On a smaller SDP relaxation for Polynomial Optimization Problems

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Joint work with Masakazu Muramatsu

Polynomial Optimization Problems

For real-valued polynomials $f_0, f_1, \ldots, f_m, g_1, \ldots, g_k$, Polynomial Optimization Problem (POP) is

$$\text{(POP)}\ f^* := \inf_{x \in \mathbb{R}^n} \left\{ f_0(x) \;\middle|\; \begin{array}{l} f_i(x) \geq 0 \ (i = 1, \ldots, m), \\ g_j(x) = 0 \ (j = 1, \ldots, k) \end{array} \right\}.$$

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SDP relaxation for POP ([Lasserre 2001] and [Parrilo 2003])

- Convert POP into a SemiDefinite Programming (SDP):

$$(\text{SDP})\ \theta := \inf_{y \in \mathbb{R}^N} \left\{ b^T y \;\middle|\; C - \sum_{j=1}^{m} A_j y_j \in \mathbb{S}^N_+ \right\}$$

- The optimal value θ of SDP is a lower bound of f^*, i.e., $f^* \geq \theta$

Aim of this talk – to overcome numerical difficulties

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- Propose a smaller SDP relaxation than Lasserre's SDP relaxation
- May not be able to tighter lower bounds than Lasserre's SDP relaxation
- Propose some techniques to improve the bounds

SDP relaxation for POPs (1)

Characterization of SOS; $\sigma = \sum_{i=1}^{k} g_i^2$

- σ is SOS with degree $2r$
- $\exists X$: p.s.d. such that $\sigma(x) = u_r(x)^T X u_r(x)$ for all $x \in \mathbb{R}^n$

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  - SDP problem is feasible $\longrightarrow$ σ is SOS
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Generalized Lagrange Function [Kim-Kojima-W (2005)]

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  $\mathbf{L(x,\sigma) := f(x) - \sum_{j=1}^{m} \sigma_j(x) f_j(x)}$, σ_j: SOS

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- Generalized Lagrange Relaxation:

$$\sup_{\sigma_j:\text{SOS}} \inf_{x \in \mathbb{R}^n} \mathbf{L(x, \sigma)} = \sup_{\sigma_j:\text{SOS}} \sup_{\rho \in \mathbb{R}} \left\{ \rho \,\middle|\, \mathbf{L(x, \sigma) - \rho \geq 0 \ (x \in \mathbb{R}^n)} \right\}$$

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- If $\exists\tilde{\sigma}_0$ and $\tilde{\sigma}_j$: SOS s.t. $L(x,\tilde{\sigma}) - \tilde{\rho} = \tilde{\sigma}_0(x)$ for all x,

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- Putinar's Lemma $\longrightarrow \sup_{\sigma_j:\text{SOS}} \inf_{x\in\mathbb{R}^n} L(x,\sigma) = f^*$

SDP relaxation for POPs (2)

Let $\mathbf{K} := \{x \in \mathbb{R}^n \mid f_j(x) \geq 0 \ (j = 1, \ldots, m)\}$.

Putinar's Lemma

Under a mild assumption, if $f(x) - \rho > 0$ on $\mathbf{K}$, then $\exists$ SOS $\sigma_0, \ldots, \sigma_m$ such that

$$f(x) - \rho - \sum_{j=1}^{m} \sigma_j(x) f_j(x) = \sigma_0(x) \ (\forall x \in \mathbb{R}^n)$$

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- Fix $\mathbf{r}$ in Generalized Lagrange Relaxation.

$$\left.\begin{array}{lll} \rho_\mathbf{r}^* := \sup_{\rho,\sigma_\mathbf{j}} & \rho & \\ \text{sub. to} & \mathbf{L}(\mathbf{x},\sigma) - \rho = \sigma_0(\mathbf{x}) \\ & \sigma_0, \sigma_\mathbf{j} : \text{SOS}, \\ & \deg(\sigma_0) \leq 2\mathbf{r}, \deg(\sigma_\mathbf{j}\mathbf{f_j}) \leq 2\mathbf{r}. \end{array}\right\} = \text{SDP}$$

