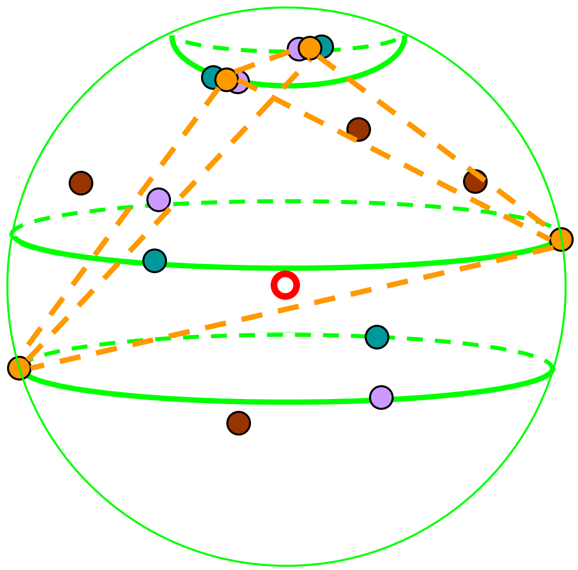
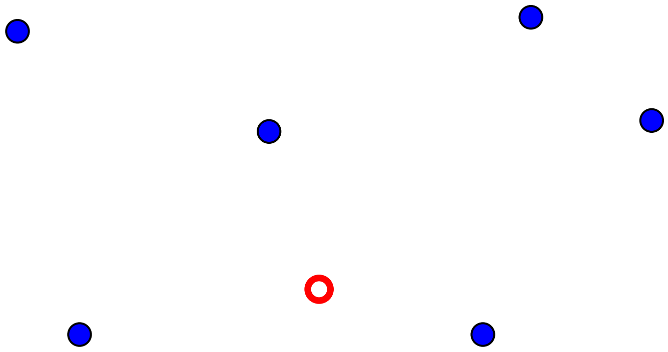
A further generalization of the colourful Carathéodory theorem

Antoine Deza (McMaster)
joint work with Frédéric Meunier (Paris VI)



McMaster University
Advanced Optimization Laboratory

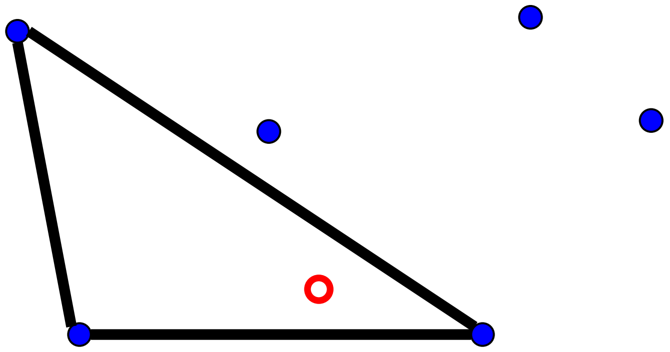
Carathéodory Theorem



Given a set S of n points in dimension d, then there exists an open simplex generated by points in S containing p

S, p general position

Carathéodory Theorem



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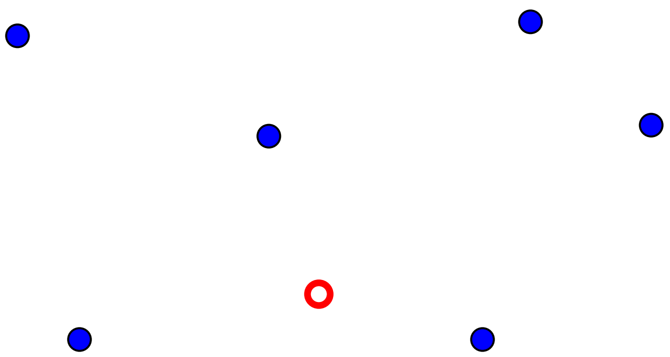
Simplicial Depth

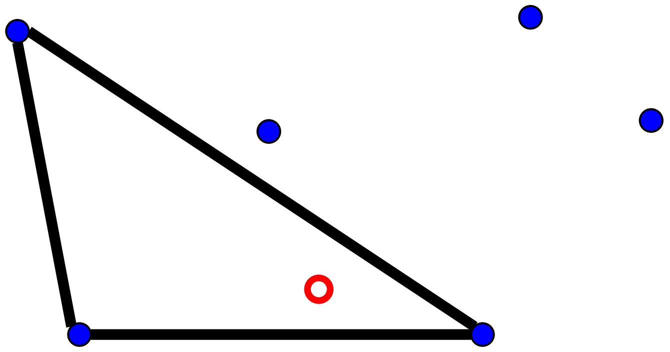


Given a set S of n points in dimension d, the *simplicial depth* of p is the number of open simplices generated by points in S containing p [Liu 1990]

S, p general position

Simplicial Depth

$depth_S(p)=1$



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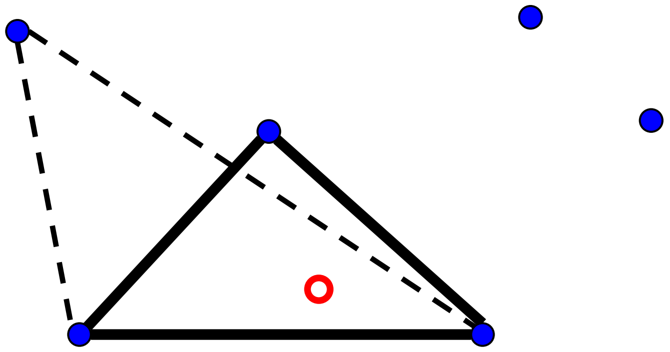
S, p general position

Simplicial Depth

depth$_S$(*p*)=2



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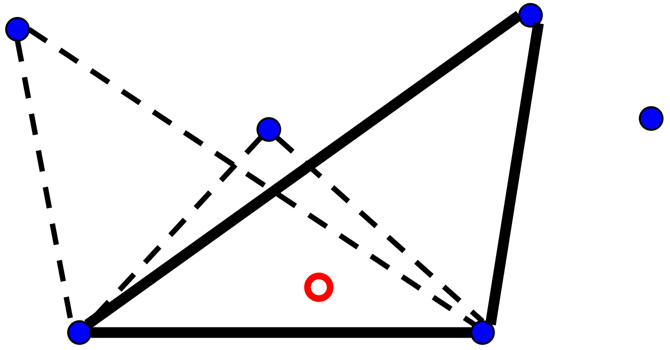
S, *p general position*

Simplicial Depth

$depth_S(p)=3$



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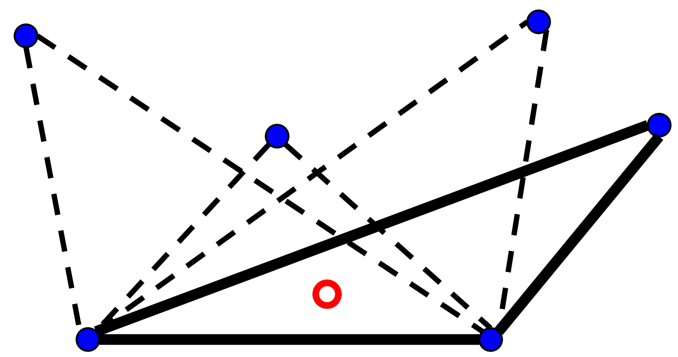
S, p *general position*

Simplicial Depth

$depth_S(p)=4$



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S, p general position

Simplicial Depth

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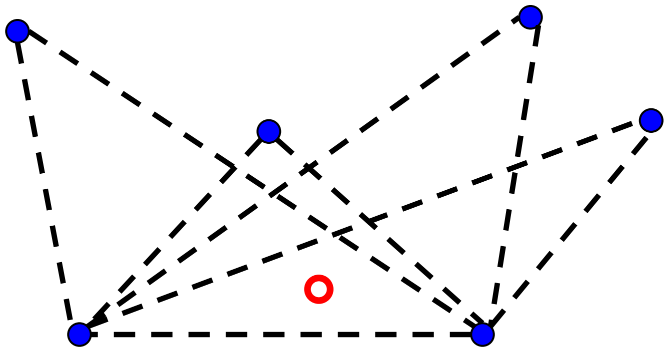
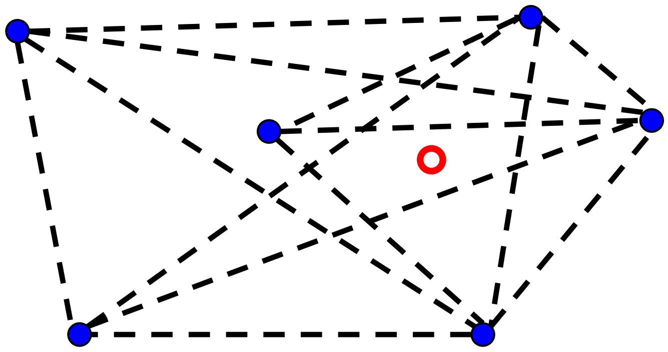


Given a set *S* of *n* points in dimension *d*, the *simplicial depth* of *p* is the number of open simplices generated by points in *S* containing *p* [Liu 1990]

S, *p general position*

Simplicial Depth

$depth_S(p)=9$



Given a set S of n points in dimension d, the *simplicial depth* of p is the number of open simplices generated by points in S containing p [Liu 1990]

S, p general position

Deepest Point in Dimension 2

Deepest point bounds in dimension 2 [Kárteszi 1955], [Boros, Füredi 1984]

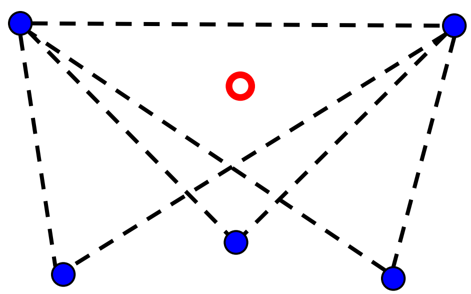
$$n^3/27 + O(n^2) \le \max_{p} depth_S(p) \le \begin{cases} (n^3 - 4n)/24 & n \text{ even} \\ (n^3 - n)/24 & n \text{ odd} \end{cases}$$

S, p general position

Deepest Point in Dimension 2

Deepest point bounds in dimension 2 [Kárteszi 1955], [Boros, Füredi 1984]

$$\frac{n^3}{27} + O(n^2) \le \max_p depth_S(p) \le \frac{n^3}{24} + O(n^2)$$



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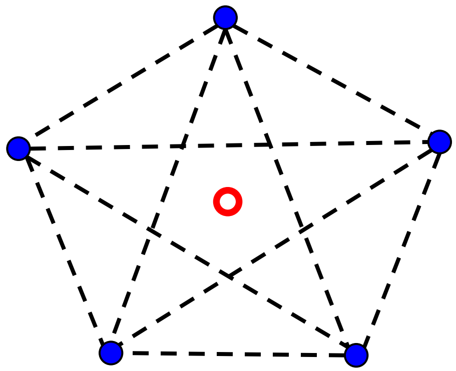


$depth_S(p)=5$

S, p general position

Deepest Point in Dimension d

Deepest point bounds in dimension *d* [Bárány 1982]

$$\frac{1}{(d+1)^{d+1}}\binom{n}{d+1}+O(n^d)\le \max_p depth_S(p)\le \frac{1}{2^d(d+1)!}n^{d+1}+O(n^d)$$

S, p general position

Deepest Point in Dimension d

Deepest point bounds in dimension *d* [Bárány 1982]

$$\frac{1}{(d+1)^{d+1}}\binom{n}{d+1} + O(n^d) \leq \max_{p} depth_S(p) \leq \frac{1}{2^d (d+1)!} n^{d+1} + O(n^d)$$

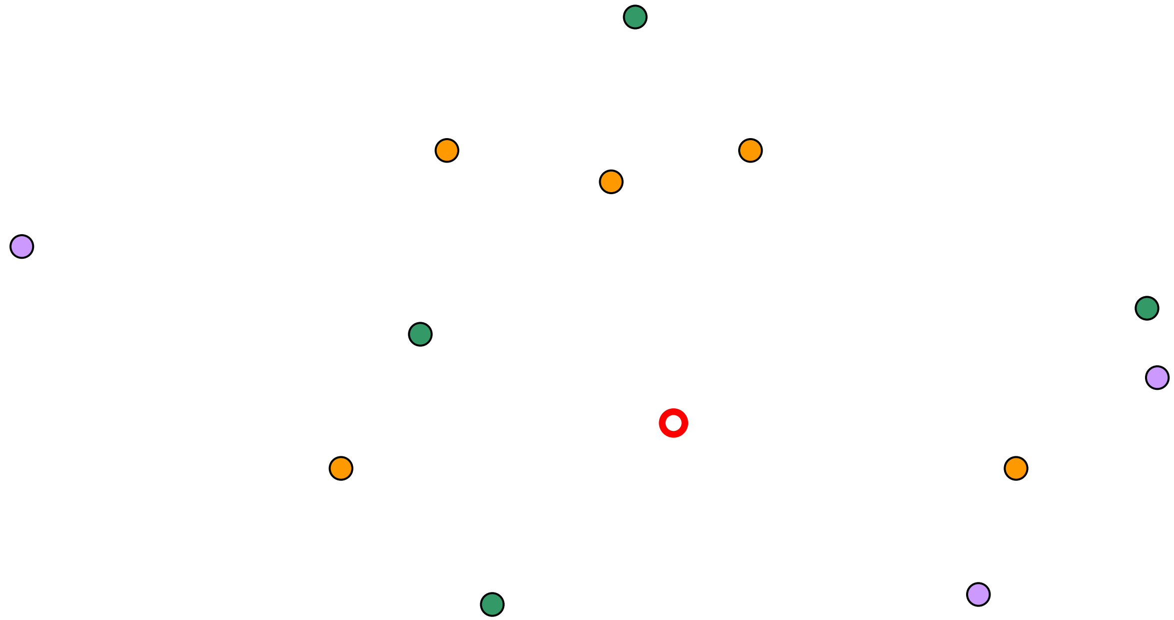
- tight upper bound
- lower bound uses Colourful Carathéodory theorem

