Universal Rigidity of bar frameworks in General Position

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Bar Frameworks

Definition

A bar framework in $\mathbb{R}^r$, denoted by $G(p)$, is a configuration $p = (p^1, \ldots, p^n)$ in $\mathbb{R}^r$ together with a graph G on the vertices $1, \ldots, n$.

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- Multidimensional scaling.
- Wireless sensor network localization problems.

Equivalent Frameworks and Affine Aotions

Definition

A framework $G(p)$ in $\mathbb{R}^r$ is equivalent to framework $G(q)$ in $\mathbb{R}^s$ if

$$||q^i - q^j|| = ||p^i - p^j|| \text{ for every edge } (i,j) \in E(G).$$

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Definition

A framework $G(p)$ in $\mathbb{R}^r$ has an affine motion if there exists a framework $G(q)$ in $\mathbb{R}^r$ that is equivalent, but not congruent, to $G(p)$ such that $q^i = Ap^i + b$ for all $i = 1, \ldots, n$.

Dimensional Rigidity

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A given framework $G(p)$ in $\mathbb{R}^r$ is said to be dimensionally rigid if there does not exist another framework $G(q)$, equivalent to $G(p)$, in $\mathbb{R}^s$ where $s \geq r + 1$.

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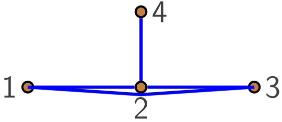
$G_1(p)$ in $\mathbb{R}^2$ is dim rigid while $G_2(q)$ is dim flexible.



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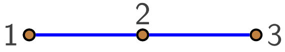

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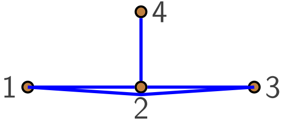

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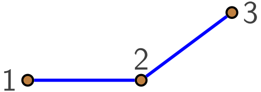

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Universal Rigidity

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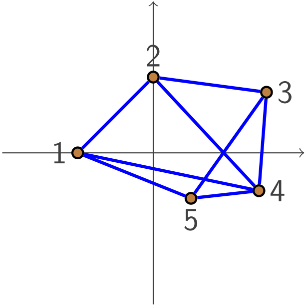

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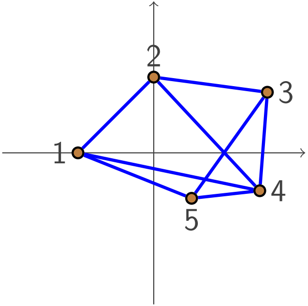
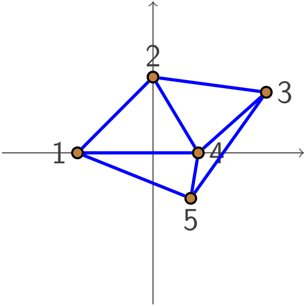


The following framework $G(q)$ is globally, but not universally rigid.



Alternative Way to Introduce Universal Rigidity:

- Given a configuration $p = (p^1, \ldots, p^n)$ in $\mathbb{R}^r$, define the matrix

$$D_p = (d_{ij} = ||p^i - p^j||^2).$$

D_p is called the Euclidean distance matrix (EDM) generated by p.

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$$D_p = (d_{ij} = ||p^i - p^j||^2).$$
D_p is called the Euclidean distance matrix (EDM) generated by p.
- Given an EDM D_p, universal rigidity is the problem of deciding whether a strict subset of the entries of D_p suffices to uniquely determine all entries of D_p, i.e., to recover p up to a rigid motion.

In This Talk:

- I'll present a sufficient condition for universal rigidity of bar frameworks in general position.

Universal and dimensional Rigidity

Theorem (A '07)

Let $G(p)$ be a given framework on n nodes in $\mathbb{R}^r$, $r \leq n-2$. Then $G(p)$ is universally rigid if and only if:

1. *$G(p)$ is dimensionally rigid, and*
2. *$G(p)$ does not have an affine motion.*

Stress Matrices

- A stress of a bar framework $G(p)$ is a real-valued function ω on $E(G)$ such that:

$$\sum_{j:(i,j)\in E(G)} \omega_{ij}(p^i - p^j) = 0 \text{ for all } i = 1, \ldots, n.$$

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- Given a stress ω, the stress matrix associated with ω is the $n \times n$ symmetric matrix $S = (s_{ij})$ where

$$s_{ij} = \begin{cases} -\omega_{ij} & \text{if } (i,j) \in E(G), \\ 0 & \text{if } (i,j) \notin E(G), \\ \sum_{k:(i,k)\in E(G)} \omega_{ik} & \text{if } i = j. \end{cases}$$

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- Note the resemblance of S to the Laplacian of G.

Gale Matrices

- A Gale matrix of $G(p)$ in $\mathbb{R}^r$ is any $n \times (\bar{r} = n - 1 - r)$ matrix Z such that the columns of Z form a basis of the null space of :
$$\begin{bmatrix} p^1 & p^2 & \cdots & p^n \\ 1 & 1 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} P^T \\ e^T \end{bmatrix}.$$

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- In Polytope theory, the rows of Z are called Gale transforms of $p^1, \ldots, p^n$.
- A Gale matrix Z encodes the affine dependencies among the points $p^1, \ldots, p^n$.