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- $\lim_{r\to\infty} \rho_r^* = f^*$ [Lasserre (2001)]

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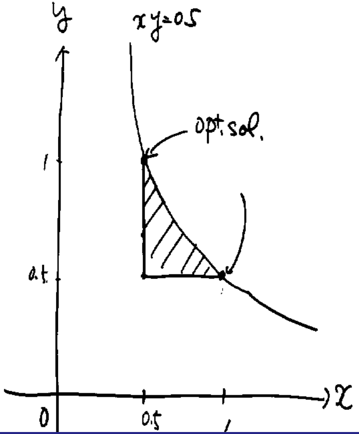
Practically, we observe,

- $\rho_2^* = f^*$ or $\rho_3^* = f^*$ without adding the new constraint
- $\rho_r^* = f^*$ if mild assumption does not hold.

An example (1) [W 2011]

$$f^* := \inf_{x,y \in \mathbb{R}} \{-x - y \mid x, y \geq 0.5, 0.5 \geq xy\}$$

- Compact feasible region
- $f^* = -1.5, (x^*, y^*) = (1, 0.5), (0.5, 1)$
- This example does not satisfy assumption



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ρ_3^*	-1.49999999e+00	-1.50011582e+00	-1.5000000e+00
ρ_4^*	-1.49999998e+00	-1.49999816e+00	-1.5000001e+00
ρ_5^*	-1.49999998e+00	-1.50000010e+00	-1.5000033e+00
ρ_6^*	-1.49999997e+00	-3.55261397e+01	-1.4999921e+00
ρ_7^*	-1.49999997e+00	-1.50000006e+00	-1.5003981e+00
ρ_8^*	-1.49999996e+00	-1.49999999e+00	-1.5000325e+00

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SDP is weakly infeasible

$\forall \rho$, $\nexists \sigma_0, \sigma_j$: SOS such that

$$L(x, y, \sigma) - \rho = \sigma_0(x, y)$$

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Conjecture for this example

$\forall \epsilon > 0$, $\exists \sigma_0, \sigma_j, \mu$: SOS such that

$$L(x, y, \sigma) - (-1.5) + \epsilon\mu = \sigma_0(x, y)$$

Perturbation Theorem for constrained POP

Theorem (W & Muramatsu 2011)

Let $\mathbf{R_j} := \max\{|f_j(x)| \mid x \in [-1, 1]^n\}$. *We assume*

- $\mathbf{K} \subseteq [-1, 1]^n$,
- $f(x) - \rho > 0$ *for all* $x \in K$.

Then $\forall \epsilon > 0$, $\exists \hat{r} \in \mathbb{N}$ *and* $\exists \sigma_0$*: SOS such that for all* $r \geq \hat{r}$

$$f(x) - \rho + \epsilon \left(1 + \sum_{i=1}^{n} x_i^{2r}\right) - \sum_{j=1}^{m} f_j(x) \left(1 - \frac{f_j(x)}{R_j}\right)^{2r} = \sigma_0(x)$$

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- Not need mild assumption
- Need a highly perturbation $\epsilon\left(1 + \sum_{i=1}^{n} x_i^{2r}\right)$
- SDP relaxation by Perturbation Theorem

Sketch of proof of Perturbation Theorem (1)

Theorem (I; Approximation of penalty function on K)

$$f(x) - \rho - \sum_{j=1}^{m} f_j(x)\left(1 - \frac{f_j(x)}{R_j}\right)^{2r} > 0 \ (x \in [-1, 1]^n)$$

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- $\mathbf{x} \in \mathbf{K}$, for sufficiently large $\mathbf{r}$,

$$-\sum_{j=1}^{m} f_j(x)\left(1 - \frac{f_j(x)}{R_j}\right)^{2r} \approx \mathbf{0}$$

