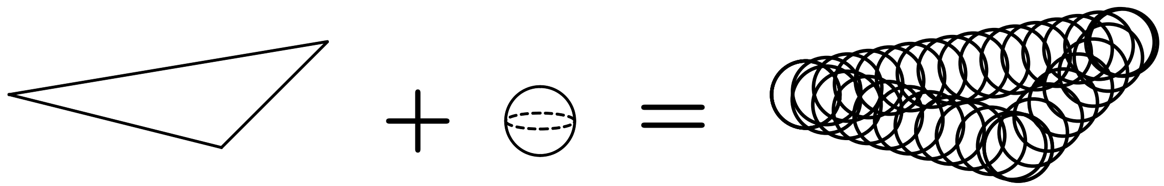


Theorem (T. Zamfirescu 1995)
The set of circle-free convex bodies in $\mathbb{R}^3$ is a nowhere dense subset of the set of all convex bodies in $\mathbb{R}^3$ with Housdorff metric.

Thus, circle-free convex bodies are rare.

Theorem (M 2011)

For every planar compact convex set X and a ball $\mathbf{B}$ in $\mathbb{R}^3$, the convex body $X+\mathbf{B}$ (Minkowski sum) is circle-free.



Problem. Is it true that for every circle-free convex body K, the set $K+\mathbf{B}$ is also circle-free?

Symmetrization Lemma. Let $\mathit{\Gamma}_0$ be a holding circle of a convex polyhedron such that

(1) $conv(\mathit{\Gamma}_0)$ divides its vertex set into U, V,

(2) both U and V are symmetric to themselves w.r.t. a fixed plane H.

Then there is a continuous family of holding circles $\mathit{\Gamma}_t$ $(0 \le t \le 1)$ such that

$$diam(\mathit{\Gamma}_t) \le diam(\mathit{\Gamma}_0),\ \mathit{\Gamma}_t \cap H = \mathit{\Gamma}_0 \cap H,$$

and $\mathit{\Gamma}_1$ is symmetric to itself w.r.t. H.

2. Holding circles of Platonic solids

Theorem (Itoh, Tanoue, Zamfirescu 2006)

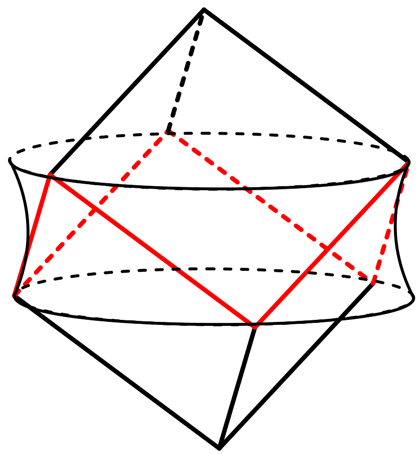
A circle of diameter d can hold a regular tetrahedron of unit edge if and only if

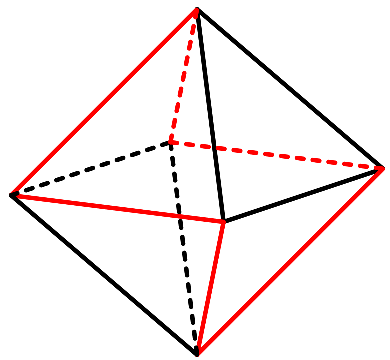
$$1/\sqrt{2} \leq d < \phi_t \approx 0.896$$

where ϕ_t is the min value of $\dfrac{2(x^2 - x + 1)}{\sqrt{3x^2 - 4x + 4}}$.

Hyperboloidal Restriction.

If the plane of a holding circle cuts the edges on a hyperboloid of revolution, the order of intersections of the circle and the edges is restricted by the alternation of inside-outside.