$\Rightarrow$ ***can we improve the lower bound?***

…recent breakthrough [Gromov 2010] & further improvements

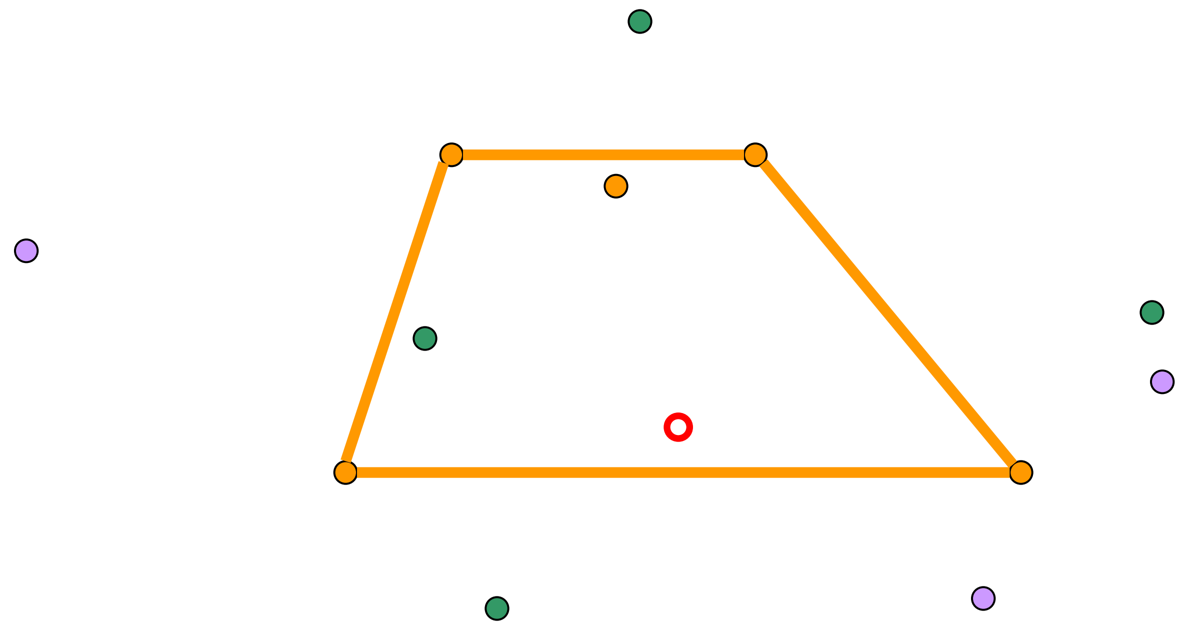
S, p general position

Colourful Carathéodory Theorem



Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension d, and $p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S_2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$, there exists a colourful simplex containing p [Bárány 1982]

Colourful Carathéodory Theorem



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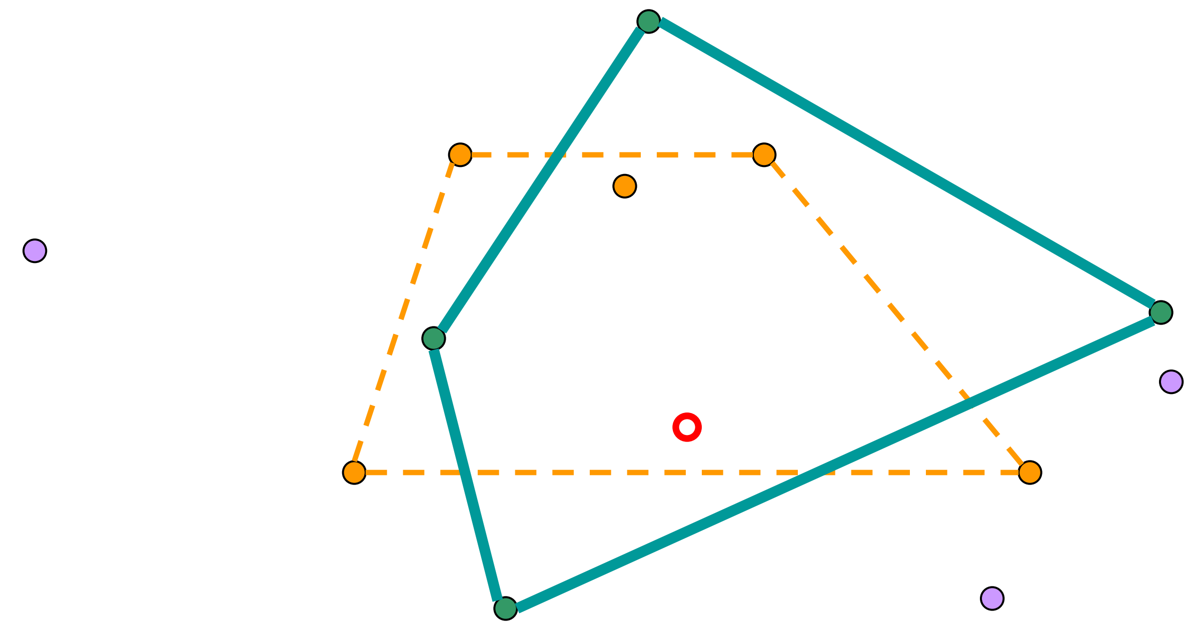

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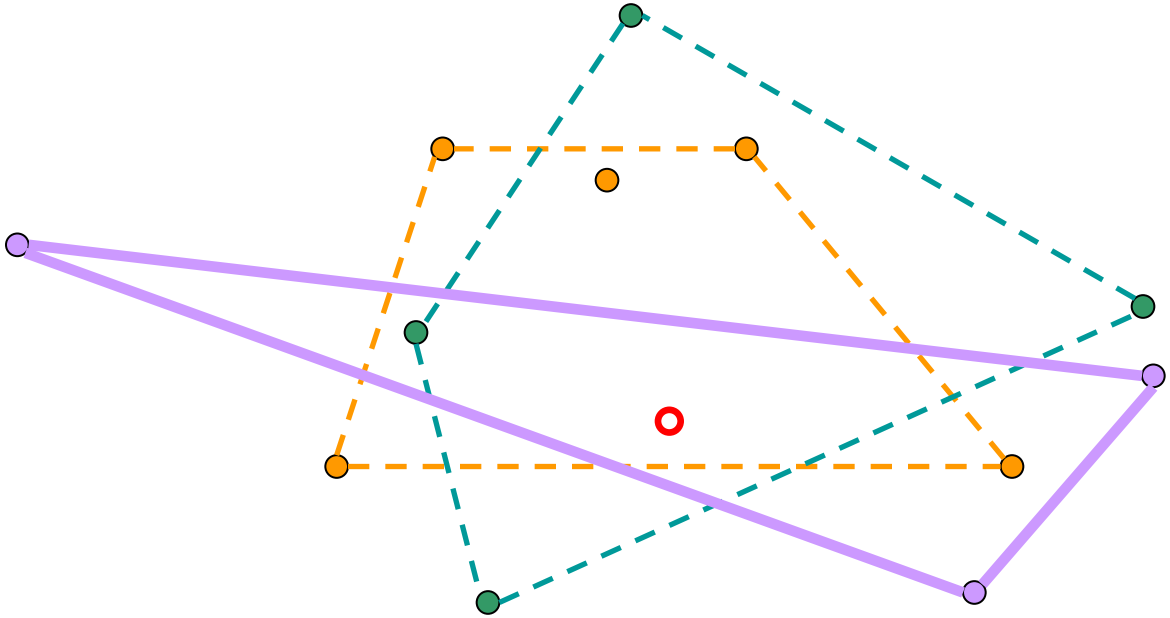

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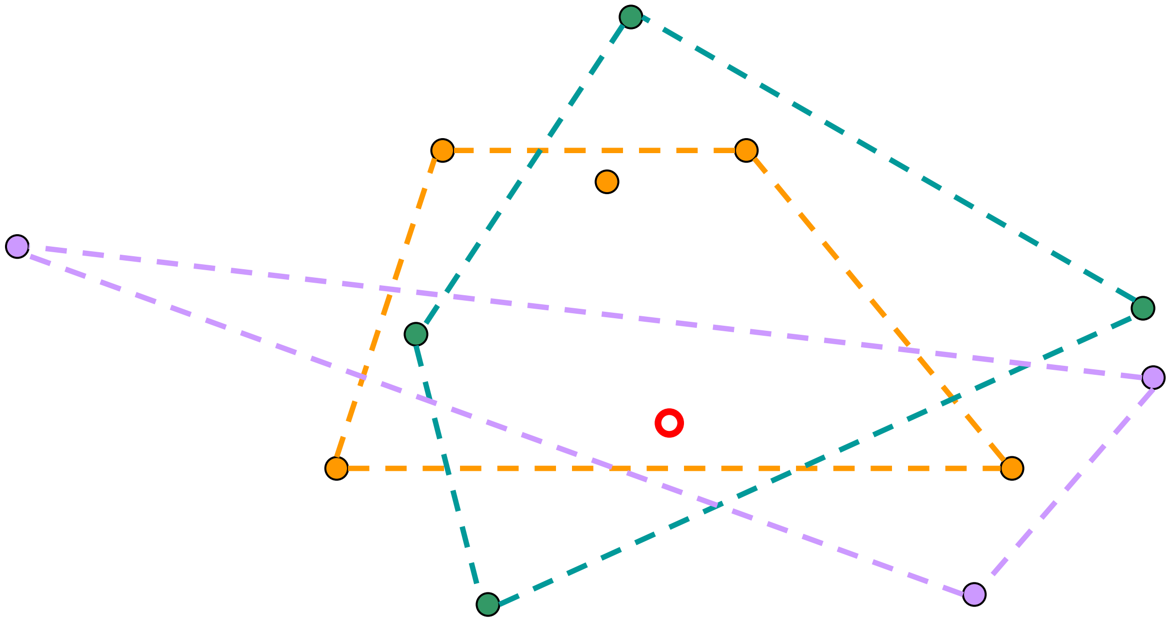

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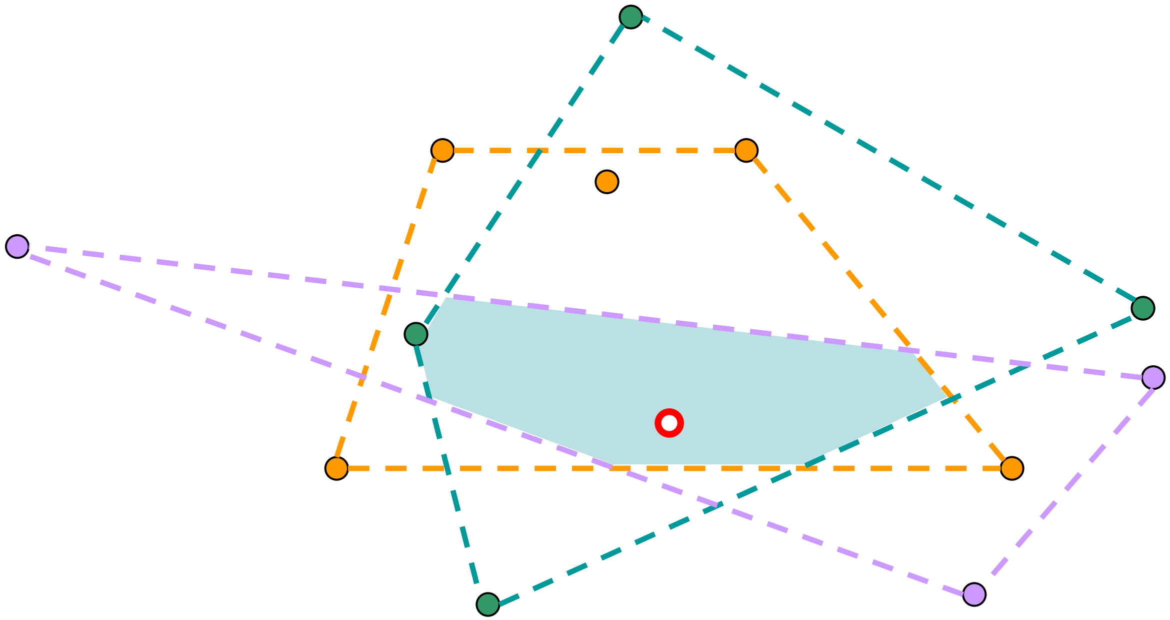

