A Robust Robust Optimization Result

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1. Problem and Goal

We consider the problem

$$\max\{v^T x : x \in C\},$$

$C \subset I\!R^n$: nonempty, compact (wlog convex),
$v \in I\!R^n$.

- Traditional viewpoint: C uncertain, model so that result computationally tractable.
- Our viewpoint: v uncertain, how much is lost?

2. A Model Case

Suppose first that C is the unit ball, v has unit norm.
The solution to our problem with the nominal objective vector v is $x = v$, with objective value 1.

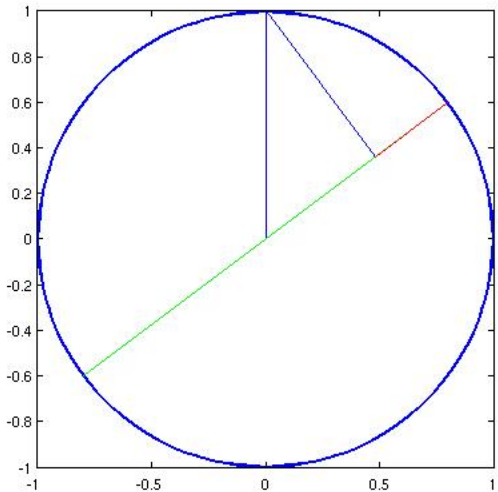
If the true objective vector is $w := w(\alpha)$, a unit vector making an angle α, $0 \leq \alpha \leq \pi$, with v, then v attains a true objective value of $\cos\alpha$, with a loss of $1 - \cos\alpha$. Since the range of $w^T x$ over C is 2 (from -1 to $+1$),

$$\text{scaled\_loss} = \frac{\text{loss}}{\text{range}} = \frac{1 - \cos\alpha}{2}.$$

We show that this scaled loss formula holds "on average" for arbitrary C.

Note that we seem to have things backward: the true objective w should be given first, while the perturbed objective v actually optimized should be a function of w. We will address this later.

Model Case, II



The loss is the length of the red line segment; the range is the combined lengths of the red and green line segments.

3. Definitions

$\max(v) := \max\{v^T x : x \in C\}$;

$\min(v) := \min\{v^T x : x \in C\}$;

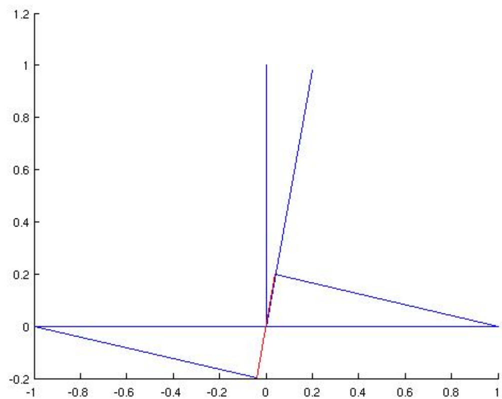
$\text{range}(v) := \max(v) - \min(v)$;

$\text{loss}(v, w) := \max(w) - \min\{w^T x : x \in C,\ v^T x = \max(v)\}$.
(The loss in the true objective $w^T x$ possible when implementing a best solution for the nominal objective $v^T x$.)

$\text{scaled\_loss}(v, w) := \frac{\text{loss}(v,w)}{\text{range}(w)}$.

4. A Very Bad Case

On the other hand, the scaled loss is terrible in the case that C is the line segment joining $[-1;0]$ and $[+1;0]$, v is $[0;1]$, and $w := w(\alpha)$ is $[\sin\alpha; \cos\alpha]$.
Then $[-1;0]$ is optimal for v but attains the worst objective value for w, so that scaled_loss(v,w) is 1.



5. Three Probabilistic Models

(i) Suppose v and u are independently drawn from the standard Gaussian distribution $N(0, I)$, and let $w := w(\alpha) := \cos\alpha\, v + \sin\alpha\, u$.

The angle between v and w is with high probability very close to α as n approaches infinity. Also, (w, v) has the same distribution as (v, w). We denote expectations with respect to this distribution by E_1.

