Iterative Valid Polynomial Inequality Generation in Polynomial Optimization

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Polynomial Optimization

Polynomial optimization problems (POPs) consist of optimizing a multivariate polynomial objective subject to multivariate polynomial constraints:

Polynomial Optimization Problem (POP)

$$
\begin{aligned}
z = \quad & \sup \quad f(x) \\
& \text{s.t.} \quad g_i(x) \geq 0 \qquad i = 1, \ldots, m.
\end{aligned}
$$

Numerous classes of problems can be modelled as POPs, including:

- Linear Problems
- Mixed-Binary Problems

$$x_i \in \{0, 1\} \quad \Leftrightarrow \quad x_i(1 - x_i) = 0$$

- Quadratic Problems (Convex / Non-convex)

Thus, solving POPs is in general NP-hard.

Relaxations of POPs

Polynomial Optimization Problem (POP)

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- Many tractable relaxations of POPs have been proposed using linear, second-order cone, and semidefinite techniques.

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- Many tractable relaxations of POPs have been proposed using linear, second-order cone, and semidefinite techniques.
- In particular, sum-of-squares (SOS) decompositions which lead to semidefinite programming (SDP) relaxations
  - are theoretically very strong:
    - Sequences of relaxations converging to the optimal value in the limit
    - Exact (exponential-sized) relaxations for pure binary POPs

Relaxations of POPs

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- Many tractable relaxations of POPs have been proposed using linear, second-order cone, and semidefinite techniques.
- In particular, sum-of-squares (SOS) decompositions which lead to semidefinite programming (SDP) relaxations
  - are theoretically very strong:
    - Sequences of relaxations converging to the optimal value in the limit
    - Exact (exponential-sized) relaxations for pure binary POPs
  - but quickly become too expensive for practical computation.

Research objective:

Improve the SDP relaxations

• without incurring an exponential growth in their size

• by iteratively generating valid polynomial inequalities.

General POP Perspective

Given a general POP problem:

$$\text{(POP)} \quad z = \sup \; f(x) \quad \text{s.t.} \; g_i(x) \geq 0 \qquad i = 1, \ldots, m.$$

If λ is the optimal value of POP, then POP is equivalent to

$$\begin{array}{ll} \inf & \lambda \\ \text{s.t.} & \lambda - f(x) \geq 0 \quad \forall x \in S := \{x : g_i(x) \geq 0, \; i = 1, \ldots, m\} \end{array}$$

which we rewrite as

$$\begin{array}{ll} \inf & \lambda \\ \text{s.t.} & \lambda - f(x) \in \mathcal{P}_d(S) \end{array}$$

where

$$\mathcal{P}_d(S) = \{p(x) \in \mathbf{R}_d[x] : p(s) \geq 0 \text{ for all } s \in S\}$$

is the cone of polynomials of degree at most d that are non-negative over S.

Understanding $\mathcal{P}_d(S)$

The set

$$\mathcal{P}_d(S) = \{p(x) \in \mathbb{R}_d[x] : p(x) \geq 0 \text{ for all } x \in S\}$$

is in general a very complex object.

- It is always a convex cone
- In most cases the decision problem for $\mathcal{P}_d(S)$ is NP-hard:

Decision problem for $\mathcal{P}_d(S)$

Given $p(x)$, decide if $p(x) \in \mathcal{P}_d(S)$
(i.e. if $p(x) \geq 0$ for all $x \in S$)

- Idea: use algebraic geometry results to approximate (or represent) $\mathcal{P}_d(S)$ in tractable ways, i.e., using only linear, second-order, and semidefinite cones.

A General Recipe for Relaxations of POP

We relax $\lambda - f(x) \in \mathcal{P}_d(S)$ to

$$\lambda - f(x) \in \mathcal{K} \text{ for a suitable } \mathcal{K} \subseteq \mathcal{P}_d(S).$$

Then

$$\begin{array}{ll} \inf & \lambda \\ \text{s.t.} & \lambda - f(x) \in \mathcal{K} \end{array}$$

provides an upper bound for the original problem.