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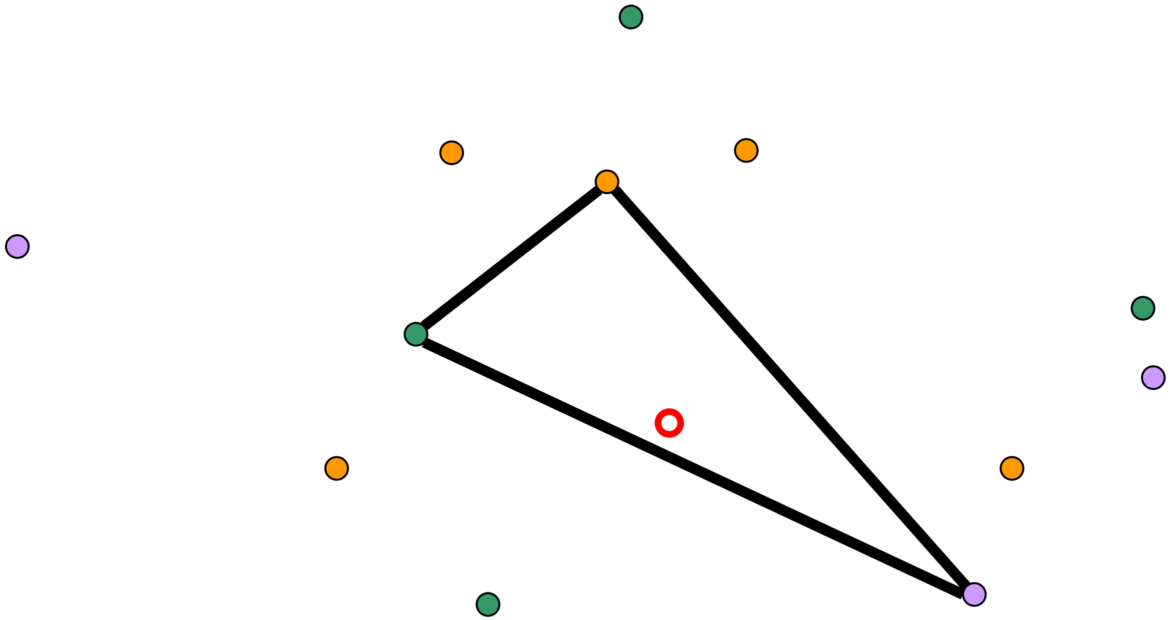
dual of Lovász Colourful Helly theorem

Colourful Simplicial Depth

$depth_S(p) = 1$



Given colourful set $S = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension d, the *colourful simplicial depth* of p is the number of open colourful simplexes generated by points in S containing p

S, p general position

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \ldots \cap \text{conv}(S_{d+1})$

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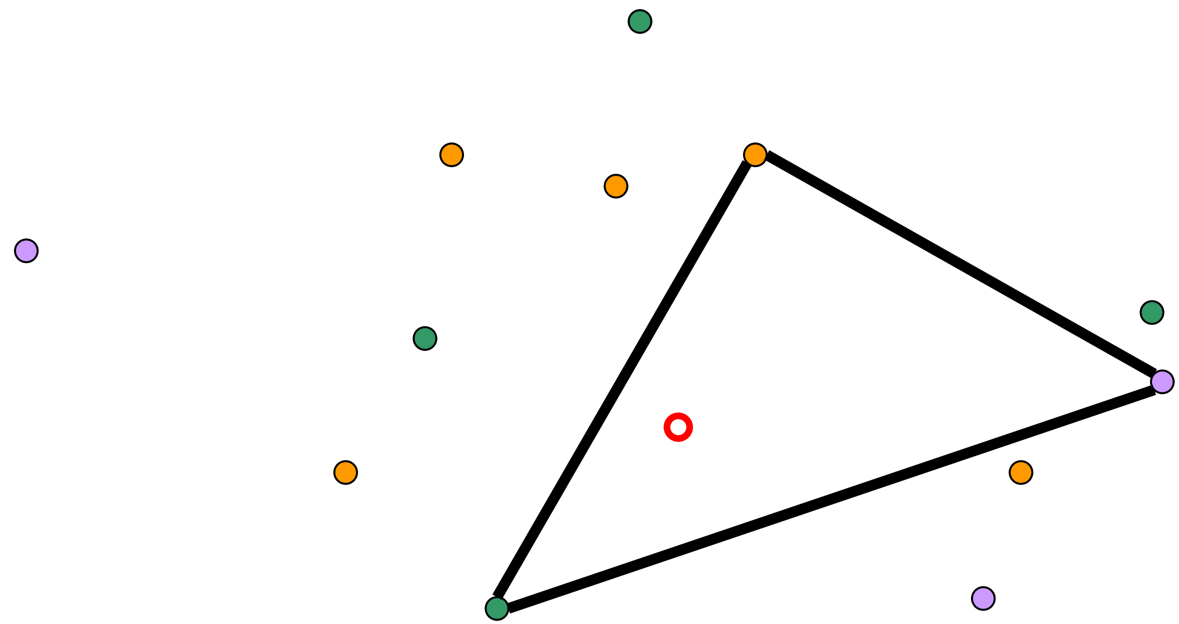

$depth_{\boldsymbol{S}}(p)=2$

Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension d, the *colourful simplicial depth* of p is the number of open colourful simplexes generated by points in $\boldsymbol{S}$ containing p

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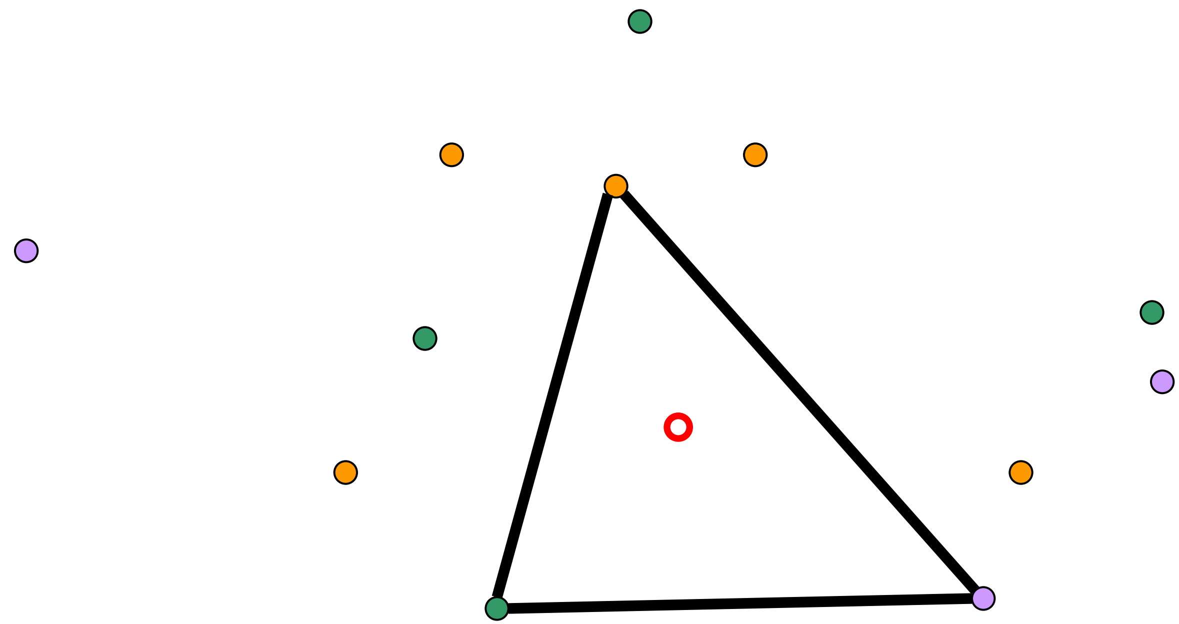

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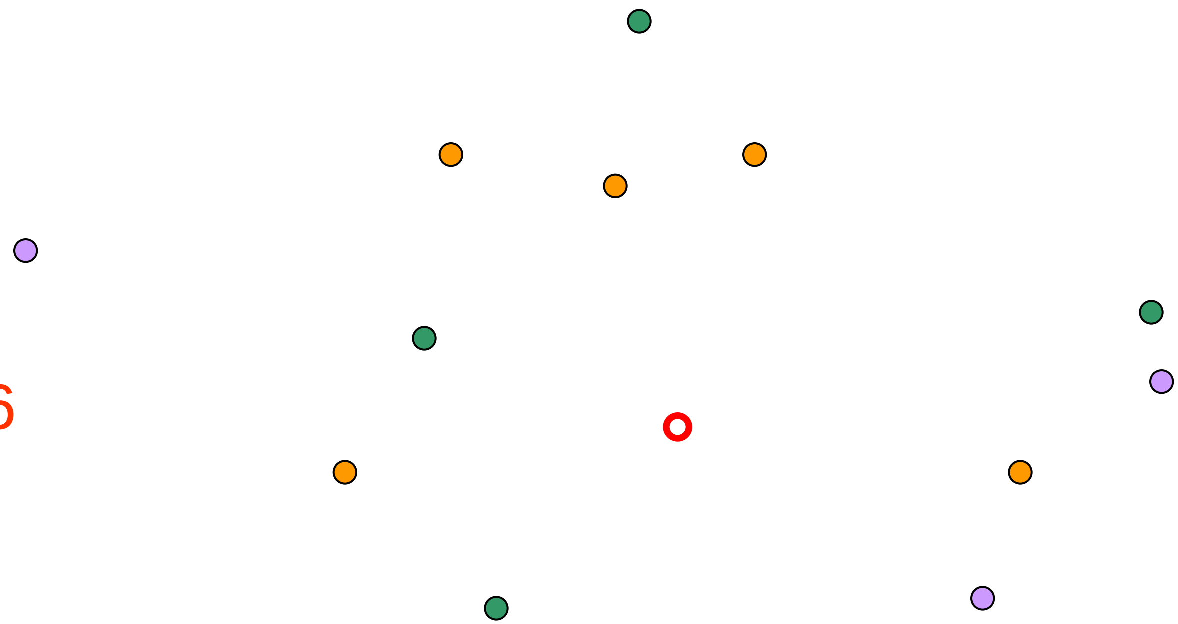
Colourful Simplicial Depth

$depth_S(p)=16$



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$$\frac{\mu(d)}{(d+1)^{d+1}}\binom{n}{d+1}+O(n^d)\leq \max_p depth_S(p)\leq \frac{1}{2^d(d+1)!}n^{d+1}+O(n^d)$$

with $\mu(d)=\min_{S,p} depth_S(p)$

[Bárány 1982]: $\mu(d)\geq 1$

…recent breakthrough [Gromov 2010] & further improvements

S, p general position

Deepest Point in Dimension d

$$\max_{p} depth_S(p) \geq c_d \binom{n}{d+1}$$

[Bárány 1982] $c_d \geq \frac{d+1}{(d+1)^{(d+1)}}$

[Wagner 2003] $c_d \geq \frac{d^2+1}{(d+1)^{(d+1)}}$

[Gromov 2010] $c_d \geq \frac{2d}{(d+1)!(d+1)}$

simpler proofs: [Karazev 2010], [Matoušek, Wagner 2011]

d=2: [Boros, Füredi 1984], [Bukh 2006]

d=3: [Král, Mach, Sereni 2011]

S, *p* *general position*

Colourful Research Directions

- Generalize the sufficient condition of Bárány for the existence of a colourful simplex
- Improve lower bound for $\mu(d) = \min_{S,p} depth_S(p)$
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S, *p general position* and $|S_1|,|S_2|,\ldots,|S_{d+1}| \geq d+1$

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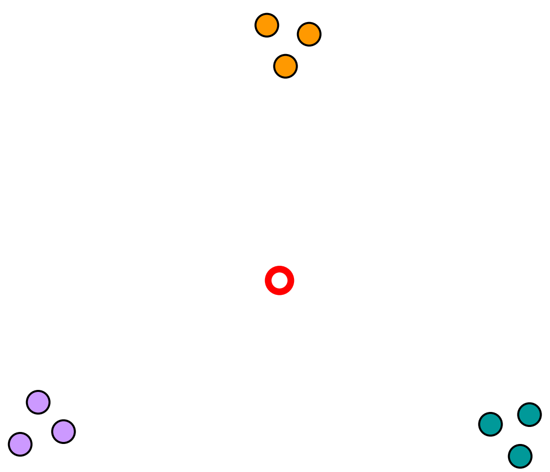
Colourful Carathéodory Theorems

[Bárány 1982] Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension d, and $p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S_2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$, then there exists a colourful simplex containing p

[Holmsen, Pach, Tverberg 2008] and [Arocha, Bárány, Bracho, Fabila, Montejano 2009] Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension d, and $p \in \mathrm{conv}(S_\mathrm{i} \cap S_\mathrm{j})$ for $1 \leq i < j \leq d+1$, then there exists a colourful simplex containing p

[Meunier, D. 2011] Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension d, if for $1 \leq i < j \leq d+1$ there exists $k \neq i$, $k \neq j$, such that for all $x_k \in S_k$ the ray $[x_k\, p)$ intersects $\mathrm{conv}(S_\mathrm{i} \cap S_\mathrm{j})$ in a point distinct from x_k, then there exists a colourful simplex containing p