(ii) Suppose $\bar{v}$ and $\bar{u}$ are independently drawn from $N(0, I)$. Let $\hat{u} := (I - \bar{v}\bar{v}^T/\bar{v}^T\bar{v})\bar{u}$, $v := \bar{v}/\|\bar{v}\|$, $u := \hat{u}/\|\hat{u}\|$, and $w := w(\alpha) := \cos\alpha\, v + \sin\alpha\, u$.

It is not hard to see that the angle between v and w is now exactly α, and again, (w, v) has the same distribution as (v, w). We denote expectations with respect to this distribution by E_2.

6. Three Probabilistic Models, continued

(iii) Our third distribution is quite general, but has a different form of perturbation. Let f_j be a symmetric probability density function on $I\!R$, $j = 1, \ldots, n$. For each j, let v_j and u_j be independently drawn from f_j, and let $w_j := w_j(\alpha)$ be v_j with probability $\cos\alpha$ and u_j with probability $1 - \cos\alpha$.

Once again, (w, v) has the same distribution as (v, w), and, under mild conditions on the f_j, the angle between v and w is concentrated around α.

Note that, in this model, a small fraction of the components is changed a possibly large amount, while in the previous model, each component is changed a small amount. We denote expectations with respect to this distribution by E_3.

7. Results

Note: $-\min(v) = \max\{-v^T x : x \in C\}$. So

Proposition:

$$E_1[\max(w)] = E_1[\max(v)],$$

$$E_1[\mathsf{range}(w)] = E_1[\mathsf{range}(v)] = 2E_1[\max(v)].$$

Let $x_v \in C$ maximize $v^T x$ over C. Then

$$w(\alpha)^T x_v = \cos\alpha\, v^T x_v + \sin\alpha\, u^T x_v.$$

Proposition:

$$E_1[\mathsf{loss}(v, w(\alpha))] = (1 - \cos\alpha) E_1[\max(v)].$$

Results, II

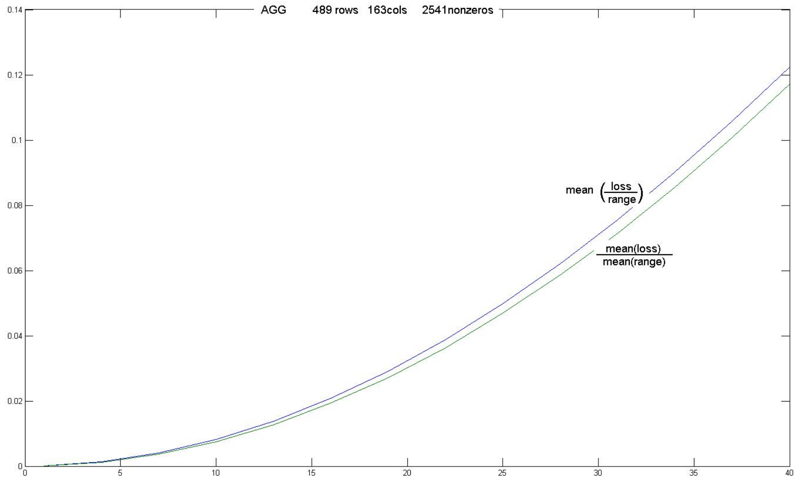
Theorem:

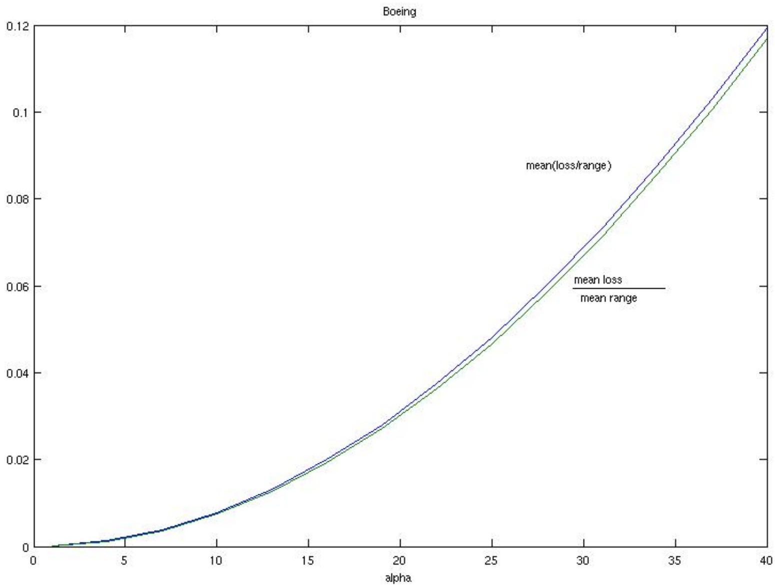
$$\frac{E_1[\mathsf{loss}(v, w(\alpha))]}{E_1[\mathsf{range}(w(\alpha))]} = \frac{1 - \cos\alpha}{2}.$$