Gale Matrix Z and Stress Matrix S

Lemma (A '07)

Let S and Z be, respectively, a stress matrix and a Gale matrix of $G(p)$. Then

$$S = Z \Psi Z^T \text{ for some symmetric matrix } \Psi.$$

On the other hand, let Ψ' be any symmetric matrix such that

$${z^i}^T \Psi' z^j = 0 \text{ for all } (i,j) \notin E,$$

where z^i is the ith row of Z. Then $Z \Psi' Z^T$ is a stress matrix of $G(p)$.

Sufficiency Results

Theorem (A '07)

Let $G(p)$ be a given framework on n nodes in $\mathbb{R}^r$, $r \leq n-2$. Let Z be a Gale matrix of $G(p)$ and let z^i be the ith row of Z.
If $\exists\ \Psi \succ 0 : {z^i}^T \Psi z^j = 0 \quad \forall (i,j) \notin E$, or equivalently,
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Corollary (A '07, Connelly '82 and '99)

Let $G(p)$ be a given framework on n nodes in $\mathbb{R}^r$, $r \leq n-2$. If
1- $\exists\ \Psi \succ 0 : {z^i}^T \Psi z^j = 0 \quad \forall (i,j) \notin E$, or equivalently,
$\exists$ a semidefinite stress matrix S of rank $= \bar{r} = n - 1 - r$, and
2- $G(p)$ does not have an affine motion.
Then $G(p)$ is universally rigid.

Frameworks with Affine Motions

Let V be such that $V^T e = 0$ and $V^T V = I_{n-1}$.
E^{ij} is the matrix with 1s in the ijth and jith entries and 0s elsewhere.
Z is a Gale matrix of $G(p)$.

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Lemma (Connelly '05, A. '07)

The following statements are equivalent:

1. *$G(p)$ in $\mathbb{R}^r$ has an affine motion.*
2. *There exists a non-zero $r \times r$ symmetric matrix Φ such that $(p^i - p^j)^T \Phi (p^i - p^j) = 0$ for all $(i,j) \in E$.*
3. *There exists $y = (y_{ij}) \neq 0$ such that $V^T \sum_{(i,j) \notin E} y_{ij} E^{ij} Z = 0$,*

Frameworks in General Position

Definition

A framework $G(p)$ in $\mathbb{R}^r$ is said to be in general position if no $r+1$ of the points $p^1, \ldots, p^n$ are affinely dependent. For example no 3 collinear points in the plane. etc.

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Theorem

Let $G(p)$ be a given framework on n nodes in general position in $\mathbb{R}^r$, $r \leq n-2$. Let Z be a Gale matrix of $G(p)$ and let ${z^i}^T$ be the ith row of Z. Then

Every subset of $\{z^1, \ldots z^n\}$ of cardinality $\leq \bar{r} = n-1-r$ is linearly independent.

Universal Rigidity in General Position

Theorem (A. and Ye '10)

Let $G(p)$ be a framework on n nodes in general position in $\mathbb{R}^r$, $r \leq n-2$. If $G(p)$ admits a positive semidefinite stress matrix S of rank $n-1-r$. Then $G(p)$ does not have an affine motion.

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Corollary (A. and Ye '10)

Let $G(p)$ be a framework on n nodes in general position in $\mathbb{R}^r$, $r \leq n-2$, and let Z be a Gale matrix of $G(p)$. If:
$\exists\ \Psi \succ 0 : {z^i}^T \Psi z^j = 0 \quad \forall (i,j) \notin E$, *or equivalently, if*
$\exists$ a positive semidefinite stress matrix S of rank $n-1-r$,
Then $G(p)$ is universally rigid.

Proof of the Theorem

- We need to prove that if $V^T \sum_{(i,j) \notin E} y_{ij} E^{ij} Z = 0$, then $y = 0$.

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- Thus $\sum_{(i,j)\notin E} y_{ij} E^{ij} Z = \sum_{j \in \overline{N}(i)} y_{ij} z^j = 0$ implies that $y = 0$.
- Therefore, if we can show that V^T is redundant, then we are done. The choice of Z is critical in this regard.

Proof of the Theorem Cont'd

- If $G(p)$ is in general position and if $G(p)$ admits a stress matrix S of rank $\bar{r}$, then there exists a Gale matrix $\hat{Z}$ with the following property:

$$\hat{z}_{ij} = 0 \text{ for all } j = 1, \ldots, \bar{r} \text{ and } i \in \overline{N}(j + r + 1).$$

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- $\hat{Z}$ is just the matrix consisting of the last $\bar{r}$ columns of S.
- $V^T \sum_{(i,j)\notin E} y_{ij} E^{ij} \hat{Z} = 0$ iff $\sum_{(i,j)\notin E} y_{ij} E^{ij} \hat{Z} = e\xi^T$.

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- To complete the proof, it suffices to show that $\xi = 0$.

Proof of the Theorem Cont'd

- The $(r+2,1)$th entry of $e\xi^T$ is $\xi_1 =$

$$[0 \ \cdots \ y_{r+2,j_1} \ \cdots \ 0 \ \cdots \ y_{r+2,j_2} \ \cdots \ ] \begin{bmatrix} \hat{z}_{11} \\ \vdots \\ \hat{z}_{r+2,1} \\ \vdots \\ \hat{z}_{n,1} \end{bmatrix} = 0,$$

since $\hat{z}_{i1} = 0$ for all $i \in \overline{N}(r+2)$.

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  since $\hat{z}_{i1} = 0$ for all $i \in \overline{N}(r+2)$.
- By considering the entries $(r+3,2)$, $(r+4,3), \ldots (n,\bar{r})$ we get $\xi = 0$.

Thank You