• The choice of $\mathcal{K}$ is a key factor in obtaining good bounds on the problem.

• We are restricted by the need for the optimization over $\mathcal{K}$ to be tractable.

SOS Approach - Lasserre (2001), Parrilo (2000)

For each $r > 0$, define the approximation $\mathcal{K}_r \subseteq \mathcal{P}_d(S)$ as

$$\mathcal{K}_r := \left(\Psi_r + \sum_{i=1}^{m} g_i(x) \Psi_{r-\deg(g_i)} \right) \cap \mathbf{R}_d[x]$$

where Ψ_d denotes the cone of real polynomials of degree at most d that are SOSs of polynomials, and $\mathbf{R}_d[x]$ denotes the set of polynomials in the variables x of degree at most d.

The corresponding relaxation can be written as

$$\begin{array}{llll} (\mathrm{L}_r) & z_r = \inf\limits_{\lambda,\sigma_i} & \lambda & \\ & \text{s.t.} & \lambda - f(x) = \sigma_0(x) + \sum_{i=1}^{m} \sigma_i(x) g_i(x) & \\ & & \sigma_0(x) \text{ is SOS of degree} \leq r & \\ & & \sigma_i(x) \text{ is SOS of degree} \leq r - \deg(g_i(x)), i = 1, \ldots, m. & \end{array}$$

Solving the SOS Relaxation

For each r, the relaxation (L_r) can be cast as an SDP problem, since $\sigma(x)$ is a SOS of degree $2k$ if and only if

$$\sigma(x) = \begin{pmatrix} 1 \\ \vdots \\ x_i \\ \vdots \\ x_i x_j \\ \vdots \\ \prod_{|k|} x \end{pmatrix}^T M \begin{pmatrix} 1 \\ \vdots \\ x_i \\ \vdots \\ x_i x_j \\ \vdots \\ \prod_{|k|} x \end{pmatrix} \quad \text{with} \quad M \succeq 0.$$

Note that $\Psi_d = \Psi_{d-1}$ for every odd degree d.

Convergence of the SOS Approach

Under mild conditions $z_r \to z$:

Lemma

Suppose that

$$\mathcal{K}_G^d \subseteq \mathcal{K}_G^{d+1} \subseteq \cdots \subseteq \mathcal{K}_G^r \subseteq \mathcal{P}_d(S),$$

where G is a compact semialgebraic set (not necessarily convex) and there exists a real-valued polynomial $u(x)$ with $u(x) \in \sum_{i=0}^m g_i(x)\Psi$ such that $\{u(x) \geq 0\}$ is compact. Then

$$\mathcal{K}_G^r \uparrow \mathcal{P}_d(S) \text{ as } r \to \infty,$$

and therefore

$$z_r \uparrow z \text{ as } r \to \infty.$$

Size of the SOS Relaxation

Good news: (L_r) can be solved using SDP techniques, and under mild conditions, $z_r \to z$.

Bad news: For a problem with n variables and m inequality constraints, the size of the relaxation is:

- One psd matrix of dimension $\binom{n+r}{r}$;
- m psd matrices, each of dimension $\binom{n+r-\deg(g_i)}{r-\deg(g_i)}$
- $\binom{n+r}{r}$ linear constraints.

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One way around this difficulty is to exploit any available structure (sparsity, symmetry) to solve smaller SDP problems.
Much progress has been made in this direction.

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Much progress has been made in this direction.

Our objective

Avoid the blow-up by keeping r constant (and small).

