Geometric Representations of Graphs, Semidefinite Optimization, and Min-Max Theorems

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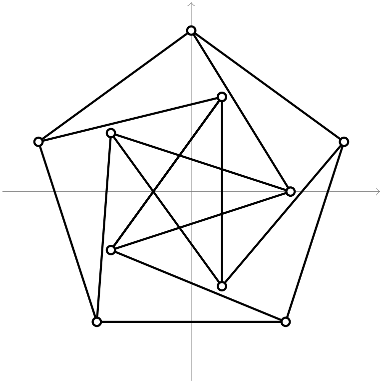
Unit-distance representations

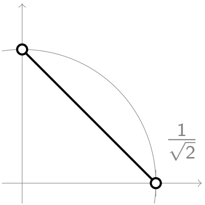
A *unit-distance representation* of $G = (V, E)$ is a map $p\colon V \to \mathbb{R}^d$ s.t.

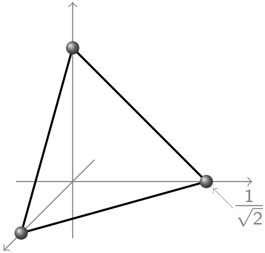
$$\|p(i) - p(j)\| = 1 \qquad \forall \{i, j\} \in E$$



Every graph has a unit-distance representation

- complete graphs have a unit-distance repr. $i \mapsto \frac{1}{\sqrt{2}} e_i \in \mathbb{R}^n$





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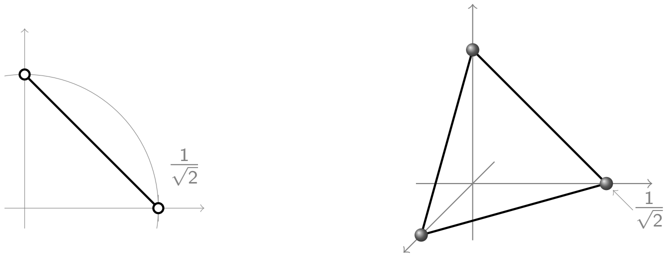
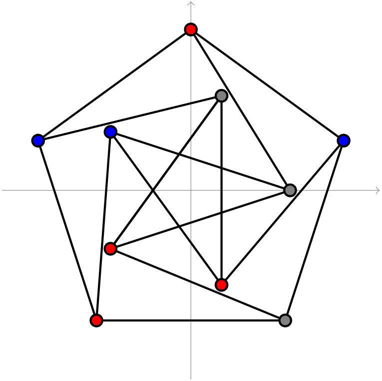


- a unit-distance repr. of G "contains" a unit-distance repr. of any subgraph of G

Petersen again



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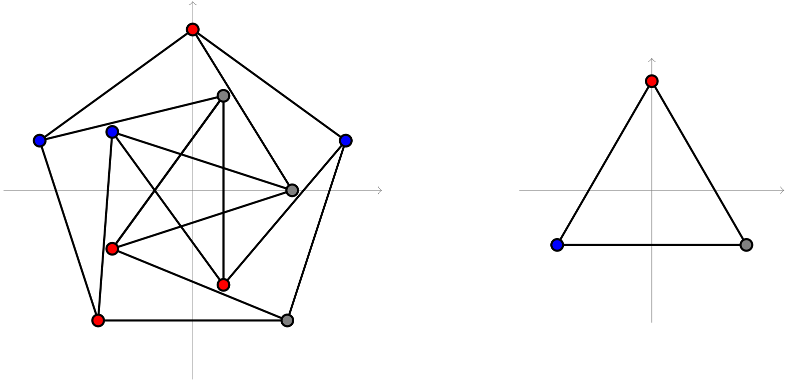

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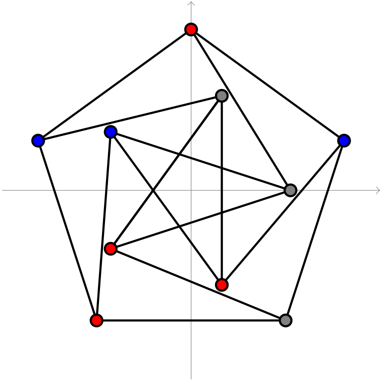

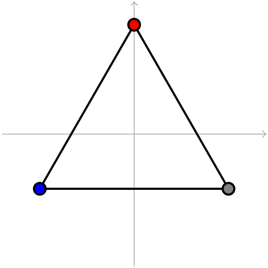

- Any unit-distance repr. of K_p contains a unit-distance repr. of any p-colorable graph.

