Bad semidefinite programs: they all look the same

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A Semidefinite Program (SDP)

$$\begin{aligned} &\sup_x \; c^T x \\ &s.t. \; \sum_{i=1}^m x_i A_i \preceq B. \end{aligned} \qquad (SDP)$$

Here

- A_i, B are symmetric matrices, $c, x \in \mathbb{R}^m$.
- $A \preceq B$ means that $B - A$ is symmetric positive semidefinite (psd).
- An $n \times n$ matrix Y is positive semidefinite, if all principal subdeterminants are nonnegative.
- Equivalently, if $v^T Y v \geq 0 \; \forall v \in \mathbb{R}^n$.

SDP in a different shape

$$\begin{aligned} \inf_Y \;\; & B \bullet Y \\ s.t. \;\; & Y \succeq 0 \\ & A_i \bullet Y = c_i \, (i = 1, \ldots, m). \end{aligned}$$

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- A_i, B are symmetric matrices, $c \in \mathbb{R}^m$.
- $A \bullet B = \sum_{i,j} a_{ij} b_{ij}$

Conic duality and SDP duality

• **Common framework for LP and SDP:** both $\mathbb{R}^n_+$ and **psd matrices are closed convex cones.**

• **A set C is a cone, if $x \in C, \lambda \geq 0 \Rightarrow \lambda x \in C$.**

• **Linear objective, affine, and conic constraint both in LP and SDP, and many other interesting problems.**

Conic duality and SDP duality

Early duality theory for conic and semi-infinite problems:

- Duffin '56
- Bellman-Fan '63
- Ben-Israel '69-'70
- Ben-Israel-Charnes-Kortanek '69-'70
- Berman '70-'73
- Duffin-Jeroslow-Karlovitz '83
- . . .

Later duality theory:

- Shapiro '85, '97
- Borwein-Wolkowicz '81-'86
- Bot-Wanka '06
- Jeyakumar, Dinh, Lee '04
- . . .

Surveys, textbooks

Surveys and textbooks (on SDP in general, and on duality theory):

- Shapiro '00
- Wolkowicz-Vandenberghe-Saigal, '00;
- Bonnans-Shapiro '00
- Renegar '01;
- Vandenberghe-Boyd '96, '04;
- Todd '01;
- Luo-Sturm-Zhang '97;
- Borwein-Lewis '00;
- Ben-Tal-Nemirovskii '01;
- Burer (talk) '07
- Güler '10
- ...

An important, related question: when is the linear image of a closed convex cone closed?

- **Classic: Theorem 9.1 in Rockafellar;**
- **Waksman-Epelman, 1976;**
- **Auslender, 1996;**
- **Bauschke-Borwein, 1999;**
- **Pataki, 2007;**
- **Borwein-Moors, 2009-11;**

SDP duality

The primal-dual pair of SDPs:

$$\sup_x \; c^T x \qquad\qquad \inf_Y \; B \bullet Y$$

$$s.t. \; \sum_{i=1}^m x_i A_i \preceq B \qquad\qquad Y \succeq 0$$

$$A_i \bullet Y = c_i \; (i = 1, \ldots, m).$$

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But: in SDP, unlike in LP pathological phenomena occur: nonattainment, positive gaps.

What are the pathologies?

Pathology # 1: nonattainment in dual

Primal:

$$\begin{array}{ll} \sup & 2x_1 \\ s.t. & x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \sup & 2x_1 \\ s.t. & \begin{pmatrix} 1 & -x_1 \\ -x_1 & 0 \end{pmatrix} \succeq 0 \end{array}$$

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Dual: Dual variable is $Y \succeq 0.$

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Here $\inf = 0$, but not attained: Any $y_{11} > 0$, $y_{22} = 1/y_{11}$ is feasible, but $y_{11} = 0$ is not.

Pathology # 2: positive duality gap

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$$\begin{array}{ll} \sup & x_2 \\ s.t. & x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

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Only feasible x_2 is $x_2 = 0$.

Dual value is 1, and it is attained.

What to do with the pathologies? Our goal

Let us find a characterization of bad SDPs, which is

- **exact**
- **efficiently verifiable**
- **aesthetic**

Terminology

Definition:

- The system $P = \{ x \mid \sum_{i=1}^m x_i A_i \preceq B \}$ is well-behaved, if for all c such that

  $$\sup\{ c^T x \mid x \in P \} \text{ is finite,}$$

  the dual program has the same value, and it attains.
- Badly behaved, otherwise.

- We would like to understand badly behaved semidefinite systems.

Motivation

The systems

$$x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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Curious similarity:

- “Hanging off” diagonals;

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are both badly behaved.

Curious similarity:

- "Hanging off" diagonals;
- if we delete 2nd row and 2nd column in all matrices in the second system, and delete the first matrix,
- we get back the first system!