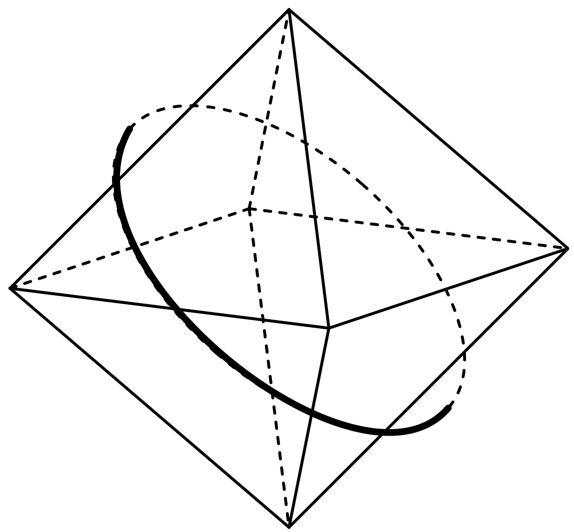
Lemma. If a circle Γ holds a cube, then $conv(\Gamma)$ cuts all faces of the cube.

Theorem (M 2010).
A circle Γ of diameter d can hold a unit cube if and only if $\sqrt{2} \le d < \phi_c \approx 1.535$, where ϕ_c is the min value of $\frac{\sqrt{2}(x^2+2)}{\sqrt{x^2+2x+3}}$.

Proof is by SL, HR, Lemma, and computations.

For a regular octahedron, the following holds.

Lemma. If Γ holds a regular octahedron, then $conv(\Gamma)$ separates its two faces.



Using the lemma, we can prove the following.

Theorem (M 2010)

A circle of diameter d can hold a regular octahedron of unit edge if and only if

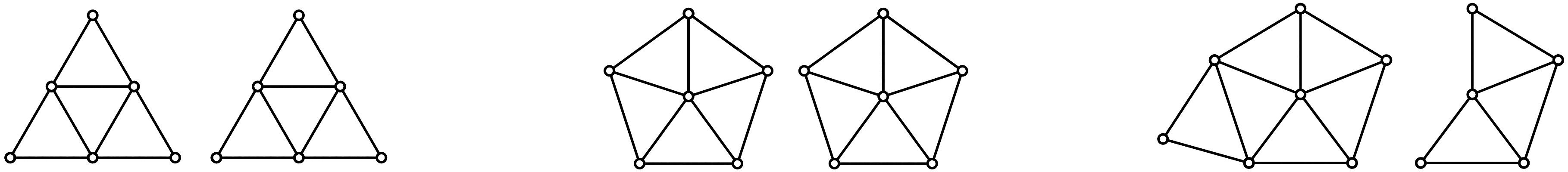
$$1 \le d < \phi_o \approx 1.1066,$$

where ϕ_o is the min value of $\frac{2(x^2+1)}{\sqrt{3x^2+2x+3}}$.

Y. Tanoue got the same result independently.

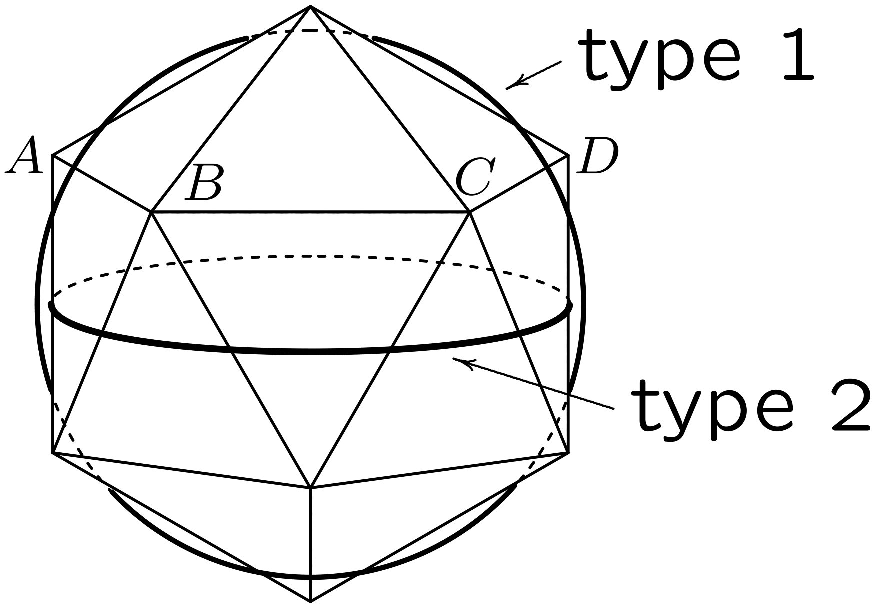
Lemma.