Chromatic number of $\mathbb{R}^n$

- The graph $(\mathbb{R}^n, \{\{x,y\} : \|x-y\| = 1\})$ "is" a unit-distance repr. of itself.
- Frankl and Wilson, Raigorodskii, Larman and Rogers: $(1+o(1))1.2^n \leq \text{chromatic}(\mathbb{R}^n) \leq (3+o(1))^n$
- The "graph" $\mathbb{R}^n$ has a unit-distance repr. in some $\mathbb{R}^d$ with finite image.
- de Bruijn, Erdős '51: $\text{chromatic}(\mathbb{R}^n) = \max \text{chromatic}(G)$ where G ranges over finite graphs with some unit-distance repr. in $\mathbb{R}^n$.

Outline

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Hypersphere number and Lovász Theta Number

Homomorphisms and Sandwich Theorems

Generalizations

Hypersphere representations

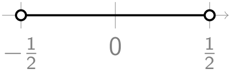
- A *hypersphere representation* of $G = (V, E)$ is a unit-distance representation of G contained in a hypersphere centered at the origin, i.e.,

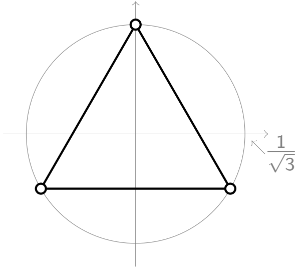
- A *hypersphere representation* of G is a map $p\colon V \to \mathbb{R}^d$ s.t.

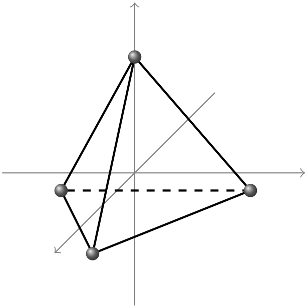
$$\begin{aligned} \|p(i)\| &= r \qquad &&\forall i \in V \\ \|p(i) - p(j)\| &= 1 \qquad &&\forall \{i, j\} \in E \end{aligned}$$

- $\text{hypersphere}(G) := \Big[\text{min. radius } r \text{ of a hypersph. repr. of } G\Big]^2$

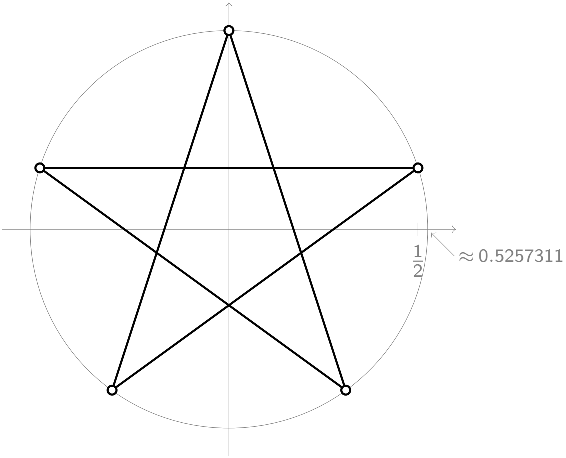
Optimal hypersphere representations of complete graphs







Optimal hypersphere representations of the 5-cycle



hypersphere(G) as an SDP

$$
\begin{aligned}
\text{hypersphere}(G) = \min \quad & t \\
& X_{ii} = t && \forall i \in V \\
& X_{ii} - 2X_{ij} + X_{jj} = 1 && \forall \{i,j\} \in E, \\
& X \succeq 0
\end{aligned}
$$

$$
\begin{aligned}
= \max \quad & \sum_{\{i,j\} \in E} z_{\{i,j\}} \\
& \mathrm{Diag}(y) \succeq \sum_{\{i,j\} \in E} z_{\{i,j\}} (e_i - e_j)(e_i - e_j)^T \\
& \textstyle\sum_{i \in V} y_i = 1
\end{aligned}
$$

- Dual may be interpreted as a problem in tensegrity theory.