Motivation

- **In fact, all badly behaved systems appearing in the literature look similar.**

Question:

- **Do all bad SDPs "look the same"?**
- **Is the first. minimal system "contained" in all of them?**

The answer is yes to both.

Technicalities

Definition: A slack matrix in P is a matrix

$$Z := B - \sum_{i=1}^{m} x_i A_i \succeq 0.$$

Fact: There is a slack matrix with maximum rank. E.g. the maximum rank slack in

$$x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ is } \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Assumption: We can replace all A_i by $T^T A_i T$ and B by $T^T B T$, where T is invertible.

Main Theorem

Assume w.l.o.g. in P the max rank slack is

$$Z = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Then P is badly behaved $\Leftrightarrow$ $\exists V$ which is a linear combination of the A_i and B of the form

$$V = \begin{pmatrix} \overbrace{V_{11}}^{r} & e_1 & \dots \\ e_1^T & 0 & \dots \\ \vdots & & \ddots \end{pmatrix},$$

where e_1 is first unit vector, and the dots arbitrary.

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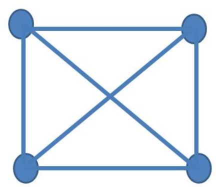
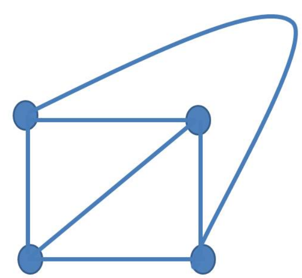
More motivation: excluded minors

Unrelated question: Given undirected graph, is it planar, i.e. can we draw the edges on the plane, so they only meet at nodes? E.g. graph below is planar,

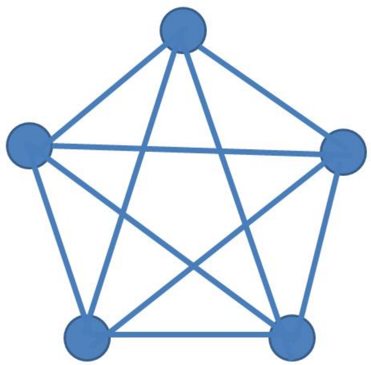


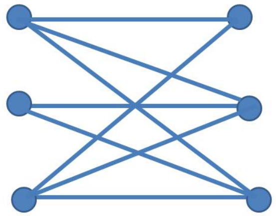
since it can be redrawn as



More motivation: excluded minors

Theorem (Kuratowski): **A graph is not planar, iff by deleting and contracting edges it can be reduced to one of the two graphs below:**





Corollary to Main Theorem

Consider the elementary operations performed on P:

- Rotation: $A_i \leftarrow T^T A_i T$ for all i and $B \leftarrow T^T BT$, where T is invertible.
- Contraction: $A_i \leftarrow \sum_{j=1}^{m} \lambda_j A_j$, for some i, where $\lambda_i \neq 0$, and $B \leftarrow B + \sum_{j=1}^{m} \mu_j A_j$.
- Deletion:
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P badly behaved $\Rightarrow$ using these we can get

$$x_1 \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

where α is some real number.

Complexity implications

We use the real number model of computing: (see e.g. Blum, Cucker, Shub, Smale '98), in which one can store, and do operations on real numbers in unit time.

Reason: SDP can have irrational solutions, or solutions with exponentially many digits.

Corollary: In this model, the question "is a semidefinite system well-behaved?" is in NP $\cap$ co-NP.

I.e., we can verify in polynomial time that a system is well-behaved, or badly behaved.

Conic LPs

- A conic linear system is

$$P = \{ x \mid Ax \leq_K b \} = \{ x \mid b - Ax \in K \},$$

where K is a closed, convex cone.

- Dual problem of $\sup \{ c^T x : x \in P \}$ involves K^*, the dual cone of K.

- Well-behaved, badly-behaved notions are defined analogously.

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Known:

K polyhedral $\Rightarrow$ P is well-behaved.

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P strictly feasible, i.e. $\exists x : b - Ax \in \text{ri } K$ $\Rightarrow$ P is well-behaved.

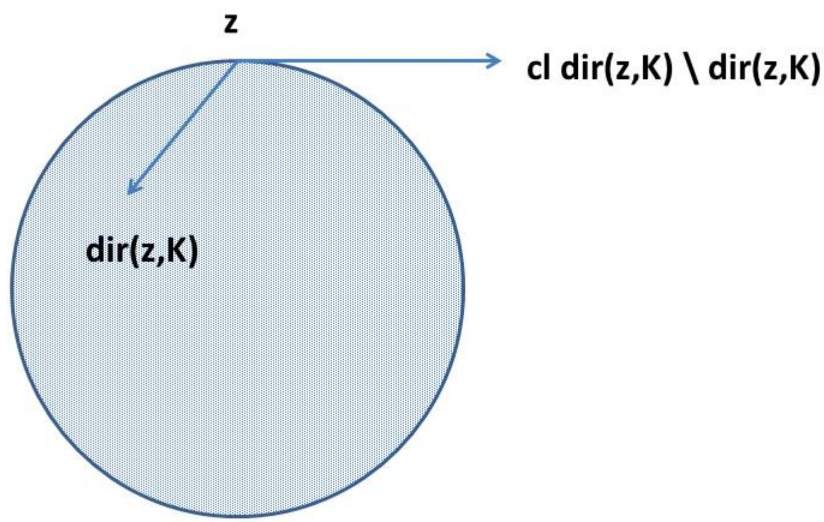
These are sufficient, but not necessary, and they have nothing to do with each other.