- $\mathbf{x} \in [-1, 1]^n \setminus \mathbf{K}$ and $\mathbf{M} > \mathbf{0}$, for sufficiently large $\mathbf{r}$,

$$-\sum_{j=1}^{m} f_j(x)\left(1 - \frac{f_j(x)}{R_j}\right)^{2r} > \mathbf{M}$$

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- $x \in [-1, 1]^n \setminus K$ and $M > 0$, for sufficiently large r,

$$-\sum_{j=1}^{m} f_j(x) \left(1 - \frac{f_j(x)}{R_j}\right)^{2r} > M$$

- $\Rightarrow$ Approximation of Penalty function on K

Sketch of proof of Perturbation Theorem (2)

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Sketch of proof of Perturbation Theorem (2)

Theorem (I; Approximation of penalty function on K)

$$f(x) - \rho - \sum_{j=1}^{m} f_j(x)\left(1 - \frac{f_j(x)}{R_j}\right)^{2r} > 0 \ (x \in [-1,1]^n)$$

Theorem (II; Netzer-Lasserre 2007)

$f(x) \geq 0$ *over* $[-1,1]^n$. *For any* $\epsilon > 0$, $\exists \hat{r} \in \mathbb{R}$ *and* σ_0: *SOS such that for all* $r \geq \hat{r}$

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- Theorem I + II $\Rightarrow$ Perturbation Theorem for constrained POP

SDP relaxation by Perturbation Theorem

- $f(x) - \rho + \epsilon \left(1 + \sum_{i=1}^{n} x_i^{2r}\right) - \sum_{j=1}^{m} f_j(x) \left(1 - \frac{f_j(x)}{R_j}\right)^{2r}$ is SOS
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Theorem

$\forall \epsilon > 0$, let $\rho(\epsilon, r)$ be the optimal value;

$$\begin{cases} \sup_{\rho,\sigma_j} & \rho \\ \textit{sub. to} & f(x) - \rho + \epsilon\left(1 + \sum_{i=1}^{n} x_i^{2r}\right) - \sum_{j=1}^{m} \sigma_j(x) f_j(x) = \sigma_0(x) \\ & \sigma_0, \sigma_j : \textit{SOS}, \\ & \deg(\sigma_0) \le 2r, \deg(\sigma_j f_j) \le 2r. \end{cases}$$

Then, $f^* - \epsilon \le \rho(\epsilon, r) \le f^* + (constant) \times \epsilon$ for large r.

A smaller SDP relaxation (1)

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- Example $f_j(x) = 1 - x_j$
  - SOS with $\mathbf{deg}(\sigma_j) \leq 2r$... $\binom{n+r}{r}$
  - monomials in $\left(1 - \frac{f_j(x)}{R_j}\right)^{r}$... $r + 1$

A smaller SDP relaxation (2)

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Theorem (A smaller SDP relaxation)

$$\begin{cases} \sup_{\rho,\sigma_j} & \rho \\ \textit{sub. to} & f(x)-\rho-\sum_{j=1}^{m}\sigma_j(x)f_j(x)=\sigma_0(x) \\ & \sigma_0 : \textit{SOS}, \sigma_j : \textit{a smaller SOS}, \\ & \deg(\sigma_0)\leq 2r, \deg(\sigma_j f_j)\leq 2r. \end{cases}$$

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- $\mathbf{K}$ is compact $\longrightarrow$ the set of optimal solutions is compact

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- e.g., Bilinear matrix inequalities

$$\begin{cases} \inf_{x \in \mathbb{R}^n_x, y \in \mathbb{R}^n_y} & f(x, y) \\ \text{sub. to} & G_{00} + \cdots + \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} G_{ij} x_i y_j \in \mathbb{S}^k_+ \end{cases}$$

- Extend Putinar's Lemma and establish SDP relaxation [Kojima-Muraamtsu, 2007]

An extension of Perturbation Theorem (2)

- Approximation of Penalty Function on **K** for POP with symmetric cones

$$\Phi_r(x) := -G(x) \bullet \left(1 - \frac{G(x)}{R}\right)^{2r},$$

where $G(x)^k := G(x) \circ G(x)^{k-1}$ for all $k \in \mathbb{N}$.