A Small Example

$$
\begin{aligned}
\inf_{x,y} \quad & (x-1)^2 + (y-1)^2 \\
\text{s.t.} \quad & x^2 - 4xy - 1 \geq 0 \\
& yx - 3 \geq 0 \\
& y^2 - 4 \geq 0 \\
12^2 - (x-2)^2 - 4(y-1)^2 & \geq 0
\end{aligned}
$$

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& 12^2 - (x-2)^2 - 4(y-1)^2 \geq 0
\end{aligned}
$$

L_2 relaxation

$$
\begin{aligned}
\sup_{\lambda, \sigma_i(\cdot)} \quad & \lambda \\
\text{s.t.} \quad & (x-1)^2 + (y-1)^2 - \lambda = \sigma_0(x,y) + \textstyle\sum_{i=1}^{4} \sigma_i(x,y) g_i(x,y) \\
& \\
& \sigma_0(x,y) \text{ is SOS of degree 2} \\
& \sigma_i(x,y) \text{ is SOS of degree 0}
\end{aligned}
$$

A Small Example

L_2 relaxation

$$\begin{array}{ll} \sup_{\lambda,\sigma_i(\cdot)} & \lambda \\ \text{s.t.} & (x-1)^2 + (y-1)^2 - \lambda = \sigma_0(x,y) + \sum_{i=1}^4 \sigma_i(x,y) g_i(x,y) \\ & \quad (6 \times 14 \text{ lin. system }) \\ & \sigma_0(x,y) \text{ is SOS of degree 2 } (3 \times 3 \text{ matrix}) \\ & \sigma_i(x,y) \text{ is SOS of degree 0 } (4 \text{ non-negative constants}) \end{array}$$

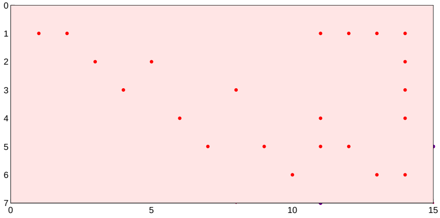

Figure: Structure of the linear system for L_2

A Small Example

L_2 relaxation (Optimal value: 9.4083)

$$\sup_{\lambda,\sigma_i(\cdot)} \quad \lambda$$

s.t. $(x-1)^2 + (y-1)^2 - \lambda = \sigma_0(x,y) + \sum_{i=1}^{4} \sigma_i(x,y) g_i(x,y)$

(6 × 14 lin. system)

$\sigma_0(x,y)$ is SOS of degree 2 (3 × 3 matrix)

$\sigma_i(x,y)$ is SOS of degree 0 (4 non-negative constants)

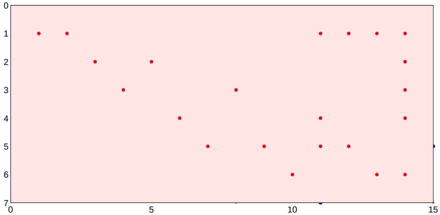

Figure: Structure of the linear system for L_2

Example (ctd)

L_4 relaxation (Optimal value: 36.0654)

$$\sup_{\lambda,\sigma_i(\cdot)} \quad \lambda$$

$$\text{s.t.} \quad (x-1)^2 + (y-1)^2 - \lambda = \sigma_0(x,y) + \sum_{i=1}^{4} \sigma_i(x,y) g_i(x,y)$$

(15×73 lin. system)

$\sigma_0(x,y)$ is SOS of degree 4 (6×6 matrix)

$\sigma_i(x,y)$ is SOS of degree 2 (3×3 SDP matrices)

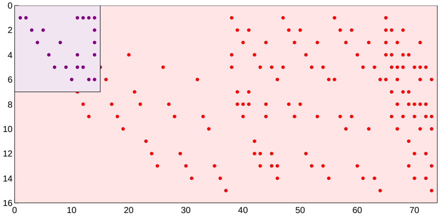

Figure: Structure of the linear system for L_4

Example (ctd)

L_6 relaxation (Optimal value: 51.7386)

$$\begin{array}{rl} \sup_{\lambda,\sigma_i(\cdot)} & \lambda \\ \text{s.t.} & (x-1)^2+(y-1)^2-\lambda = \sigma_0(x,y)+\sum_{i=1}^4 \sigma_i(x,y)g_i(x,y) \end{array}$$

(28×245 lin. system)

$\sigma_0(x,y)$ is SOS of degree 6 (10×10 matrix)