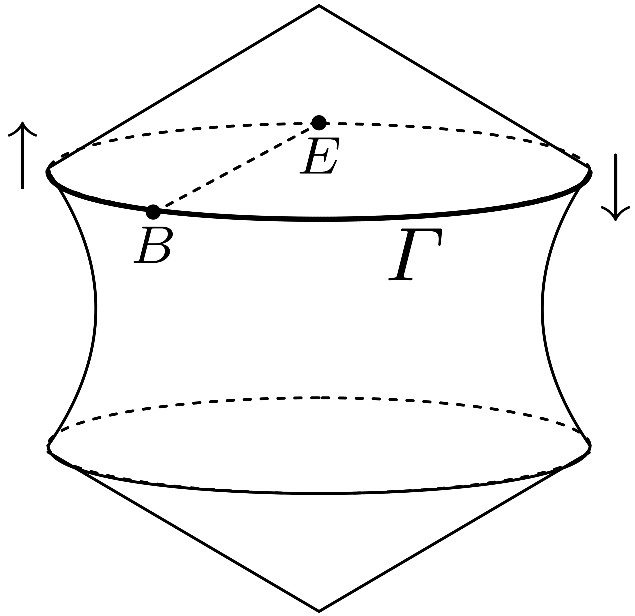
If a circle Γ holds a regular icosahedron I, then $conv(\Gamma)$ divides the graph of I into one of the following pairs:



(1) (2) (3)

Corresponding to (1), (2), (3), holding circles are called type 1, type 2, type 3, respectively.





To get a type 3 circle, rotate the circum-circle Γ of $ABCDE$ around $\overleftrightarrow{BE}$ so that $conv(\Gamma)$ goes above A, push down the circle a bit, squeeze it.

Theorem.

A regular icosahedron of unit edge has three types holding circles. The min diameter of type 1 circle is $\sqrt{3} \approx 1.7320$, and the min diameter of type 2 circle is $\tau := \frac{1+\sqrt{5}}{2} \approx 1.6180$.

There is a holding circle that can vary between types 2 and 3 by going over a vertex. The min diameter of such circle is $d_* \approx 1.6816$, the min value of $\frac{\tau^2+(1-x)^2}{\sqrt{\tau^2+(1-x)^2-(x/2)^2}}$.

Theorem.

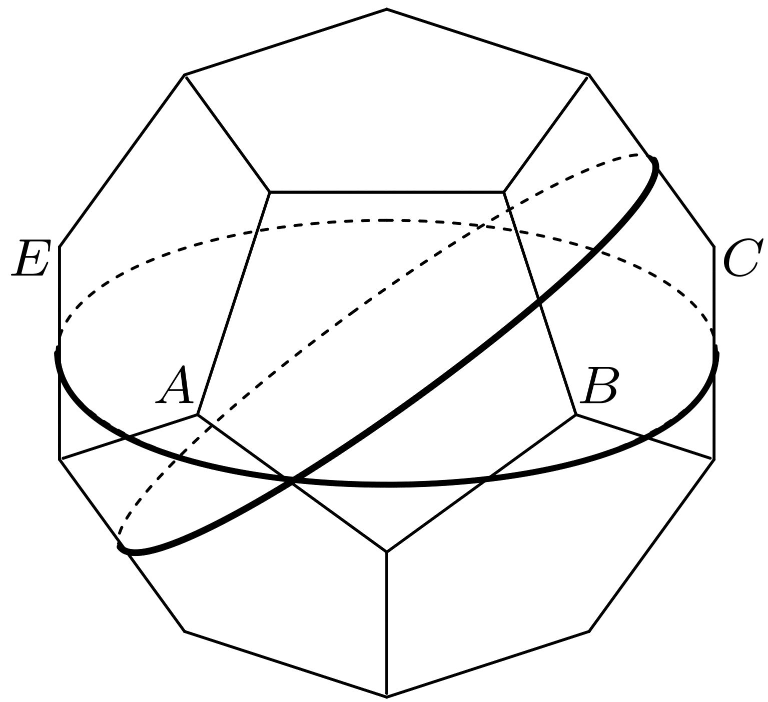
The circum-circle of a regular pentagon of unit side (diameter $2\sqrt{\tau/\sqrt{5}} \approx 1.7013$) cannot hold a regular icosahedron of unit edge.