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Theorem (W & Muramatsu 2011)

Let $R_j := \max\{\lambda_{\max}(G(x)) \mid x \in [-1, 1]^n\}$. *We assume*

- *set of opt. sol.* $\subseteq [-1, 1]^n$ *and* $f(x) - \rho > 0$ *for all* $x \in K$.

Then $\forall \epsilon > 0$, $\exists \hat{r} \in \mathbb{N}$ *and* $\exists \sigma_0$: *SOS such that for all* $r \geq \hat{r}$

$$f(x) - \rho + \epsilon\left(1 + \sum_{i=1}^{n} x_i^{2r}\right) + \Phi_r(x) = \sigma_0(x)$$

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$$\Rightarrow \mathbf{f}(\mathbf{x}) = \sum_{p=1}^{k} \sigma_p(\mathbf{x}_{C_p}), \sigma_p : \text{SOS with } \mathbf{x}_{C_p}$$

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- In a smaller SDP relaxation,

$$\mathbf{f}(\mathbf{x}) - \rho - \sum^{m} \sigma_j(\mathbf{x})\mathbf{f}_j(\mathbf{x}) = \sum^{k} \tilde{\sigma}_p(\mathbf{x}_{C_p}), \tilde{\sigma}_p : \text{SOS with } \mathbf{x}_{C_p}$$

Addition of valid inequalities (1)

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Its dual of SDP relaxation

$$(\mathbf{POP}) \equiv \begin{cases} \inf_x & f(x) \\ \text{sub.to} & f_j(x)u_{r_j}(x)u_{r_j}(x)^T \succeq O, \\ & u_r(x)u_r(x)^T \succeq O, \ (\text{moment matrix}) \end{cases}$$

where $\mathbf{u}_k(x) = (1, x_1, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_n^2, \ldots, x_n^k)^T$.
Replacing $\mathbf{x}^\alpha$ by $\mathbf{y}_\alpha$, obtain its dual of SDP relax.

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- Assume $\mathbf{K} \subseteq [0,1]^n$
- Add valid inequalities $\mathbf{0} \leq \mathbf{x}^\alpha \leq \mathbf{1}$ for all monomials
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- Assume $\mathbf{K} \subseteq [0, 1]^n$
- Add valid inequalities $0 \le x^\alpha \le 1$ for all monomials $\rightarrow 0 \le y_\alpha \le 1$
- Stronger than the original SDP relaxation

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- For example, $\mathbf{n = 2, r = 2}$

$$\left(\begin{array}{c|cc|ccc}
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 & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\
 & & y_{02} & y_{21} & y_{12} & y_{03} \\ \hline
 & & & \color{red}{y_{40}} & y_{31} & y_{22} \\
 & & & & y_{22} & y_{13} \\
 & & & & & \color{red}{y_{04}}
\end{array}\right) \in \mathbb{S}^6_+$$

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- $\color{red}{y_{40}}, \color{red}{y_{04}} \leq 1 \Rightarrow y_\alpha \leq 1$
- because principal matrices

$$\begin{pmatrix} 1 & y_{20} \\ y_{20} & \color{red}{y_{40}} \end{pmatrix}, \begin{pmatrix} 1 & y_{02} \\ y_{02} & \color{red}{y_{04}} \end{pmatrix}, \begin{pmatrix} \color{red}{y_{04}} & y_{22} \\ y_{22} & \color{red}{y_{04}} \end{pmatrix} \in \mathbb{S}^2_+$$