Colourful Carathéodory Theorems



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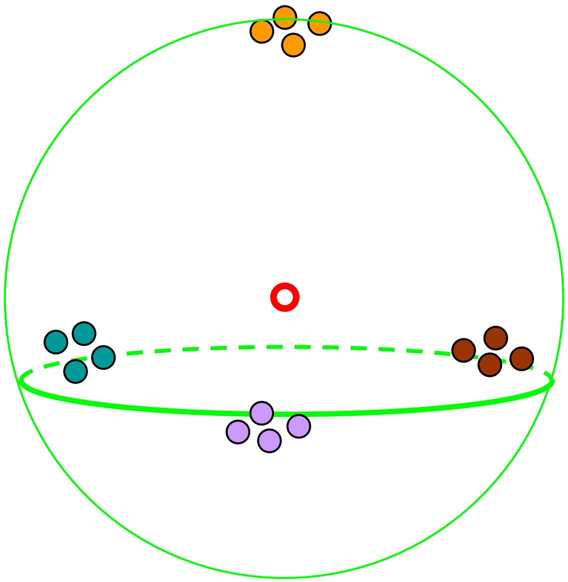

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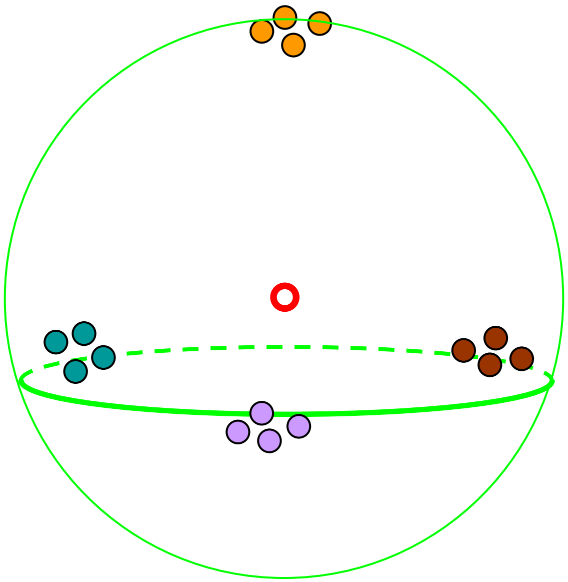
Colourful Carathéodory Theorems



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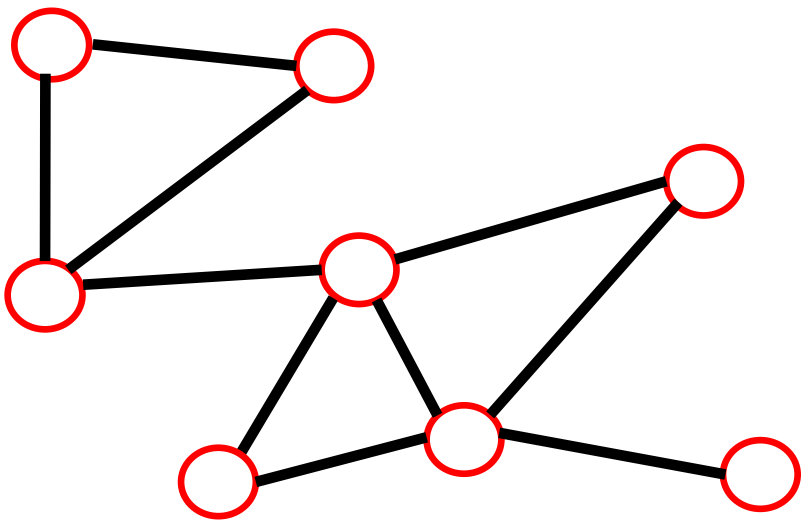
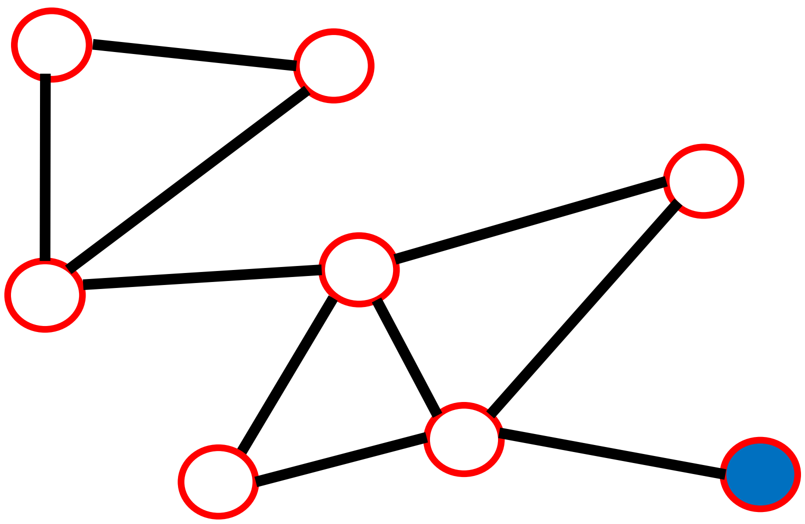
❖ *further generalization in dimension 2*

Given One, Get Another One



In a graph, if there is a vertex with an odd degree…

Given One, Get Another One



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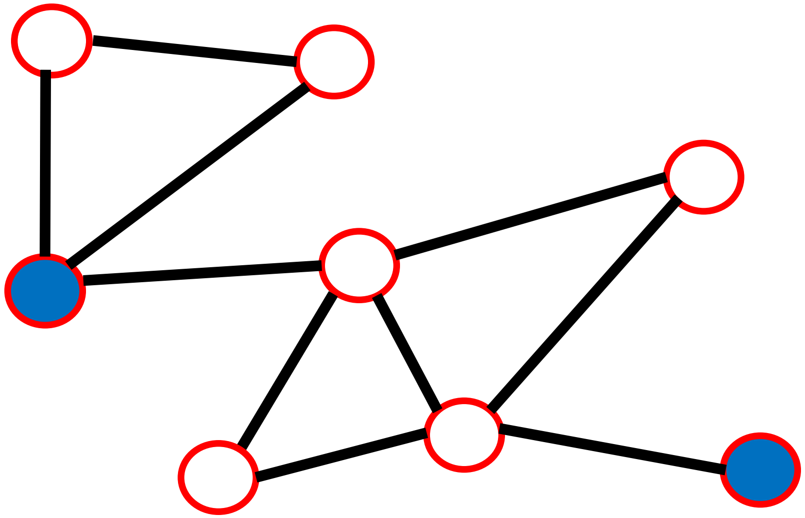

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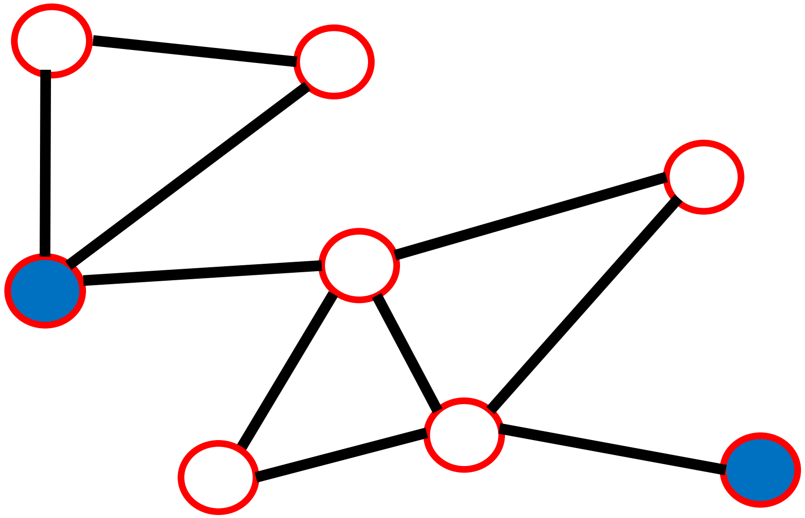

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- *Polynomial Parity Argument* PPA(D) [Papadimitriou 1994]

(Hamiltonian circuit in a cubic graph, Borsuk-Ulam, …)

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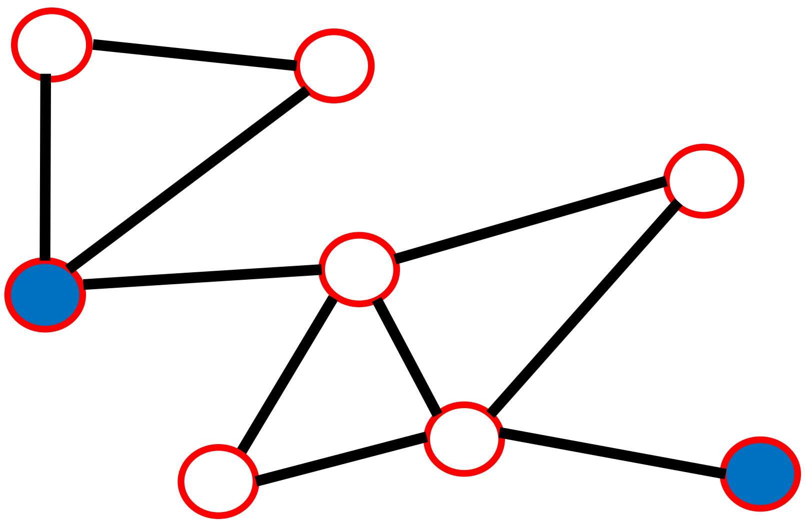

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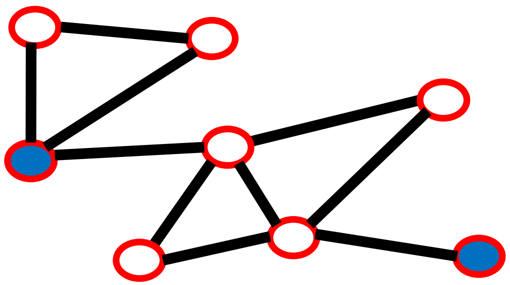
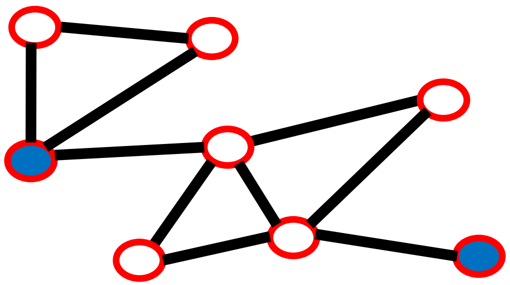


[Meunier, D. 2011] Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension d with $|S_i|=2$, if there is a colourful simplex containing p then there is another one

Given One, Get Another One



[Meunier, D. 2011] Any condition implying the existence of a colourful simplex containing p actually implies that the number of such simplices is at least d+1

$\boldsymbol{S}$, p *general position* and $|S_1|,|S_2|,\ldots,|S_{d+1}| \geq d+1$

Colourful Research Directions

- Generalize the sufficient condition of Bárány for the existence of a colourful simplex
- Improve lower bound for $\mu(d) = \min_{S,p} depth_S(p)$
- Computational approaches for $\mu(d)$ for small *d*.
- Obtain an efficient algorithm to find a colourful simplex : *Colourful Linear Programming Feasibility* problem

$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S_2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$

S, *p* *general position* and $|S_1|, |S_2|, \ldots, |S_{d+1}| \geq d+1$

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

[Bárány 1982] $$1 \leq \mu(d)$$

$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

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Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

[Bárány 1982]

$$d + 1 \leq \mu(d)$$

$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

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Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

[Bárány 1982] $d+1 \le \mu(d)$

[D.,Huang,Stephen,Terlaky 2006] $2d \le \mu(d) \le d^2+1$

$\mu(d)$ even for odd d

$$\max_p depth_S(p) \ge \frac{\mu(d)}{(d+1)^{d+1}}\binom{n}{d+1} + O(n^d)$$

$p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S_2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$

S, p *general position* and $|S_1|,|S_2|,\ldots,|S_{d+1}| \ge d+1$

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

[Bárány 1982] $d+1 \le \mu(d)$

[D.,Huang,Stephen,Terlaky 2006] $2d \le \mu(d) \le d^2+1$

$\mu(d)$ even for odd d

[Bárány, Matoušek 2007] $\max(3d, \frac{d^2+d)}{5}) \le \mu(d)$ for $d \ge 3$

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[D.,Stephen, Xie 2011] $\left\lceil \frac{(d+1)^2}{2} \right\rceil \le \mu(d) \text{ for } d \ge 4$

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S, *p* *general position* and $|S_1|, |S_2|, \ldots, |S_{d+1}| \ge d+1$