Relation with Lovász Theta Number

- Lovász proved $2\big[\text{hypersphere}(G)\big] + \dfrac{1}{\theta(\overline{G})} = 1$

Sketch of Proof.

Rewrite dual:

$$\begin{aligned} 2\big[\text{hypersphere}(G)\big] = \max \quad & \langle J - I, S\rangle \\ & S \succeq 0 \\ & S_{ij} = 0 \qquad \forall \{i,j\} \in \overline{E} \\ & \langle J, S\rangle = 1 \end{aligned}$$

□

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$$

□

Reciprocal SDPs

$$1/\theta(\overline{G}) = \begin{array}{lll} \min & \langle I, S \rangle & \\ & S \succeq 0 & \\ & S_{ij} = 0 & \forall \{i,j\} \in \overline{E} \\ & \langle J, S \rangle = 1 & \end{array}$$

$$\theta(\overline{G}) = \begin{array}{lll} \max & \langle J, X \rangle & \\ & X \succeq 0 & \\ & X_{ij} = 0 & \forall \{i,j\} \in \overline{E} \\ & \langle I, X \rangle = 1 & \end{array}$$

Min-Max Interpretation

- hypersphere(G) $= \Big[\text{min. radius of a hypersph. repr. of } G\Big]^2$
- $\theta(G) = \max \left\{ \sum_{i \in V} x_i : x \in \underbrace{\mathrm{TH}(G)}_{\text{theta body}} \right\}$

Theorem

- *Let p be a hypersphere repr. of G with radius r*
- *Let $x \in \mathrm{TH}(\overline{G})$ with $x \neq 0$*

Then
$$2r^2 + \frac{1}{\sum_{i \in V} x_i} \geq 1.$$

Equality holds $\iff$ $r^2 = \text{hypersphere}(G)$ *and* $\sum_{i \in V} x_i = \theta(\overline{G})$

SDP-free Interpretation

- An *orthonormal representation* of $G = (V, E)$ is a map u from V to the unit sphere in $\mathbb{R}^V$ s.t. non-adjacent nodes are orthogonal

Theorem

- *Let p be a hypersphere repr. of G with radius r*
- *Let c be a unit vector and u an orthonormal repr. of G*

Then
$$2r^2 + \frac{1}{\sum_{i \in V} (c^T u(i))^2} \geq 1.$$

Equality $\iff$ $r^2 = \text{hypersphere}(G)$ *and* $\sum_{i \in V} (c^T u(i))^2 = \theta(\overline{G})$

Characterization of bipartite graphs

$$G \text{ is bipartite} \iff \theta(\overline{G}) \leq 2$$

$$(\text{proof reduces to } \theta(\overline{C_{2k+1}}) > 2)$$

Equivalently $$G \text{ is bipartite} \iff \text{hypersphere}(G) \leq 1/4$$

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($\Longrightarrow$): already done.

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Proof.

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($\Longleftarrow$): In a hypersphere with radius $1/2$, the only pairs of points at distance 1 are the pairs of antipodal points. □

Unit-distance representations in Euclidean balls

Recall:

- $\text{hypersphere}(G) := \left[\begin{array}{c} \text{min. radius of a hypersphere} \\ \text{containing a unit-distance repr. of } G \end{array} \right]^2$

Next:

- $\text{ball}(G) := \left[\begin{array}{c} \text{min. radius of a ball containing} \\ \text{a unit-distance repr. of } G \end{array} \right]^2$

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- $\text{ball}(G) \stackrel{?}{=} \text{hypersphere}(G)$

Another min-max relation

$$2\text{hypersphere}(G) + \frac{1}{\theta(\overline{G})} = 1$$

where $\theta(\overline{G}) =$

$$\begin{aligned} \max \quad & \langle J, X \rangle \\ & X \succeq 0 \\ & X_{ij} = 0 \qquad \forall \{i,j\} \in \overline{E} \\ & \langle I, X \rangle = 1 \end{aligned}$$

$$2\,\text{ball}(G) + \frac{1}{\theta_b(\overline{G})} = 1$$

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$$\text{hypersphere}(\cdot) = \text{ball}(\cdot) \iff \theta(\cdot) = \theta_b(\cdot)$$

Hypersphere = Euclidean Balls?