Similar arguments show that the same result holds with E_1 replaced by E_2 or E_3.

Note that all results refer to the ratio of expectations, rather than the expectation of the ratio, the scaled loss.

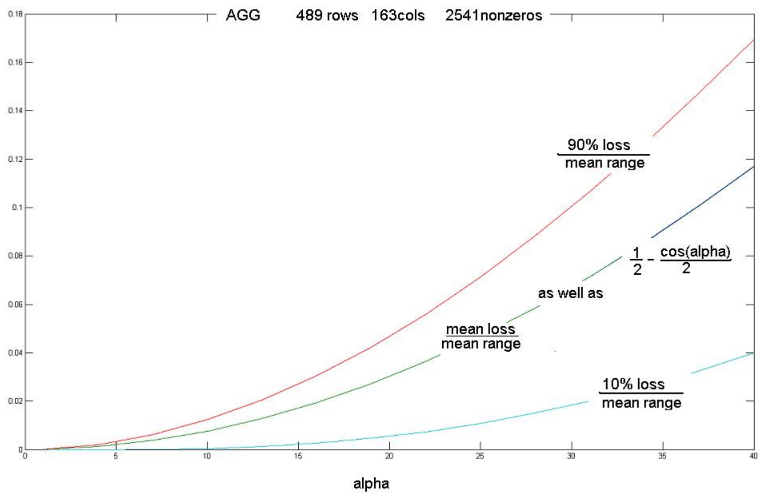
8. Comparison of Ratio of Expectations to Expectation of Ratio

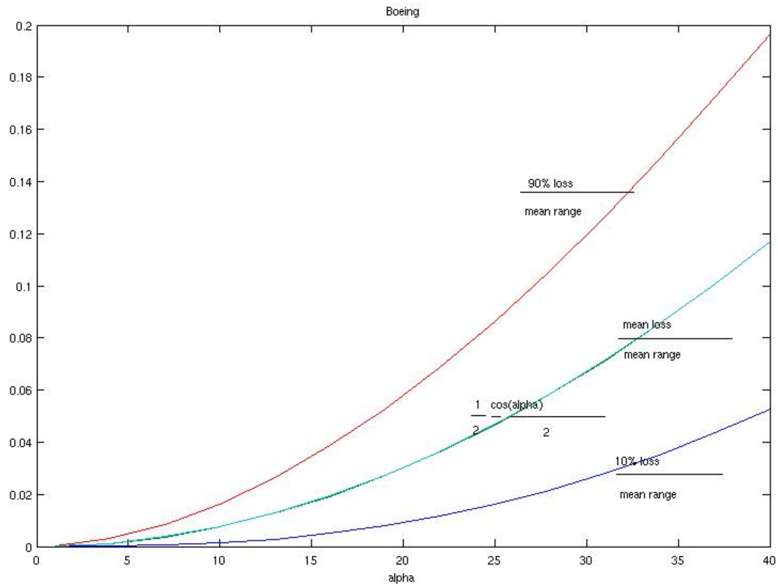




Boeing
0.12
0.1
0.08
0.06
0.04
0.02
0
0
5
10
15
20
25
30
35
40
alpha
mean(loss/range)
mean loss
mean range

9. Graphs of Percentiles





Boeing
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
5
10
15
20
25
30
35
40
alpha
90% loss
mean range
mean loss
mean range
1
2
cos(alpha)
2
10% loss
mean range

10. Conclusion

Under three probabilistic models, the loss in objective value from even a fairly large misspecification of a linear objective function is likely to be quite modest, for any nonempty compact feasible region.