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

$$\mu(1) = 2 \qquad \mu(2) = 5 \qquad \mu(3) = 10$$

$$\left\lceil \frac{(d+1)^2}{2} \right\rceil \le \mu(d) \le d^2 + 1 \quad \text{for } d \ge 4$$

$$\mu(d) \text{ even for odd } d$$

$$\text{conjecture: } \mu(d) = d^2 + 1$$

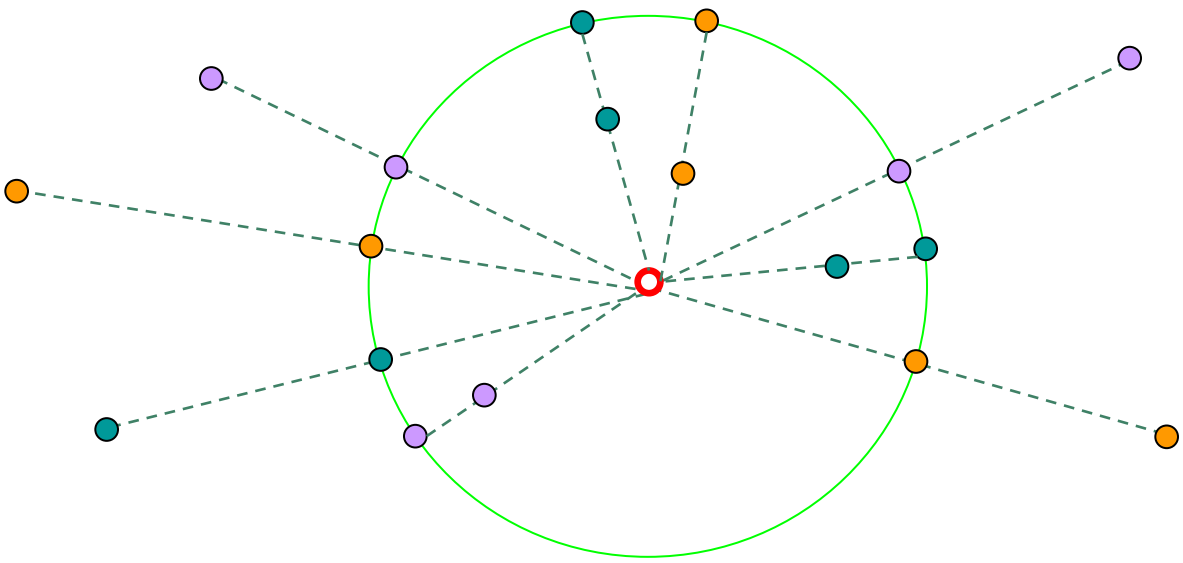
$$\max_p depth_S(p) \ge \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \ldots \cap \text{conv}(S_{d+1})$

S, *p* *general position* and $|S_1|, |S_2|, \ldots, |S_{d+1}| \ge d+1$

Normalization

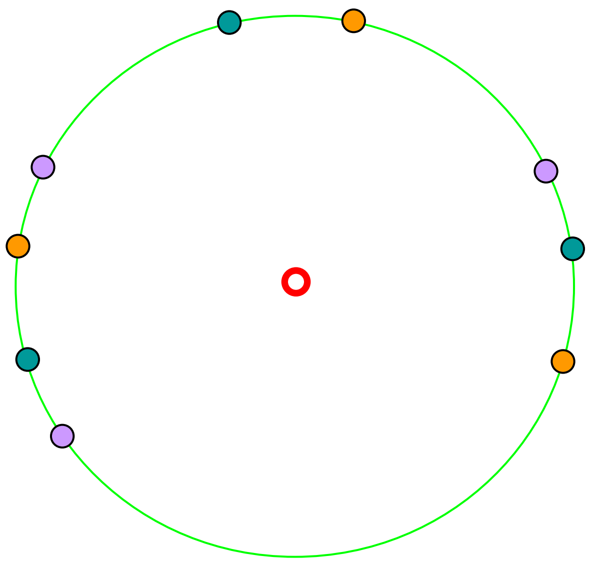
$$\mu(d) = \min_{S,p} depth_S(p)$$



assume p is the origin, scale the points to the unit sphere (perturb if degeneracy)

Normalization

$$\mu(d) = \min_{S,p} depth_S(p)$$

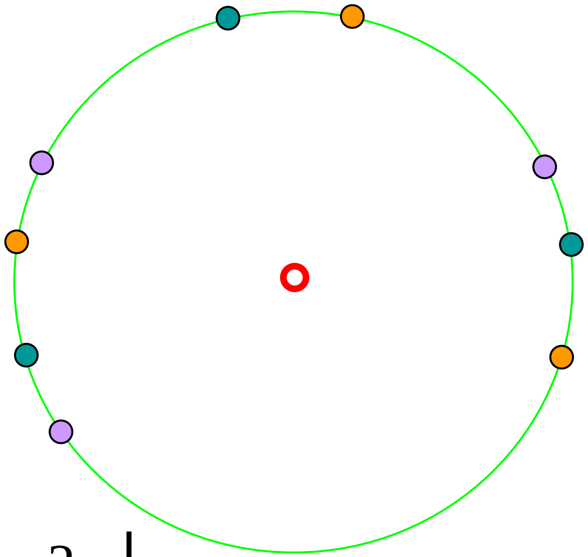


scale the points to the unit sphere

$d+1$ points for each color (convex hull contain origin p)

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$



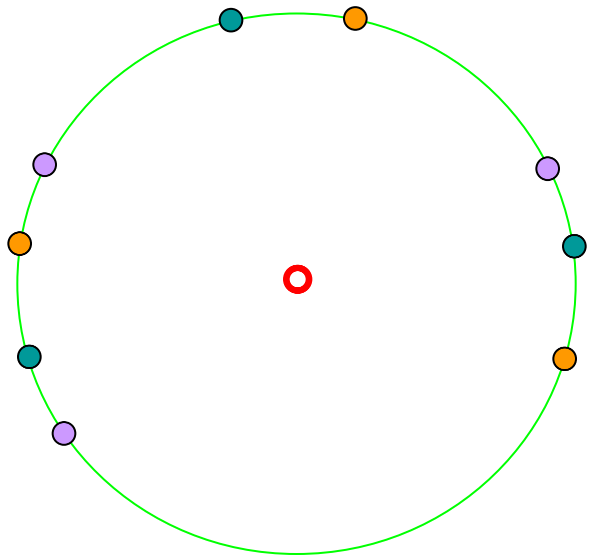
$$\left\lfloor \frac{(d+1)^2}{2} \right\rfloor \le \mu(d) \le d^2 + 1$$

$p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S_2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$

S, *p* *general position* and $|S_1|, |S_2|, \ldots, |S_{d+1}| \ge d+1$

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$



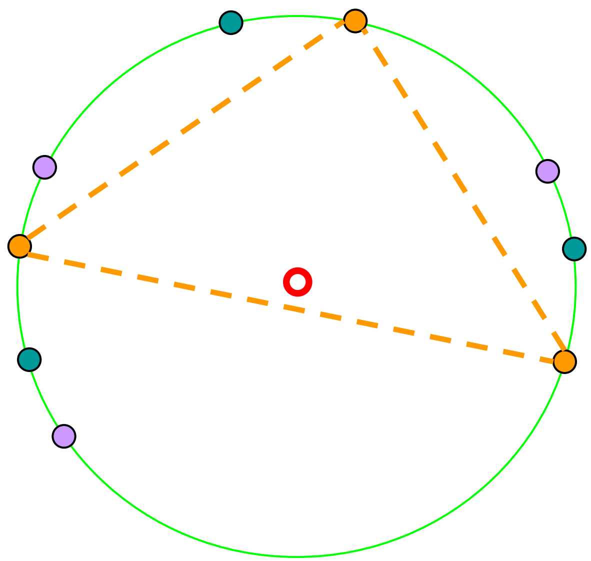
$$5 \leq \mu(2) \leq 5$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \ldots \cap \text{conv}(S_{d+1})$

S, p *general position* and $|S_1|,|S_2|,\ldots,|S_{d+1}| \geq d+1$

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$



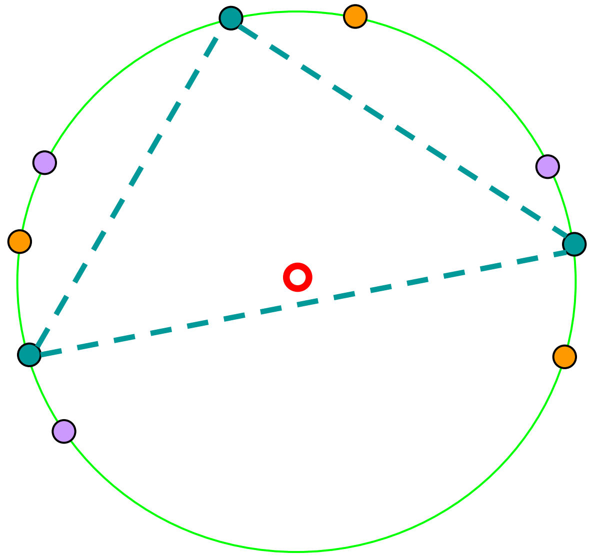
$$\mu(2) = 5$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \ldots \cap \text{conv}(S_{d+1})$

$\mathbf{S}$, p *general position* and $|S_1|,|S_2|,\ldots,|S_{d+1}| \geq d+1$

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$



$$\mu(2) = 5$$

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Colourful Simplicial Depth Bounds

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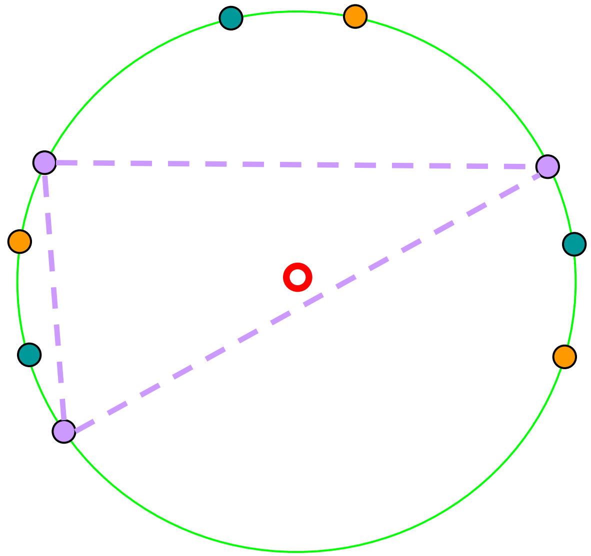

$$\mu(2) = 5$$

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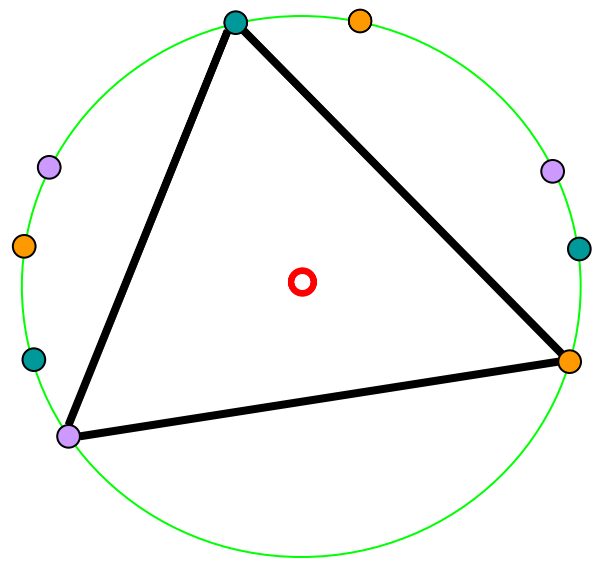
S, *p* *general position* and $|S_1|, |S_2|, \ldots, |S_{d+1}| \geq d+1$

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

$depth_S(p) = 1$



$$\mu(2) = 5$$

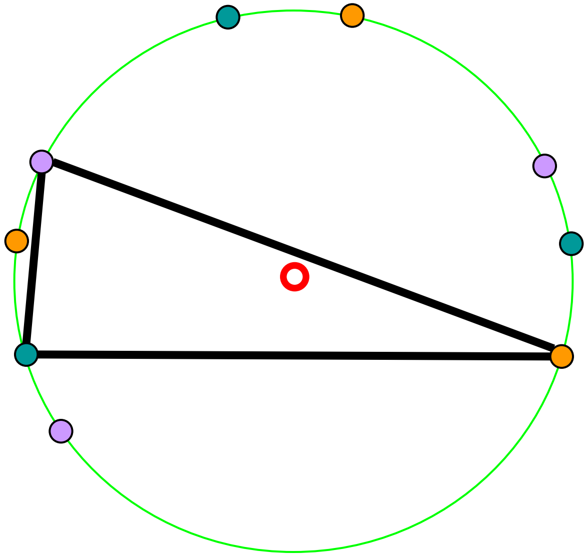
$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \ldots \cap \text{conv}(S_{d+1})$

$\boldsymbol{S}$, p *general position* and $|S_1|, |S_2|, \ldots, |S_{d+1}| \geq d+1$

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

$depth_S(p)=2$



$$\mu(2) = 5$$

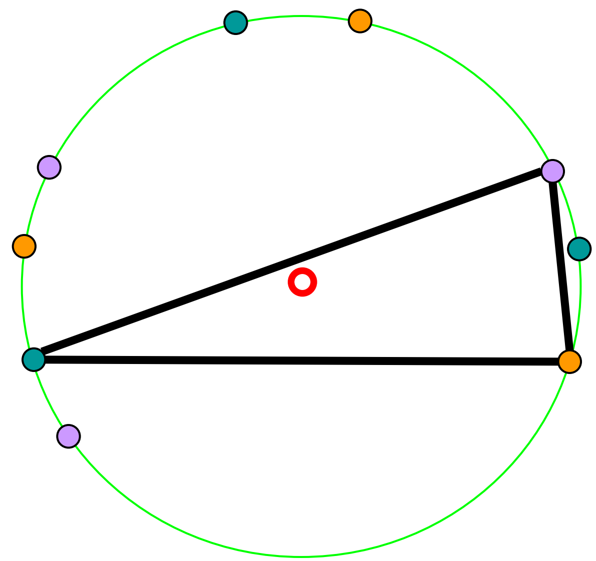
$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \ldots \cap \text{conv}(S_{d+1})$