$\sigma_i(x,y)$ is SOS of degree 4 (6×6 SDP matrices)

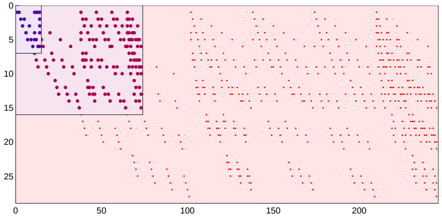

Figure: Structure of the linear system for L_6

Lasserre's Hierarchy for our Example

To solve

$$\begin{array}{ll} \inf_{x,y} & (x-1)^2+(y-1)^2 \\ \text{s.t.} & x^2-4xy-1 \geq 0 \\ & yx-3 \geq 0 \\ & y^2-4 \geq 0 \\ & 12^2-(x-2)^2-4(y-1)^2 \geq 0 \end{array}$$

r	2	4	6
# vars	14	73	245
# constraints	6	15	28
Bound	9.40	36.06	51.73

Lasserre's Hierarchy for our Example

To solve

$$\begin{array}{ll} \inf_{x,y} & (x-1)^2 + (y-1)^2 \\ \text{s.t.} & x^2 - 4xy - 1 \geq 0 \\ & yx - 3 \geq 0 \\ & y^2 - 4 \geq 0 \\ \multicolumn{2}{c}{12^2 - (x-2)^2 - 4(y-1)^2 \geq 0} \end{array}$$

r	2	4	6
# vars	14	73	245
# constraints	6	15	28
Bound	9.40	36.06	51.73

There is no need to run relaxations for $r > 6$, because an optimal solution (and optimality certificate) can be extracted from solution to L_6.

Improving the approximation without growing r

Recall

$$\text{(POP)} \quad z = \sup \; f(x) \quad \text{s.t.} \; x \in S := \{x : g_i(x) \geq 0,\ i = 1, \ldots, m\}$$

$$(\mathrm{L}_r(G)) \quad z_r(G) = \inf_{\lambda, \sigma_i} \; \lambda$$
$$\text{s.t.} \quad \lambda - f(x) = \sigma_0(x) + \sum_{i=1}^m \sigma_i(x) g_i(x)$$
$$\sigma_0(x) \text{ is SOS of degree} \leq r$$
$$\sigma_i(x) \text{ is SOS of degree} \leq r - \deg(g_i(x)),$$
$$i = 1, \ldots, m.$$

Observe that

- (L_r) is defined in terms of the functions used to describe S
- Call this set $G = \{g_i(x) : i = 1, \ldots, m\}$

Goal

Improve our description of S by growing G in such a way that the bound obtained from L_r improves, for fixed r.

Back to our Example

We start with

$$G = \{x^2 - 4xy - 1, yx - 3, y^2 - 4, 12^2 - (x-2)^2 - 4(y-1)^2\}$$

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$$G = \{x^2 - 4xy - 1, yx - 3, y^2 - 4, 12^2 - (x-2)^2 - 4(y-1)^2\}$$

- For all $(x, y) \in S$,

$$p_1(x, y) = 0.079x^2 + 0.072xy + 0.325x - 0.850y^2 - 0.339y - 0.213 \geq 0$$

- We say that $p_1(x, y)$ is a valid (polynomial) inequality for S.

Back to our Example

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- We say that $p_1(x, y)$ is a valid (polynomial) inequality for S.

Let $G_1 = G \cup \{p_1(x, y)\}$

Then

$$z_2(G_1) = 22.8393 > 9.4083 = z_2(G)$$

Why Stop at p_1?

Add valid inequalities iteratively

- Start with $G_0 = G$.
- Given G_i, generate p_i valid (inequality) for S. Let $G_{i+1} = G_i \cup \{p_i\}$.

Why Stop at p_1?