Since $\tau < 1.7013 < \sqrt{3}$, the set of diameters of all holding circles (**DHC**) of a regular icosahedron has 2 connected components.

Theorem.

A regular dodecahedron of unit edge can be held by a circle of diameter $\tau^2 \approx 2.6180$ in two different ways, where $\tau = \frac{1+\sqrt{5}}{2}$.

There are holding circles Γ of a dodecahedron such that $conv(\Gamma)$ divides the vertex set unevenly.



Two holding circles of a regular dodecahedron.

Problem. Find all types of circles that hold a regular dodecahedron. Determine DHC for a regular icosahedron/dodecahedron of unit edge.

Problem. Is there a convex body whose DHC has more than 2 connected components?

3. The case of regular pyramids

- For a regular pyramid, define its **slope** ρ by

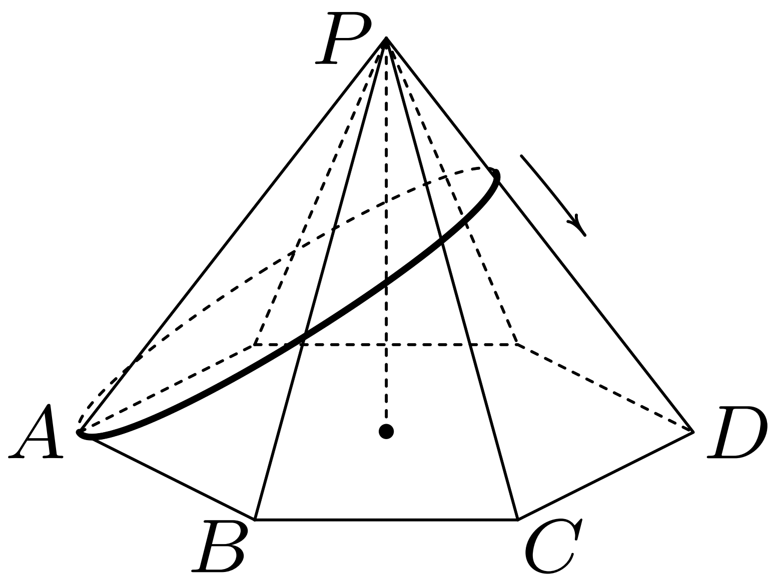
$$\rho := \frac{\text{height}}{\text{circum-radius of the base}}$$

Theorem (M 2011)

Every regular pyramid with $\rho \geq 1$ can be held by a circle.

Proof (Outline).

Slide the circle slightly in the direction $\overrightarrow{PD}$, and then squeeze its radius. The resulting circle cannot slip out of the pyramid.



Theorem (M 2011)

For every $0 < \varepsilon < 1$, there is a circle-free regular pyramid with $\rho = 1 - \varepsilon$.

If $m > 2\pi/\varepsilon^2$, then a regular pyramid with base regular $4m$-gon and $\rho = 1 - \varepsilon$ is circle-free.

Theorem (Tanoue 2009, M 2011)
A regular pyramid with equilateral triangular base can be held by a circle if and only if

$$\rho > \sqrt{(3\sqrt{17} - 5)/32} \approx 0.4799.$$

Y. Tanoue proved the if part.

Theorem (M 2011)

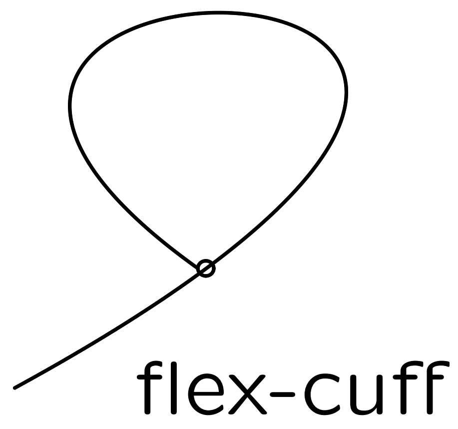
A regular pyramid with square base can be held by a circle if and only if

$$\rho > \sqrt{(\sqrt{33} - 3)/4} \approx 0.828.$$

The Great Pyramid of Giza has base-edge $230m$, height $140m$. Since $140\sqrt{2}/230 \approx 0.860 > 0.828$, it can be held by a circle.