S, p *general position* and $|S_1|,|S_2|,\ldots,|S_{d+1}| \geq d+1$

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

$depth_S(p)=3$



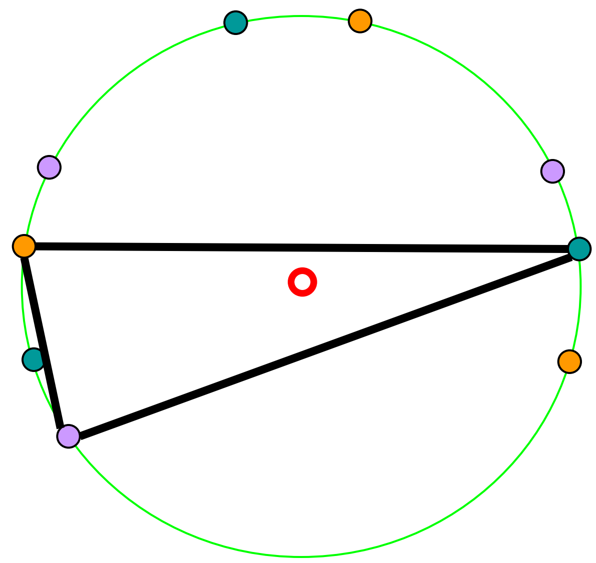
$$\mu(2) = 5$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \ldots \cap \text{conv}(S_{d+1})$
S, p *general position* and $|S_1|,|S_2|,\ldots,|S_{d+1}| \geq d+1$

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

$depth_S(p)=4$



$$\mu(2) = 5$$

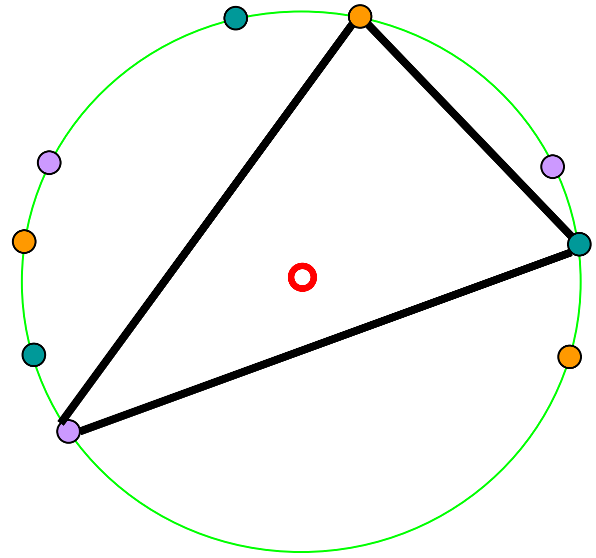
$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \ldots \cap \text{conv}(S_{d+1})$

S, p *general position* and $|S_1|,|S_2|,\ldots,|S_{d+1}| \geq d+1$

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

$depth_S(p) = 5$



$$\mu(2) = 5$$

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S, *p general position* and $|S_1|, |S_2|, \ldots, |S_{d+1}| \geq d+1$

Colourful Simplicial Depth Bounds

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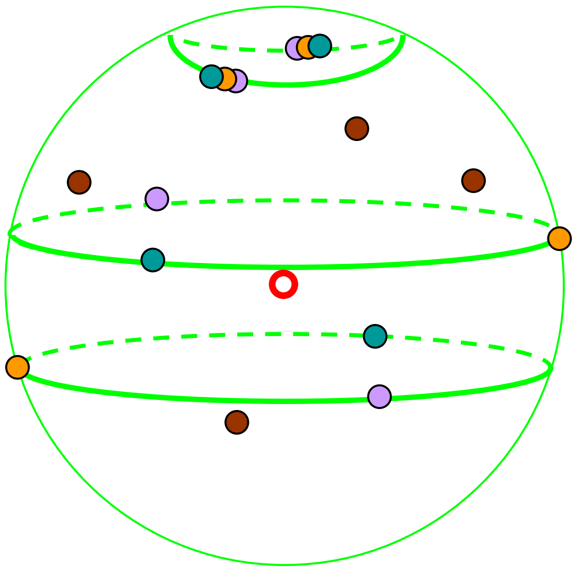

$$3d \le \mu(d) \le d^2 + 1$$

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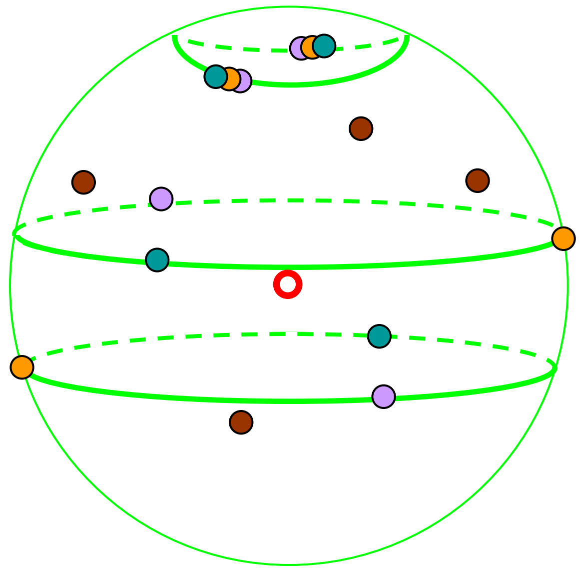


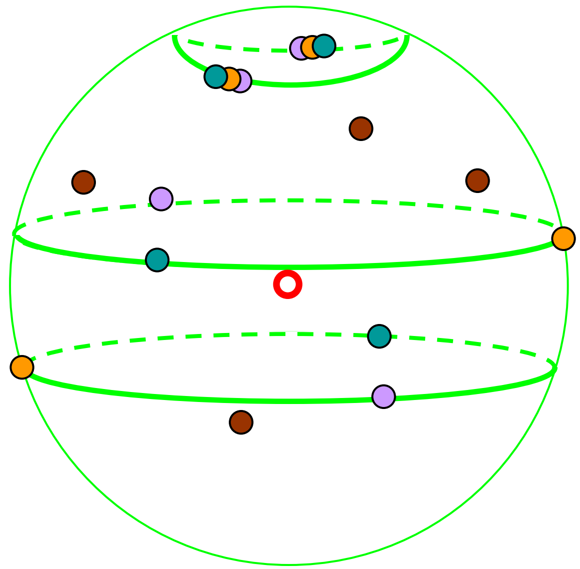
$$9 \leq \mu(3) \leq 10$$

$p \in conv(S_1) \cap conv(S_2) \cap \ldots \cap conv(S_{d+1})$

S, p *general position* and $|S_1|, |S_2|, \ldots, |S_{d+1}| \geq d+1$

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S, p} depth_S(p)$$



$$9 \le \mu(3) \le 10$$

$$\mu(d) \text{ even for odd } d$$

$p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S_2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$

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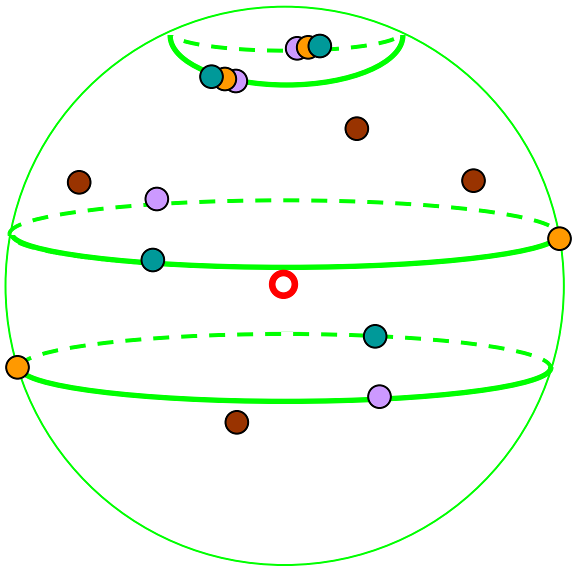

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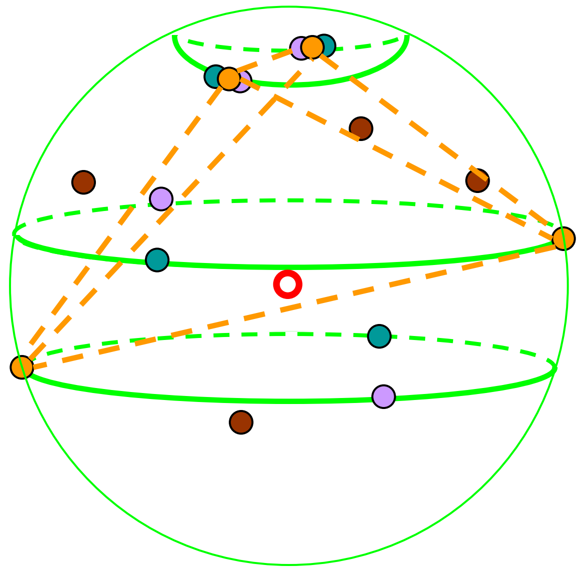

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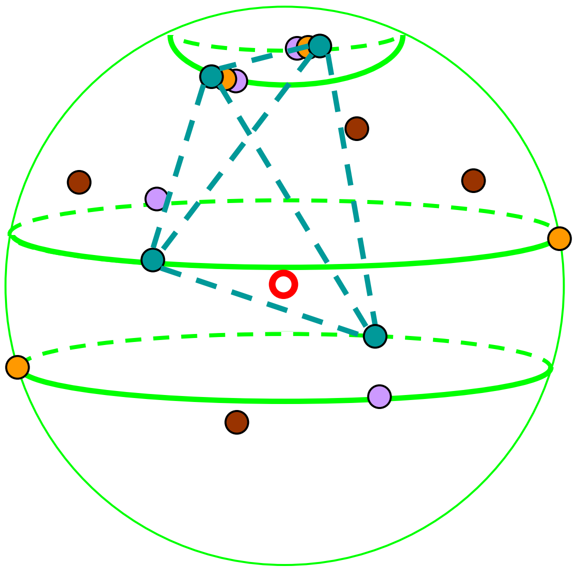

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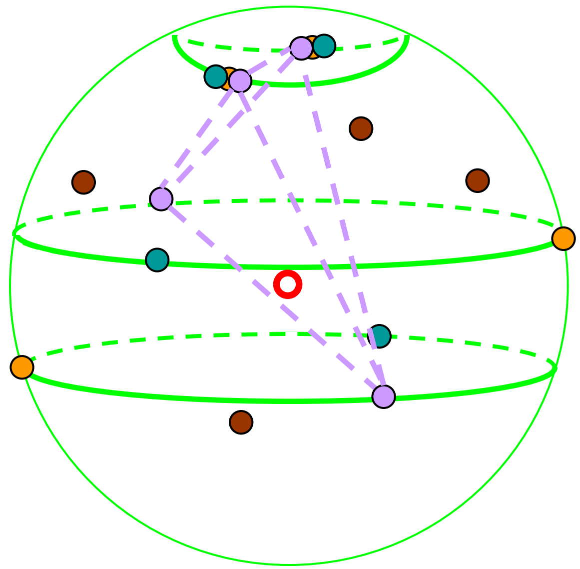

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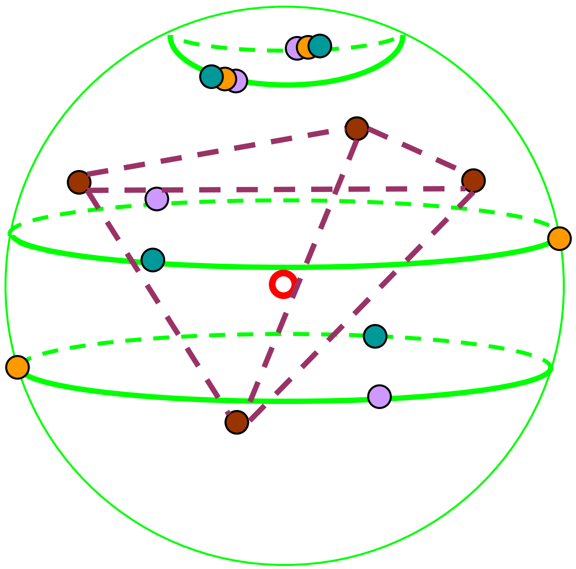

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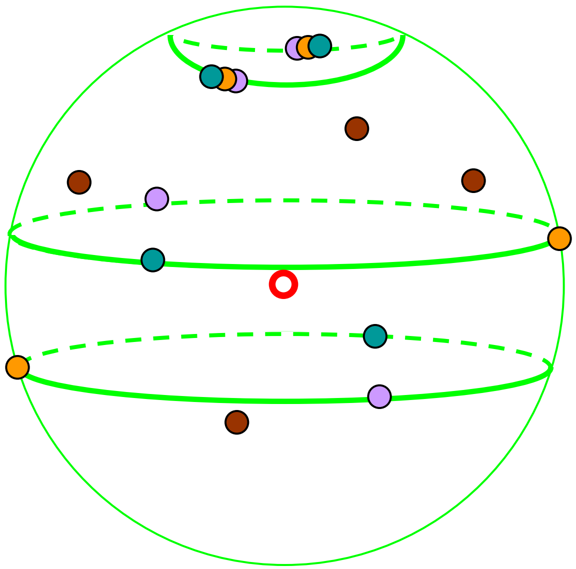
Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