Add valid inequalities iteratively

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- Given G_i, generate p_i valid (inequality) for S. Let $G_{i+1} = G_i \cup \{p_i\}$.

$$p_1(x, y) = 0.079x^2 + 0.072xy + 0.325x - 0.850y^2 - 0.339y - 0.213$$

i	0	1
	9.4083	22.8393

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- Start with $G_0 = G$.
- Given G_i, generate p_i valid (inequality) for S. Let $G_{i+1} = G_i \cup \{p_i\}$.

$$p_2(x, y) = 0.053x^2 + 0.082xy + 0.205x - 0.764y^2 - 0.533y - 0.282$$

i	0	1	2
	9.4083	22.8393	30.1062

Why Stop at p_1?

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- Start with $G_0 = G$.
- Given G_i, generate p_i valid (inequality) for S. Let $G_{i+1} = G_i \cup \{p_i\}$.

$$p_3(x, y) = 0.069x^2 + 0.002xy - 0.239x - 0.770y^2 + 0.551y - 0.200$$

i	0	1	2	3
	9.4083	22.8393	30.1062	32.2653

Why Stop at p_1?

Add valid inequalities iteratively

- Start with $G_0 = G$.
- Given G_i, generate p_i valid (inequality) for S. Let $G_{i+1} = G_i \cup \{p_i\}$.

$$p_4(x,y) = -0.019x^2 + 0.338xy + 0.097x - 0.691y^2 - 0.577y - 0.254$$

i	0	1	2	3	4
	9.4083	22.8393	30.1062	32.2653	40.1754

Why Stop at p_1?

Add valid inequalities iteratively

- Start with $G_0 = G$.
- Given G_i, generate p_i valid (inequality) for S. Let $G_{i+1} = G_i \cup \{p_i\}$.

$$p_5(x, y) = 0.070x^2 + 0.071xy - 0.158x - 0.858y^2 - 0.425y - 0.214$$

i	0	1	2	3	4
	9.4083	22.8393	30.1062	32.2653	40.1754

i	5
	43.1587

Why Stop at p_1?

Add valid inequalities iteratively

- Start with $G_0 = G$.
- Given G_i, generate p_i valid (inequality) for S. Let $G_{i+1} = G_i \cup \{p_i\}$.

$$p_6(x, y) = 0.052x^2 + 0.047xy + 0.012x - 0.935y^2 - 0.130y - 0.321$$

i	0	1	2	3	4
	9.4083	22.8393	30.1062	32.2653	40.1754

i	5	6
	43.1587	49.3414

Why Stop at p_1?

Add valid inequalities iteratively

- Start with $G_0 = G$.
- Given G_i, generate p_i valid (inequality) for S. Let $G_{i+1} = G_i \cup \{p_i\}$.

$$p_7(x, y) = 0.046x^2 + 0.006xy - 0.182x - 0.707y^2 + 0.652y - 0.195$$

i	0	1	2	3	4
	9.4083	22.8393	30.1062	32.2653	40.1754

i	5	6	7
	43.1587	49.3414	51.5485

Why Stop at p_1?

Add valid inequalities iteratively

- Start with $G_0 = G$.
- Given G_i, generate p_i valid (inequality) for S. Let $G_{i+1} = G_i \cup \{p_i\}$.

$$p_8(x, y) = 0.023x^2 + 0.093xy + 0.116x - 0.566y^2 - 0.621y - 0.519$$

i	0	1	2	3	4
	9.4083	22.8393	30.1062	32.2653	40.1754

i	5	6	7	8
	43.1587	49.3414	51.5485	51.7135

Why Stop at p_1?