Feasible directions

K closed convex cone, $z \in K$. The set of feasible directions at z in K is

$$\mathrm{dir}(z, K) = \{\, y \mid \exists \epsilon > 0 \text{ s.t. } z + \epsilon y \in K \,\}$$

The set $\mathrm{dir}(z, K)$ is convex, but may not be closed.



A geometric result on conic LPs

Let

- z be a maximum slack in $P = \{\, x \mid Ax \leq_K b \,\}$, and
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If $K^* + \mathrm{face}(z, K)^{\perp}$ is closed, then $\Leftrightarrow$.

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Corollaries

K polyhedral $\Rightarrow$ P well-behaved.

P strictly feasible, i.e. $z \in \text{ri} K$ $\Rightarrow$ P well-behaved.

I.e. we unify the two unrelated conditions.

A geometric result, if K is nice

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Then P well-behaved $\Leftrightarrow$ $(*)$

Hence, P badly behaved $\Leftrightarrow$
z and some $v \in R(A, b) \cap (\text{cl dir}(z, K) \setminus \text{dir}(z, K))$
are certificates of the bad behavior.

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Polyhedral, semidefinite, and second-order cones are nice.

In the "Bad SDP" theorem:

$$Z = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \; V = \begin{pmatrix} \overbrace{V_{11}}^{r} & e_1 & \dots \\ e_1^T & 0 & \dots \\ \vdots & & \ddots \end{pmatrix} \in \text{cl dir}(Z, \mathcal{S}_+^n) \setminus \text{dir}(Z, \mathcal{S}_+^n).$$

Background

- **When $b = 0$, we have that P is well-behaved $\Leftrightarrow$ A^*K^* closed.**
- **$\rightarrow$ we get back characterization of closedness of A^*K^* in P 2007.**

Second order conic systems

- A second order conic system is $P = \{x \mid Ax \leq_K b\}$ with $K = K_1 \times \cdots \times K_t$, where

$$K_i = \{ x \in \mathbb{R}^{m_i} \mid x_1 \geq \sqrt{x_2^2 + \ldots x_{m_i}^2} \}.$$

A badly behaved system:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} x_1 \leq_{K_1} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Second order conic systems

Theorem Suppose w.l.o.g. the maximum slack in P is of the form

$$z = \left(\underbrace{0, \ldots, 0}_{O}; \underbrace{\begin{pmatrix} 1 \\ e_1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ e_1 \end{pmatrix}}_{R}; \underbrace{e_1, \ldots, e_1}_{I} \right).$$

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Then P is badly behaved $\Leftrightarrow \exists v \in R(A, b)$ s.t.

- $v_i \in K_i \, \forall \, i \in O$,
- $v_{i,1} \geq v_{i,2} \; \forall \; i \in R$,
- $v_j = (\alpha, \alpha, 1, \ldots)^T \; \exists \; j \in R$.

where $\alpha \in \mathbb{R}$, and the dots are arbitrary components.

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In the example: $O = I = \emptyset$, $\underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{v} x_1 \leq_{K_1} \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{z}.$

Well-behaved semidefinite systems

The system

$$x_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is well-behaved (though not strictly feasible).

Can we characterize well-behaved systems?

Theorem on good SDPs

S.t. (w.l.o.g.) in $P = \{ x \mid \sum_{i=1}^m x_i A_i \preceq B \}$ **the max rank slack is**

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$$Z = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Then P is well behaved $\Leftrightarrow$ (1) and (2) below hold.

(1) $\exists U = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix}$ s.t. $B \bullet U = A_i \bullet U = 0 \; \forall i.$

(2) $\forall V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^T & 0 \end{pmatrix} \in \text{lin}\{A_1, \dots, A_m, B\}$ we have $V_{12} = 0$.

- These are easy to verify.

Conclusion

• Duality in SDP: similar to LP, and similarly important: a dual solution gives a certificate of optimality.

• However: pathologies occur: nonattainment, duality gaps, etc.

• Main result: all pathologies have a very simple underlying structure, i.e. "all bad SDPs look the same" (Hanging off "1"s structure).

• An "excluded minor" type theorem for SDPs.

• A general, geometric result for conic LPs (cl dir $\setminus$ dir . . .)

• Characterization of good SDPs, bad SOCPs, . . .

Thank you!