- But $\theta(\cdot) = \theta_b(\cdot)$!
- It was pointed out by Fernando Mario de Oliveira Filho that the following result can be used to prove $\theta(\cdot) = \theta_b(\cdot)$

Theorem (Prop. 9 in Gijswijt's PhD thesis, 2005)

Let $\mathbb{K} \subseteq \mathbb{S}^n$ *s.t.* $\mathrm{Diag}(h) X \,\mathrm{Diag}(h) \in \mathbb{K}$ *whenever* $X \in \mathbb{K}$ *and* $h \in \mathbb{R}^n_+$*. If* X^* *is an optimal solution to*

$$\max \left\{ \langle J, X \rangle : \mathrm{Tr}(X) = 1,\, X \in \mathbb{K} \cap \mathbb{S}^n_+ \right\}, \tag{1}$$

then $\exists \mu > 0$ *s.t.* $\mathrm{diag}(X^*) = \mu X^* \mathbb{1}$*.*

The Sandwich Theorem

- $\text{clique}(G) \leq \theta(\overline{G}) \leq \text{chromatic}(G)$
- $\equiv \begin{cases} \text{hypersphere}(K_{\text{clique}(G)}) \leq \text{hypersphere}(G) \\ \text{hypersphere}(G) \leq \text{hypersphere}(K_{\text{chromatic}(G)}) \end{cases}$
- $H \subseteq G \implies \text{hypersphere}(H) \leq \text{hypersphere}(G)$

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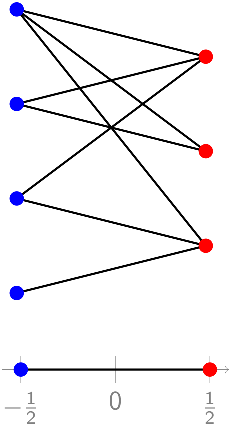
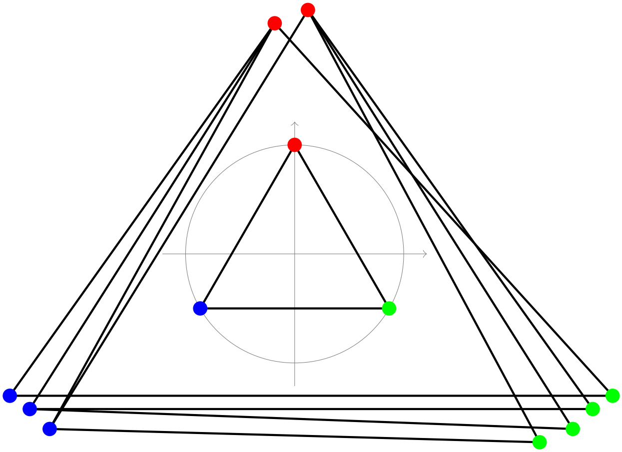
- $H \subseteq G \implies \text{hypersphere}(H) \leq \text{hypersphere}(G)$
- if $c\colon V(G) \to \{1, \ldots, n\}$ is a n-colouring of G, and p is a hypersphere repr. of the complete graph on $\{1, \ldots, n\}$, then $p \circ c$ is a hypersphere repr. of G

Hypersphere representations of 2-colourable graphs



Hypersphere representations of 3-colourable graphs



Outline

Graph Homomorphisms

- A homomorphism from a graph G to a graph H is a function $f\colon V(G) \to V(H)$ that preserves edges, i.e., if $\{i, j\} \in E(G)$, then $\{f(i), f(j)\} \in E(H)$.