Add valid inequalities iteratively

- Start with $G_0 = G$.
- Given G_i, generate p_i valid (inequality) for S. Let $G_{i+1} = G_i \cup \{p_i\}$.

$$p_9(x, y) = 0.049x^2 + 0.002xy + 0.117x - 0.647y^2 - 0.592y - 0.463$$

i	0	1	2	3	4
	9.4083	22.8393	30.1062	32.2653	40.1754

i	5	6	7	8	9
	43.1587	49.3414	51.5485	51.7135	51.7382

Generating Valid Inequalities for POPs

Recall

$$\begin{aligned} z = \inf \quad & \lambda \\ \text{s.t.} \quad & \lambda - f(x) \in \mathcal{P}_d(S) \\ z_r(G) = \inf \quad & \lambda \\ \text{s.t.} \quad & \lambda - f(x) \in \mathcal{K}_r(G) \end{aligned}$$

Lemma

Let G be a description for S, and let p(x) be a valid inequality for S. Then

$$z_r(G \cup \{p(x)\}) \geq z_r(G)$$

Generating Valid Inequalities for POPs

Recall

$$\begin{aligned} z = \inf \quad & \lambda \\ \text{s.t.} \quad & \lambda - f(x) \in \mathcal{P}_d(S) \end{aligned}$$

$$\begin{aligned} z_r(G) = \inf \quad & \lambda \\ \text{s.t.} \quad & \lambda - f(x) \in \mathcal{K}_r(G) \end{aligned}$$

Lemma

Let G be a description for S, and let $p(x)$ be a valid inequality for S. Then

$$z_r(G \cup \{p(x)\}) \geq z_r(G)$$

How to generate a valid improving inequality?

Given a description G of S, find $p(x)$ valid for S such that

$$z_r(G \cup \{p(x)\}) > z_r(G)$$

Valid Inequality Generation for POPs

Goal

Given a description G of S, find $p(x) \in \mathcal{P}_d(S) \setminus \mathcal{K}_r(G)$

Issues to address:

1. Generate $p(x) \in \mathcal{P}_d(S)$.
2. Ensure that $p(x) \subsetneq \mathcal{K}_r(G)$

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Issue 1: Generate $p(x) \in \mathcal{P}_d(S)$

There is no tractable representation for $\mathcal{P}_d(S)$

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Issue 1: Generate $p(x) \in \mathcal{P}_d(S)$

There is no tractable representation for $\mathcal{P}_d(S)$

- Sol: Generate $p(x) \in \mathcal{K}_{r+2}(G) \cap \mathbf{R}_r[x] \subset \mathcal{P}_d(S)$.

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Issue 2: Ensure $p(x) \notin \mathcal{K}_r(G)$

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Issue 2: Ensure $p(x) \notin \mathcal{K}_r(G)$

- Let Y be the dual optimal solution of $\mathrm{L}_r(G)$
- Then $Y \in \mathcal{K}_r(G)^*$
- and therefore $p \in \mathcal{K}_r(G) \Rightarrow \langle p, Y \rangle \geq 0$.

$\Rightarrow$ Look for $p(x)$ such that $\langle p, Y \rangle < 0$.

Inequality Generating Subproblem

Given G and Y

$$\begin{array}{ll} \min & \langle p, Y \rangle \\ \text{s.t.} & p(x) \in \mathcal{K}_{r+2}(G) \\ & \|p\| = 1 \end{array}$$

Inequality Generating Subproblem

Given G and Y

$$\begin{array}{ll} \min & \langle p, Y \rangle \\ \text{s.t.} & p(x) \in \mathcal{K}_{r+2}(G) \\ & \|p\| = 1 \end{array}$$

The normalization is necessary, otherwise the problem is unbounded

- For any $c > 0$, $p(x) \geq 0 \Leftrightarrow cp(x) \geq 0$.

Another Small Example

Optimal value = 0

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Iter.	0	1	2	3	4	5	10	50
Objective val.	unb.	-0.109	-0.073	-0.069	-0.068	-0.066	-0.057	-0.014
Time (s)								
Subproblem	-	1.5	1.8	2.1	1.9	2.0	2.5	
Master problem	0.2	0.3	0.3	0.4	0.5	0.5	0.6	
Cumulative	0.2	2.0	4.2	6.7	9.2	11.8	26.1	200.1