4. Strings and frames of other shapes

Theorem (A. Fruchard 2009)
A loop of string winding on a convex body can slip out of the convex body. (Hence a flex-cuff cannot hold a convex body.)

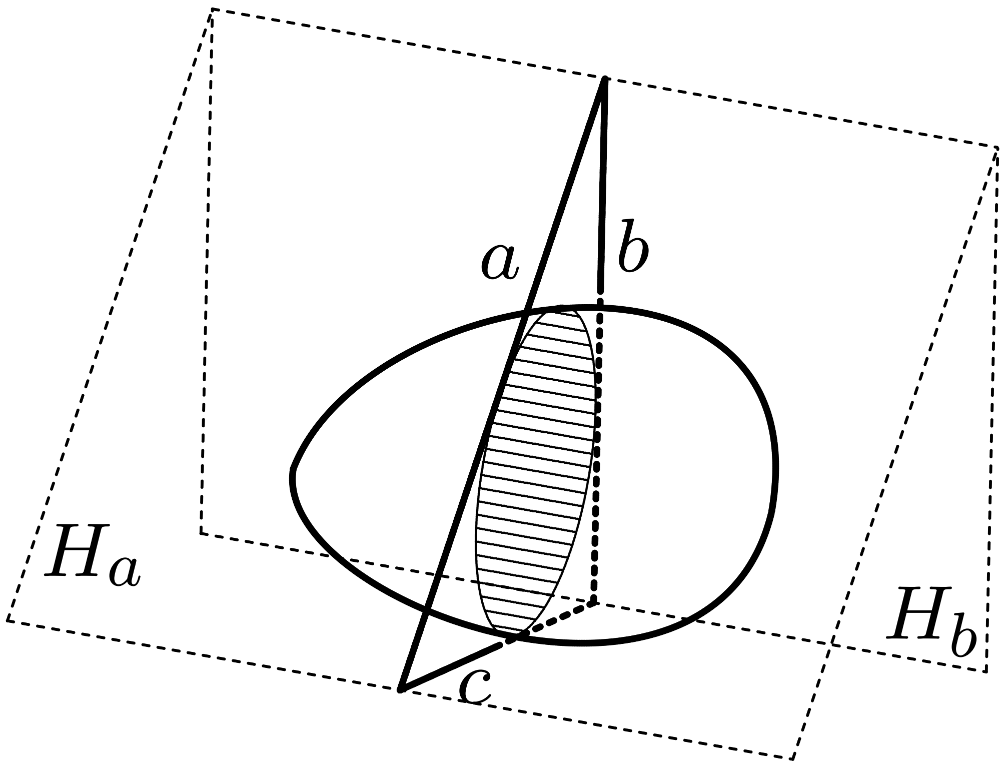


Proof by Fruchard (For convex polyhedron P): By contradiction. Let L be the shortest loop that cannot slip out of P. Then, L draws a geodesic on ∂P, and never passes through a vertex of P. It is possible to develop (unfold) ∂P on the plane so that the track of L becomes a line-segment between two parallel edges. Slide this line-segment between the parallel edges. This corresponds to a motion of L on ∂P. Then, L eventually hits a vertex, a contradiction. $\square$

Theorem (Bárány, Tokushige, M. 2011)
No triangular frame can hold a convex body.

Proof.

H_a, H_b, H_c enclose either a triangular cone or a triangular prism. Hence the convex body can slip out. □



- A frame F is said to **fix** a convex body K if F holds K in such a way that when K is fixed, F is confined in a fixed plane.

Theorem (Bárány, Tokushige, M 2011)
Every non-triangular frame can fix some tetrahedron.