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- Notation: $G \to H \equiv \exists$ a homomorphism from G to H
- "$\to$" is transitive (compose homs.)
- G is a subgraph of $H \implies G \to H$
- $G \to K_p \iff \text{chromatic}(G) \leq p$
- $\text{chromatic}(G) = \min\{\, p : G \to K_p\}$

Homomorphism-monotone invariants

A real-valued graph invariant f is hom-monotone if

- $f(G) \leq f(H)$ if $G \to H$

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- $f(G) \leq f(H)$ if $G \to H$ and
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Examples:

- clique$(\cdot)$
- chromatic$(\cdot)$
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- chromatic$(\cdot)$
- chromatic$^*(\cdot)$
- hypersphere$(\cdot)$
- some variants of hypersphere$(\cdot)$

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Then $\mathrm{clique}(G) \leq g(f(G)) \leq \mathrm{chromatic}(G)$.

Proof.

$$
\begin{aligned}
& K_{\mathrm{clique}(G)} \to G \to K_{\mathrm{chromatic}(G)} \\
\Longrightarrow\ & f(K_{\mathrm{clique}(G)}) \leq f(G) \leq f(K_{\mathrm{chromatic}(G)}) \\
\Longrightarrow\ & \mathrm{clique}(G) = g(f(K_{\mathrm{clique}(G)})) \leq g(f(G)) \leq g(f(K_{\mathrm{chromatic}(G)})) \\
& \qquad = \mathrm{chromatic}(G)
\end{aligned}
$$

□

Yet another variant

Define hypersphere$'(G)$ similarly as hypersphere(G), but require edges to be at distance ≥ 1.

$$2\,\text{hypersphere}(G) + \frac{1}{\theta(\overline{G})} = 1$$

where $\theta(\overline{G}) =$

$$\begin{aligned} \max \quad & \langle J, X \rangle \\ & X \succeq 0 \\ & X_{ij} = 0 \qquad \forall \{i,j\} \in \overline{E} \\ & \langle I, X \rangle = 1 \end{aligned}$$

$$2\,\text{hypersphere}'(G) + \frac{1}{\theta'(\overline{G})} = 1$$

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$$\begin{aligned} \max \quad & \langle J, X \rangle \\ & X \succeq 0 \\ & X_{ij} = 0 \qquad \forall \{i,j\} \in \overline{E} \\ & \langle I, X \rangle = 1 \\ & X \geq 0 \end{aligned}$$

$\theta'(\cdot)$ was introduced by McEliece, Rodemich, Rumsey, and, independently, by Schrijver

An aside: sparse solutions to SDPs

Theorem (de C.S., Harvey, Sato 2011)

Let $B_1, \ldots, B_m$ be psd $n \times n$ matrices. Set $B := \sum_i B_i$. Then $\forall \varepsilon \in (0,1)$ there exists $y \in \mathbb{R}_m^+$ with $\leq 4n/\varepsilon^2$ nonzero entries and

$$(1-\varepsilon)B \preceq \sum_i y_i B_i \preceq (1+\varepsilon)B.$$

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When applied to the dual SDP for hypersphere$'(G)$, we get:

for all $\varepsilon \in (0,1)$ and every graph G, there exists a spanning subgraph H of G such that

$$|E(H)| \leq 8\frac{|V(G)|}{\varepsilon^2}$$

and

$$\frac{\text{hypersphere}'(G)}{1+\varepsilon} \leq \text{hypersphere}'(H) \leq \text{hypersphere}'(G).$$

Unit-Distance Dimension

- $\dim(G) :=$ smallest d s.t. $\exists$ a unit-distance repr. of G in $\mathbb{R}^d$.
- if $G \to H$ and H has a unit-distance repr. in $\mathbb{R}^d$, then so does G
- $\dim(K_n) = n - 1$
- so $\dim(\cdot)$ is hom-monotone
- Sandwich: $\text{clique}(G) \leq \dim(G) + 1 \leq \text{chromatic}(G)$