$depth_S(p) = 10$



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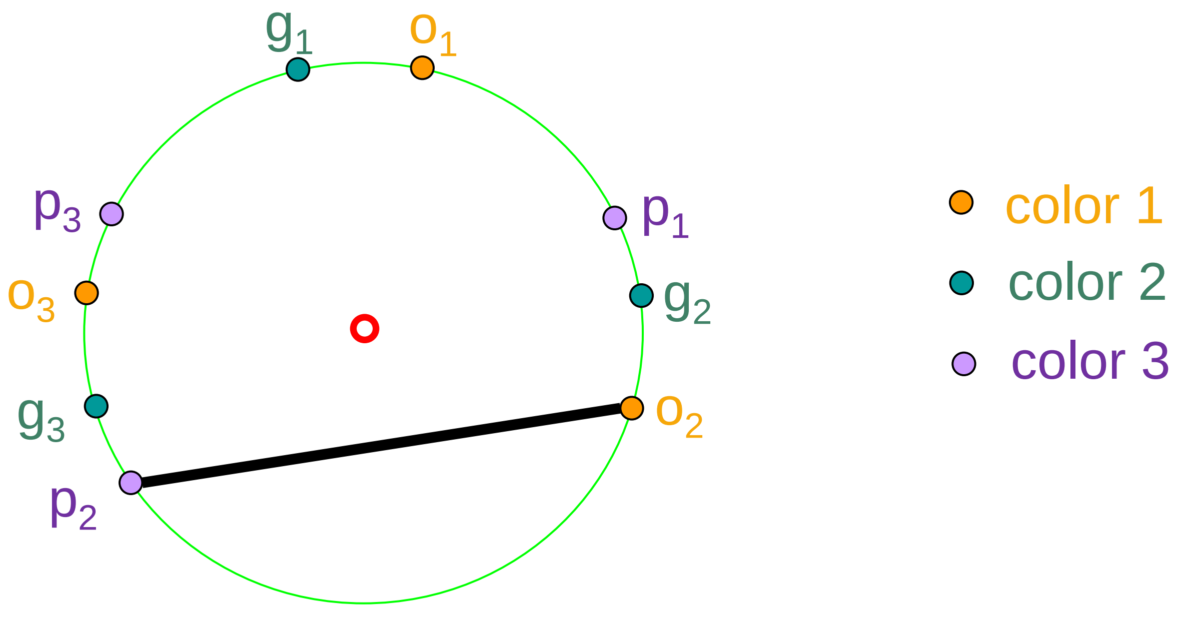
S, p *general position* and $|S_1|, |S_2|, \ldots, |S_{d+1}| \geq d+1$

Transversal

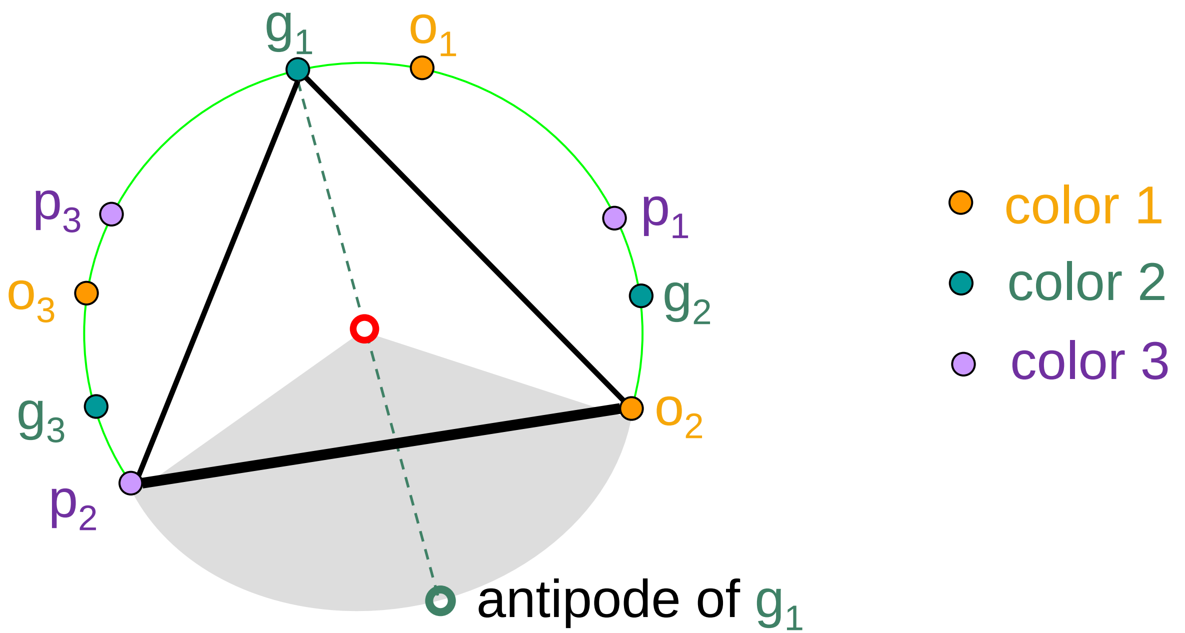
colourful set of *d* points (one colour missing)



$\hat{2}$-transversal (o_2,p_2)

Transversal

colourful set of d points (one colour missing)

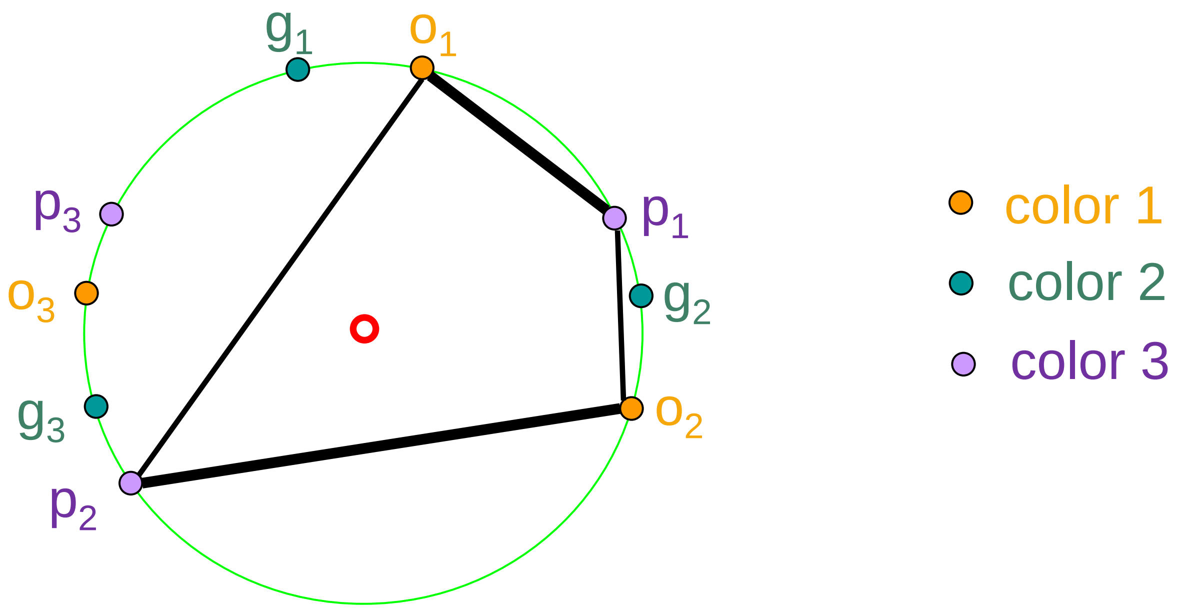


$\hat{2}$-transversal (o_2,p_2) *spans* the antipode of g_1

iff (o_2,p_2,g_1) is a colourful simplex

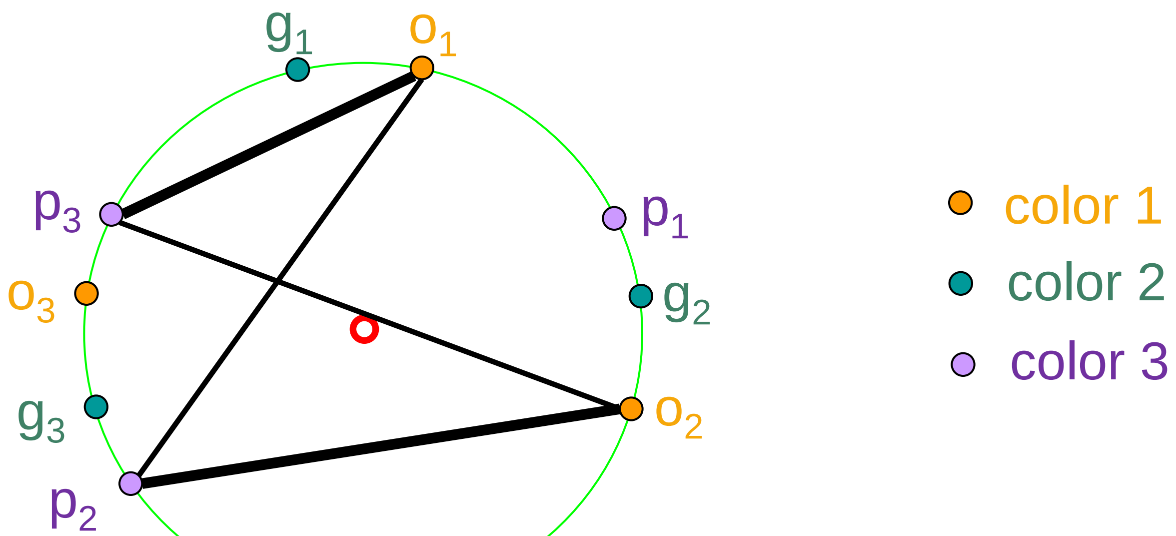
Combinatorial (topological) Octahedra

pair of disjoint $\hat{i}$-transversals



octahedron $[(o_1,p_1),(o_2,p_2)]$

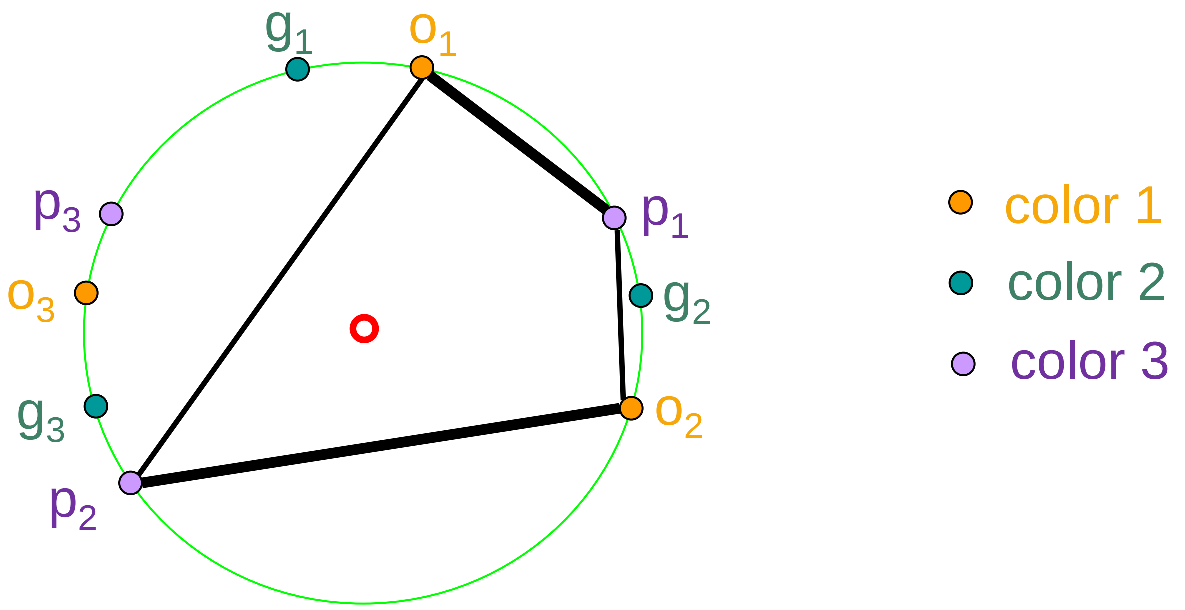
Combinatorial (topological) Octahedra



octahedron [(o_1,p_3),(o_2,p_2)]

Octahedron Lemma

origin-containing octahedra

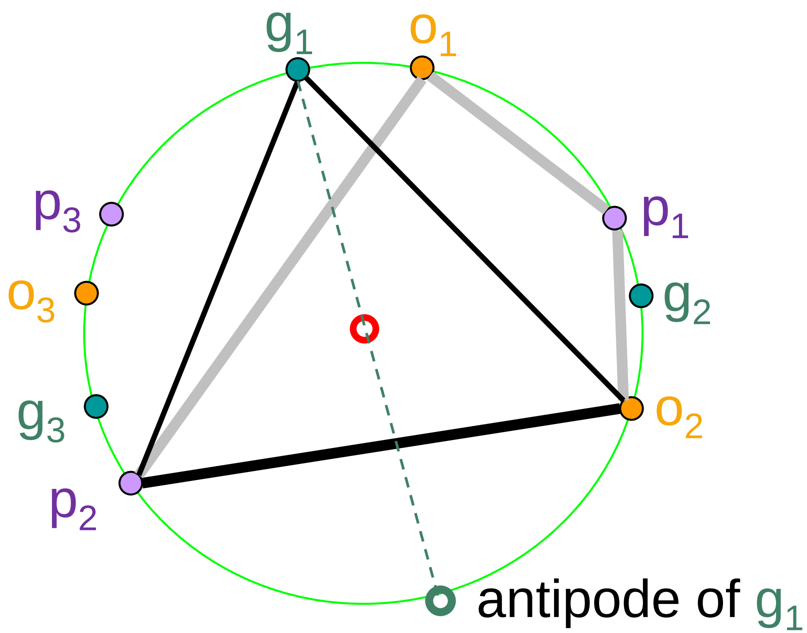


octahedron $[(o_1,p_1),(o_2,p_2)]$

2^d colourful faces span the whole sphere if it contains the origin (creating d+1 colourful simplexes)