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- Not need to add $y_{20}, y_{02}, y_{40}, y_{22}, y_{04} \geq 0$

Refine approximated solution by SDP relaxation

$$\text{(POP)} \inf_{\mathbf{x}\in\mathbb{R}^n} \{f_0(\mathbf{x}) \mid f_j(\mathbf{x}) \geq 0 \ (j = 1, \ldots, m)\}$$

- In its dual of SDP relaxation: $\mathbf{x}^{\alpha} \longrightarrow \mathbf{y}_{\alpha}$

$$\begin{cases} x_1 & \longrightarrow & y_{(1,0,\ldots,0)} \\ x_2 & \longrightarrow & y_{(0,1,\ldots,0)} \\ & \vdots & \\ x_n & \longrightarrow & y_{(0,0,\ldots,1)} \end{cases}$$

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- No guarantee on the global optimality for refined solutions

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- By using local solvers, refine solutions obtained by SDP relaxation

Indices for checking the quality of SDP relaxation

$$\text{(POP)}\ \mathbf{f}^* := \inf_{\mathbf{x}\in\mathbb{R}^n} \{\mathbf{f}_0(\mathbf{x}) \,|\, \mathbf{f}_j(\mathbf{x}) \geq 0\ (j = 1, \ldots, m)\}$$

- Let $\hat{\mathbf{x}} = (\mathbf{y}^*_{(1,0,\ldots,0)}, \ldots, \mathbf{y}^*_{(0,\ldots,0,1)})^T$ be a solution obtained by SDP relaxation.

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- Check the feasibility in POP: $\epsilon_{\mathrm{feas}} := \min_{j=1,\ldots,m}\{f_j(\hat{\mathbf{x}})\}$

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- ϵ_{feas} and ϵ_{obj} $\approx \mathbf{0} \Rightarrow$ (opt. val. of SDP relax.) $= \mathbf{f^*}$

GLOBAL Library (1)

Test problems in
http://www.gamsworld.org/global/globallib/globalstat.htm & SDP solver is SeDuMi

- Feasibility in POP: $\epsilon_{\text{feas}} := \min_{j=1,\ldots,m}\{f_j(\hat{x})\}$
- Optimality: $\epsilon_{\text{obj}} :=$ (opt.va. of SDP relax.) $-$ $f(\hat{x})$
- **ex2_1_8**... Quadratic Optimization Problem ($n = 24$, $m = 58$, $r = 2$)

	sizeA	nnzA
Lasserre	[2924,38223]	68788
Sparse	[1789, 15059]	23144
Our	[565, 2409]	3495

	SDP relaxation			Refine by fmincon	
	ϵ_{obj}	ϵ_{feas}	time[**sec**]	obj.val.	ϵ_{feas}
Lasserre	3.9e-09	-5.7e-11	286.62	1.5639e+04	-5.7e-11
Sparse	3.5e-09	-1.1e-12	12.68	1.5639e+04	-3.5e-11
Our	4.7e-02	-1.3e-09	9.94	1.5639e+04	-3.6e-15

GLOBAL Library (2)

- Feasibility in POP: $\epsilon_{\text{feas}} := \min_{j=1,\ldots,m}\{f_j(\hat{x})\}$
- Optimality: $\epsilon_{\text{obj}} :=$ (opt.va. of SDP relax.) $- f(\hat{x})$
- **meanvarx**... contains 0-1 constraints with ($n = 35$, $m = 66$, $r = 3$)
- No improvement by local solver (fmincon)

	sizeA	nnzA
Lasserre	[37597,258331]	34220
Sparse	[526, 2980]	4482
Our	[406, 1791]	2887

	obj.val.	ϵ_{obj}	ϵ_{feas}	time[**sec**]
Lasserre	Out of memory			
Sparse	1.4327558e+01	2.3e-04	-1.0e-01	161.90
Our	1.4327495e+01	2.3e-04	-1.0e-01	69.30