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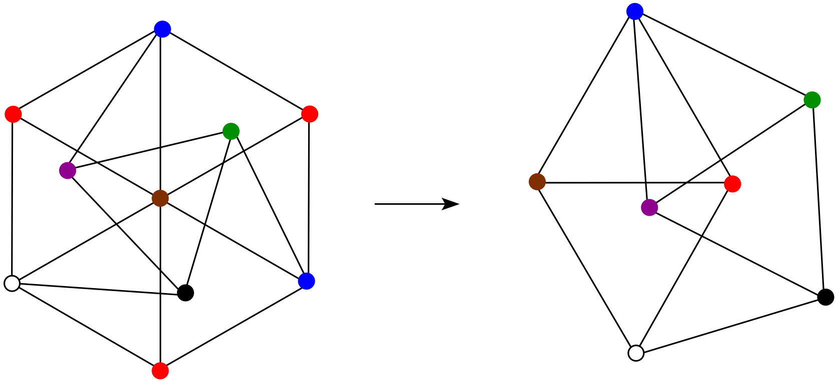
- $\dim(K_n) = n - 1$
- so $\dim(\cdot)$ is hom-monotone
- Sandwich: $\text{clique}(G) \leq \dim(G) + 1 \leq \text{chromatic}(G)$
- $\dim(G) \leq \text{maxdegree}(G)$
- Brooks' Theorem $\implies$ G is connected and not complete nor an odd cycle, then $\dim(G) \leq \text{maxdegree}(G) - 1$.

Golomb graph and Mosers' spindle



Golomb graph and Mosers' spindle

- Golomb $\rightarrow$ Moser
- Assume $G \rightarrow H$
- $\dim(G) \leq \dim(H)$ and $\mathrm{chromatic}(G) \leq \mathrm{chromatic}(H)$
- $\mathrm{chromatic}(\mathbb{R}^{\dim(H)}) \geq \mathrm{chromatic}(H) \geq \mathrm{chromatic}(G)$, i.e., G cannot improve the lower bound of $\mathrm{chromatic}(\mathbb{R}^{\dim(H)})$ given by H.

Hardness

Deciding whether $\dim(G) = 2$ is NP-complete.

Proof.

- k-Embeddability Problem:
  - input: graph $G = (V, E)$ and prescribed edge lengths $\ell\colon E \to \mathbb{R}_+$
  - decide if $\exists p\colon V \to \mathbb{R}^k$ such that $\|p(i) - p(j)\| = \ell_{ij}$ for all $ij \in E$.

□

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  - decide if $\exists p \colon V \to \mathbb{R}^k$ such that $\|p(i) - p(j)\| = \ell_{ij}$ for all $ij \in E$.
- Saxe '79: $\forall k \geq 1$, the problem k-Embeddability is NP-complete, even if $\ell(E) \subseteq \{1, 2\}$.

□

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  - input: graph $G = (V, E)$ and prescribed edge lengths $\ell \colon E \to \mathbb{R}_+$
  - decide if $\exists p \colon V \to \mathbb{R}^k$ such that $\|p(i) - p(j)\| = \ell_{ij}$ for all $ij \in E$.
- Saxe '79: $\forall k \geq 1$, the problem k-Embeddability is NP-complete, even if $\ell(E) \subseteq \{1, 2\}$.
- We show 2-Embeddability with $\ell(E) \subseteq \{1, 2\}$ reduces to deciding if $\dim(G) = 2$.

□

Hardness

Deciding whether $\dim(G) = 2$ is NP-complete.