Table: Inequality Generation

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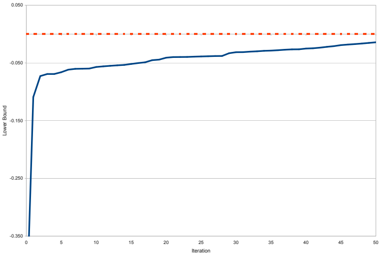

The Motzkin Polynomial

Optimal value = 0

$$\min_{x,y,z\in\mathbb{R}} \quad x^2y^2(x^2+y^2-3z^2)+z^6$$

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Iter.	0	1	2	3	4	5	10
Objective val.	unb.	-8591.8	-5687.1	-663.8	-643.8	-640.7	-613.5
Time (s)							
Subproblem	-	0.4	0.3	0.3	0.4	0.4	0.5
Master problem	0.3	0.3	0.3	0.3	0.3	0.3	0.3
Cumulative	0.3	1.0	1.6	2.2	2.9	3.6	7.5

Table: Inequality Generation

Special Case: Binary Quadratic POPs

Consider the general binary quadratic POP:

$$\begin{array}{lll} \max & f(x) & \\ \text{s.t.} & g_i(x) \geq 0 & \forall i \in I = \{1, \ldots, m\} \\ & x \in \{-1, 1\}^n. & \end{array}$$

where $f(x)$ and $g_i(x)$ are polynomials of degree at most 2.

We write the following equivalent formulation:

$$\begin{array}{ll} \min & \lambda \\ \text{s.t.} & \lambda - f(x) \in \mathcal{P}_2(S \cap \{-1, 1\}^n) \end{array}$$

where $S = \{x : g_i(x) \geq 0\}$.

Valid Inequality Generation for Binary Quadratic POPs

We make use of the following theorem:

Theorem (Peña-Vera-Zuluaga (2006))

Let S be a compact set. For any degree d,

$$p(x) \in \mathcal{P}_d(x \in S : x_j \in \{-1, 1\})$$

$$\Leftrightarrow$$

$$p(x) = (1 + x_j)r_+(x) + (1 - x_j)r_-(x) + (1 - x_j^2)c(x),$$

where $r_+(x), r_-(x) \in \mathcal{P}_d(S)$ and $c(x) \in \mathbf{R}_{d-1}[x]$.

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We can approximate $\mathcal{P}_2(S \cap \{-1, 1\}^n)$ by

$$\mathcal{Q}_2^j(G) = \{(1 + x_j)r_+(x) + (1 - x_j)r_-(x) + (1 - x_j^2)c(x) : r^+(x), r^-(x) \in \mathcal{K}_2(G), \quad c(x) \in \mathbf{R}_1[x]\}$$

and we have that

$$\mathcal{K}_2(G) \subset \mathcal{Q}_2^j(G) \subset \mathcal{P}_2(S \cap \{-1, 1\}^n)$$

Valid Inequality Generation for Binary Quadratic POPs

Goal

Given G describing S, find $p(x) \in \mathcal{P}_2(S \cap \{-1,1\}^n) \setminus \mathcal{K}_2(G)$

Given G and Y

$$\begin{aligned} \min \quad & \langle p, Y \rangle \\ \text{s.t.} \quad & p(x) \in \mathcal{Q}_2^j(G) \\ & \|p\| = 1 \end{aligned}$$

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$$\begin{array}{ll} \min & \langle p, Y \rangle \\ \text{s.t.} & p(x) \in \mathcal{Q}_2^j(G) \\ & \|p\| = 1 \end{array}$$

Note that there is exactly one subproblem per binary variable j.
Moreover,

- the size of $\mathcal{Q}_2^j(G)$ is only twice size of $\mathcal{K}_2(G)$
- while the size of $\mathcal{K}_4(G)$ is $\sim n^2$ times size of $\mathcal{K}_2(G)$

Convergence Result

Theorem

When the polynomial inequality generation scheme is applied to a binary quadratic optimization problem with linear constraints $Ax = b$, and the initial set is

$$G_0 = \left\{ n - \|x\|^2, \sum_i (A_i^T x - b_i)^2, -\sum_i (A_i^T x - b_i)^2 \right\},$$

then if all the subproblems have an optimal value 0, then the algorithm has converged to a global optimal solution.