GLOBAL Library (3)

- Feasibility in POP: $\epsilon_{\text{feas}} := \min_{j=1,\ldots,m}\{f_j(\hat{x})\}$
- Optimality: $\epsilon_{\text{obj}} :=$ (opt.va. of SDP relax.) $- f(\hat{x})$
- **st_fp7a**... QOP ($n = 20$, $m = 20$, $r = 2$)

Table: Numerical Result for st_fp7a by SeDuMi

	SDP relaxation					
	ϵ_{obj}	ϵ_{feas}	time[**sec**]	sizeA	nnzA	# iter.
Lasserre	1.5e-08	0.0e+00	156.37	[1770, 15421]	87294	32
Our	5.7e-03	0.0e+00	217.87	[1770, 5313]	55450	48

GLOBAL Library (3)

- Feasibility in POP: ϵ_{feas} $:= \min_{j=1,\ldots,m}\{f_j(\hat{x})\}$
- Optimality: ϵ_{obj} $:=$ (opt.va. of SDP relax.) $- f(\hat{x})$
- **st_fp7a**... QOP ($\mathbf{n = 20}$, $\mathbf{m = 20}$, $\mathbf{r = 2}$)

Table: Numerical Result for st_fp7a by SDPT3

	SDP relaxation					
	ϵ_{obj}	ϵ_{feas}	time[**sec**]	sizeA	nnzA	# iter.
Lasserre	4.1e-09	0.0e+00	49.31	[1770, 15421]	87294	29
Our	5.6e-03	0.0e+00	49.67	[1770, 5313]	55450	42

GLOBAL Library (4)

- Feasibility in POP: $\epsilon_{\mathrm{feas}} := \min_{j=1,\ldots,m}\{f_j(\hat{x})\}$
- Optimality: $\epsilon_{\mathrm{obj}} :=$ (opt.va. of SDP relax.) $- f(\hat{x})$
- **ex5_3_2**... QOP with ($\mathbf{n = 22}$, $\mathbf{m = 60}$, $\mathbf{r = 2}$ & $\mathbf{r = 3}$)

Table: Numerical Result for ex5_3_2

		SDP relaxation			Refine by fmincon	
	r	ϵ_{obj}	ϵ_{feas}	time[sec]	obj.val.	ϵ_{feas}
Lasserre	2	Out of memory				
Sparse	2	1.1e-07	-6.5e-01	170.05	1.8746711	-2.4e-15
Our	2	4.5e-09	-3.3e-01	13.93	1.8641595	-3.6e-15
Lasserre	3	Out of memory				
Sparse	3	Out of memory				
Our	3	1.1e-06	-1.3e-01	168.37	1.8641595	3.6e-15

Bilinear Matrix Inequality

$$\mathbf{B_k(x, y)} := \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij} x_i y_j + \sum_{i=1}^{n} B_{i0} x_i + \sum_{j=1}^{n} B_{0j} y_j + B_{00}$$

$$\text{(BMI)} \quad \inf_{s\in\mathbb{R}, x, y \in [0,1]^n} \left\{ s \,\middle|\, sI_k - B_k(x, y) \in \mathbb{S}^k_+, \right\}.$$

- No correlative sparsity in (BMI)
- # of var. in (BMI) $= \mathbf{2n+1}$ and $\mathbf{r = 2}$

(n, k)	Lasserre			Our		
	ϵ_{obj}	ϵ_{feas}	time[**sec**]	ϵ_{obj}	ϵ_{feas}	time[**sec**]
(2, 5)	2.7e-09	0.0e+00	6.88	1.5e-09	0.0e+00	1.33
(4, 5)	9.2e-09	-1.4e-10	3846.19	6.4e-09	-1.2e-10	137.89
(2, 10)	Out of memory			6.9e-10	0.0e+00	2.58
(4, 10)	Out of memory			1.7e-09	-5.4e-12	331.87

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- Thank you for your attention