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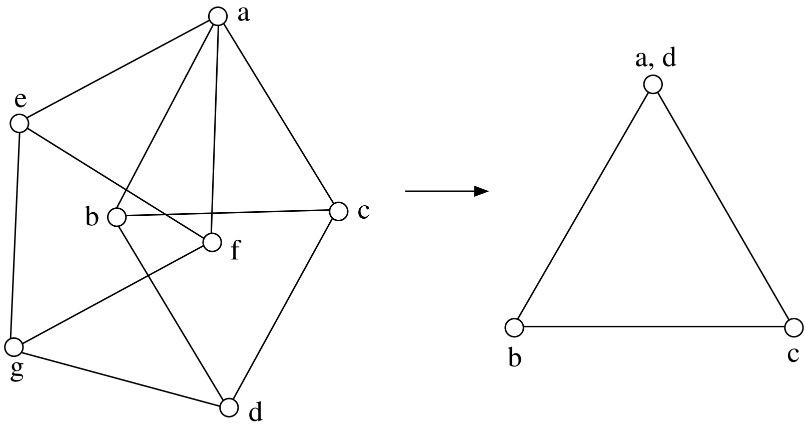
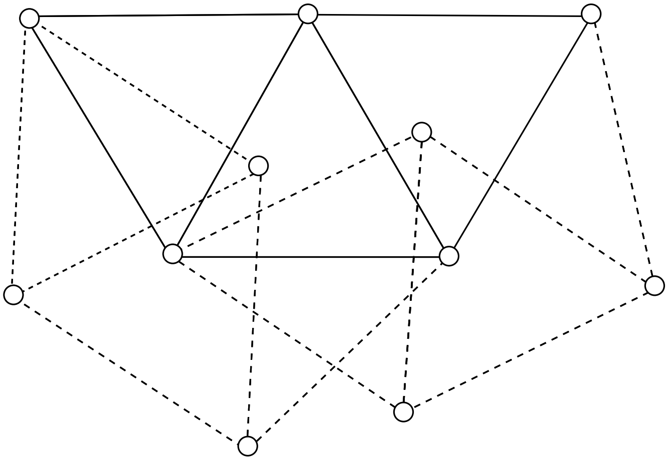
- We show 2-Embeddability with $\ell(E) \subseteq \{1, 2\}$ reduces to deciding if $\dim(G) = 2$.
- We need a gadget to force distance 2 using only distance 1 requirements.

□

Unique embedding of Mosers' spindle



Gadget



Outline

Weighted Hypersphere Number

We want to define hypersphere(G, w) for $w \in \mathbb{R}_+^V$ so that

$$2\,\text{hypersphere}(G, w) + \frac{1}{\theta(\overline{G}, w)} = 1.$$

$$\begin{array}{lll} & \text{hypersphere}(G, w) = & \\ \min & t & \\ & X_{ii} = w_i t + (1 - w_i)/2 & \forall i \in V \\ & X_{ii} - 2X_{ij} + X_{jj} = 1 + (t - 1/2)(w_i - 2\sqrt{w_i w_j} + w_j) & \forall \{i, j\} \in E \\ & X \succeq 0 & \end{array}$$

Solutions encode hypersphere repr. for graph obtained from G by "blowing up" each node i into a clique of size w_i.

Objective Function as Norm

- $\text{ball}(G) = \min \left\{ \left\| (u_i^T u_i)_{i \in V} \right\|_\infty : u \text{ a unit-distance repr. of } G \right\}$

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- for a fixed $A \succeq 0$, the invariant $\mathrm{ellipse}_\infty(\cdot, A)$ satisfies the first condition of hom-monotonicity, i.e.,

$$G \to H \implies \mathrm{ellipse}_\infty(G, A) \leq \mathrm{ellipse}_\infty(H, A)$$

Action of the Orthogonal Group

$$\text{ellipse}_1(G, A) = \min \left\{ \sum_{i \in V} \|A^{1/2} u_i\|_2^2 : u \text{ a unit-distance repr. of } G \right\}$$

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Complete Graphs

$$\begin{aligned} \text{ellipse}_1(G, A) = \min \quad & \sum_{i=1}^{n} \lambda_i(A)\lambda_{n-i+1}(X) \\ \text{s.t.} \quad & X_{ii} - 2X_{ij} + X_{jj} = 1 \quad \forall \{i, j\} \in E \\ & X \succeq 0 \end{aligned}$$

- for $G = K_n$, a matrix X is feasible iff X is of the form $(\mathbb{1}y^T + y\mathbb{1}^T + 2I)/4$ with $\|\mathbb{1}\|\|y\| \leq 2 + \mathbb{1}^T y$
- $\text{ellipse}_1(K_n, A) = \text{Tr}(A) - \lambda_{\max}(A)$ using SOC program

Hardness

- if A is $n \times n$ diagonal with $n - 2$ one entries and 2 zeroes on the diagonal, then $\text{ellipse}_1(G, A) = 0$ if and only if $\dim(G) \leq 2$
- Computing $\text{ellipse}_1(G, A)$ given G and $A \succeq 0$ as inputs is NP-hard.
- For any fixed $p \in [1, \infty]$, computing $\text{ellipse}_p(G, A)$ given G and $A \succeq 0$ as inputs is NP-hard.