Computational Results

Quadratic Knapsack Problem

$$\max x^T P x$$
$$\text{s.t. } w^T x \le c$$
$$x \in \{-1, 1\}^n$$

		Lasserre $r = 4$		Lasserre $r = 2$		Poly. Ineq. Gen.				
n	Optimal	Obj.	Time (s)	Obj.	Time (s)	Iter. 0	Iter. 1	Iter. 5	Iter. 10	Time (s)
10	1653	1707.3	28.1	1857.7	0.8	1857.7	1821.9	1797.4	1784.8	5.8
20	8510	8639.7	17269.1	9060.3	2.9	9060.3	9015.3	8925.9	8850.3	35.4
30	18229	-	-	19035.9	4.3	19035.9	18920.2	18791.7	18727.2	196.6
40	2679	-	-	4735.9	6.8	4735.9	4590.7	4248.2	4126.7	1009.7
50	16192	-	-	21777.9	19.2	21777.9	21390.3	20162.1	19407.1	7014.3
60	58451	-	-	62324.4	126.6	62324.4	62019.1	60906.0	60585.5	17961.1
70	16982	-	-	23884.9	231.4	23884.9	23484.0	22852.8	-	15582.2
80	-	-	-	80482.7	365.4	80482.7	79738.9	-	-	11072.3

(5-hour time limit)

Computational Results

Quadratic Assignment Problem

$$\begin{aligned} \min \ & \textstyle\sum_{i\neq k, j\neq l} f_{ik} d_{jl} x_{ij} x_{kl} & \\ \text{s.t. } & \textstyle\sum_i x_{ij} = 1 & 1 \leq j \leq n \\ & \textstyle\sum_j x_{ij} = 1 & 1 \leq i \leq n \\ & x \in \{0,1\}^{n\times n}. & \end{aligned}$$

		Lasserre $r=4$		Lasserre $r=2$		Poly. Ineq. Gen.				
n	Optimal	Obj.	Time (s)	Obj.	Time (s)	Iter. 0	Iter. 1	Iter. 5	Iter. 10	Time (s)
3	46			**46.0**	0.3	**46.0**				0.3
4	52	**52.0**	1154.8	50.8	1.0	50.8	51.8	**52.0**		6.3
5	110	-	-	104.3	3.4	104.3	105.1	106.3	106.8	68.5
6	272	-	-	268.9	9.3	268.9	269.4	269.8	270.2	404.4
7	356	-	-	344.2	18.1	344.2	344.9	345.6	346.0	3331.3
8	100	-	-	77.2	73.2	77.2	77.8	78.9	-	11413.9
9	280	-	-	247.5	281.7	247.5	248.6	-	-	13171.5

(5-hour time limit)

Computational Results

Degree Three Binary POPs

$$\max \sum_{|\alpha|\leq 3} c_\alpha x^\alpha$$
$$\text{s.t. } a^T x \leq b$$
$$x \in \{-1, 1\}^n$$

		Lasserre $r = 6$		Lasserre $r = 4$		Poly. Ineq. Gen.				
n	Optimal	Obj.	Time (s)	Obj.	Time (s)	Iter. 0	Iter. 1	Iter. 5	Iter. 10	Time (s)
5	58	**58.00**	9.6	59.37	2.1	67.16	58.45	**58.00**		5.2
10	139	**139.00**	4866.0	148.97	35.9	154.59	148.85	143.41	139.12	75.3
15	1371	-	-	1524.71	1436.2	1582.04	1575.49	1519.88	1494.01	1319.9
20	1654	-	-	1707.95	18106.6	1718.53	1716.00	1708.66	1705.15	15763.9
25	-	-	-	-	-	3967.12	3960.78	-	-	14287.3

(5-hour time limit)

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  - Find ways to reduce size of SDP subproblems
  - Avoid SDP altogether: second-order cone optimization?