Octahedron Lemma

origin-containing octahedron



Colourful simplexes:

(g_1, o_2, p_2)

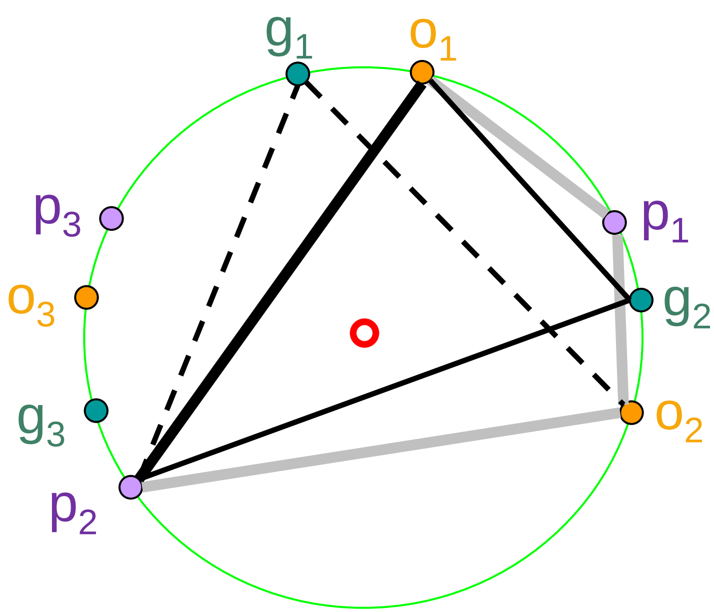
octahedron $[(o_1,p_1),(o_2,p_2)]$

2^d colourful faces span the sphere if it contains *p*

$\Rightarrow$ creating *d*+1 colourful simplexes

Octahedron Lemma

origin-containing octahedron



Colourful simplexes:

(g_1, o_2, p_2)

(g_2, o_1, p_2)

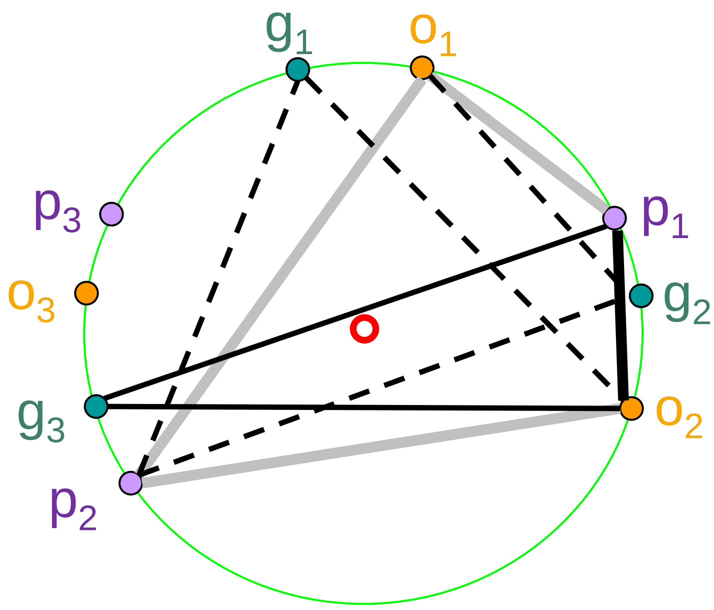
octahedron $[(o_1, p_1), (o_2, p_2)]$

2^d colourful faces span the sphere if it contains *p*

$\Rightarrow$ creating d+1 colourful simplexes

Octahedron Lemma

origin-containing octahedron



Colourful simplexes:

(g_1, o_2, p_2)

(g_2, o_1, p_2)

(g_3, o_2, p_1)

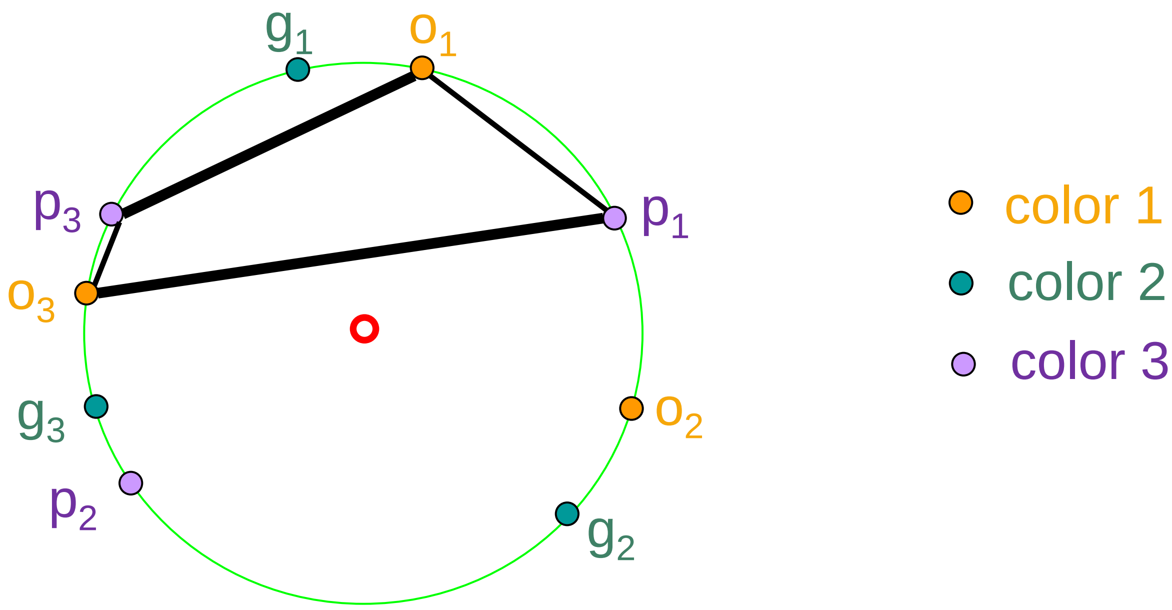
octahedron $[(o_1, p_1), (o_2, p_2)]$

2^d colourful faces span the sphere if it contains p

$\Rightarrow$ creating d+1 colourful simplexes

Octahedron Lemma

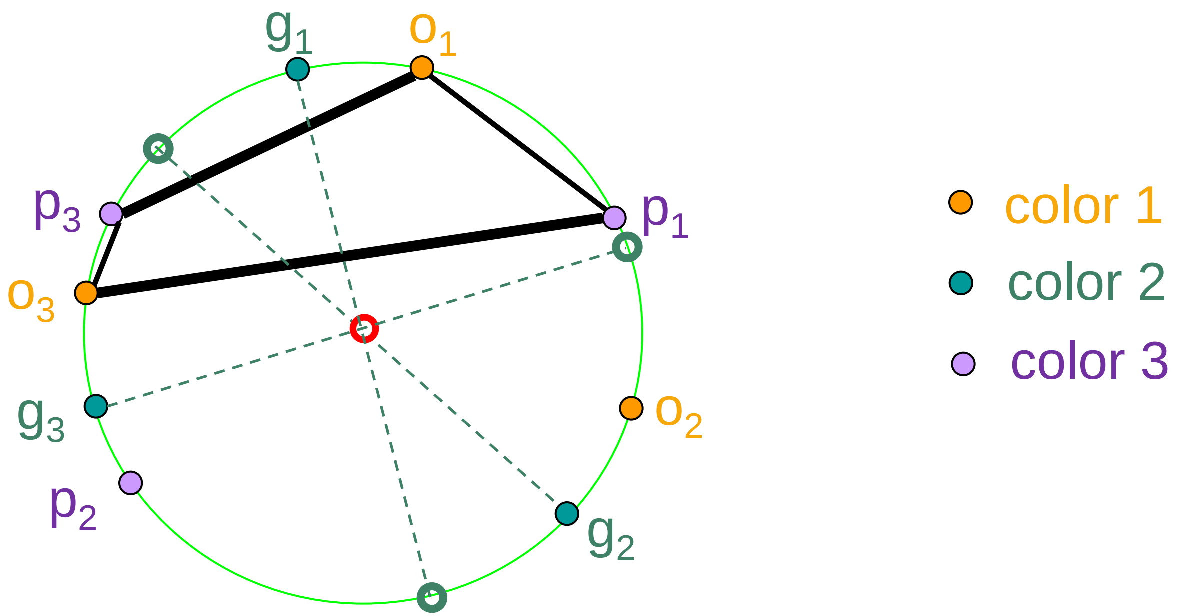
octahedron not containing the origin



octahedron [(o_1,p_3),(o_3,p_1)] does not contain *p*

Octahedron Lemma

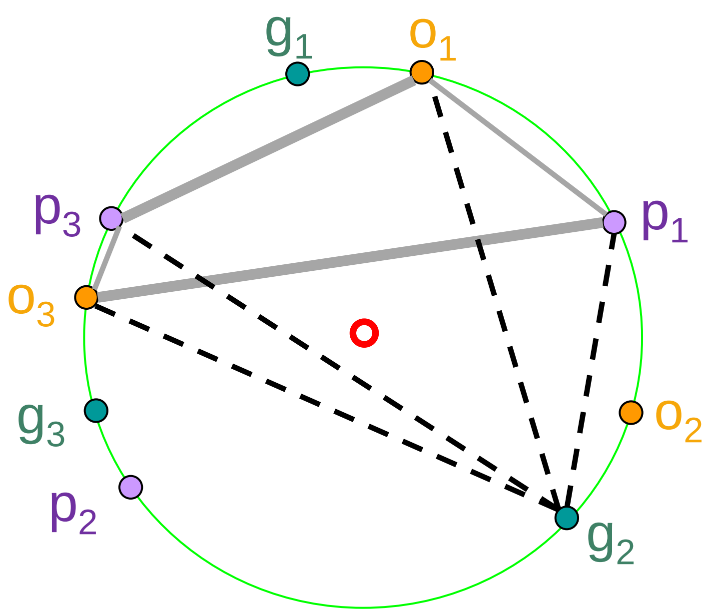
octahedron not containing the origin



octahedron [(o_1,p_3),(o_3,p_1)] spans any antipode an *even* number of times

Octahedron Lemma

octahedron not containing the origin



Colourful simplexes:

(g_2,o_1,p_3)

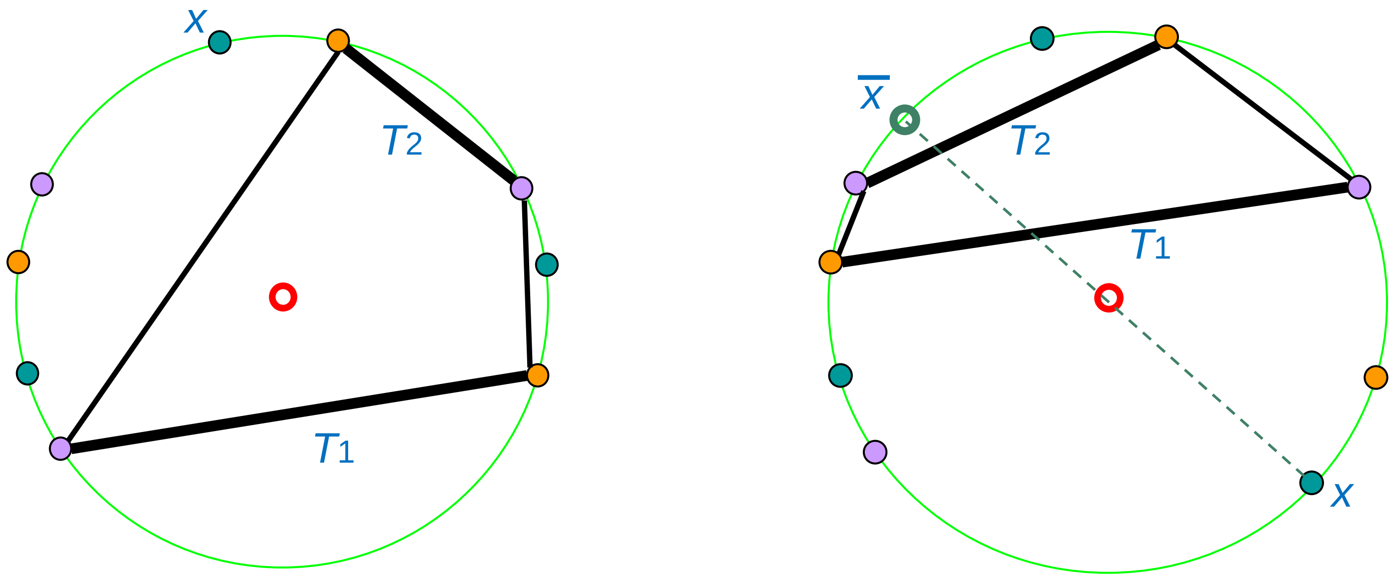
(g_2,o_3,p_1)

octahedron [(o_1,p_3),(o_3,p_1)] spans any antipode an even number of times $\Rightarrow$ creating an even number of colourful simplexes (might be zero)

Octahedron Lemma

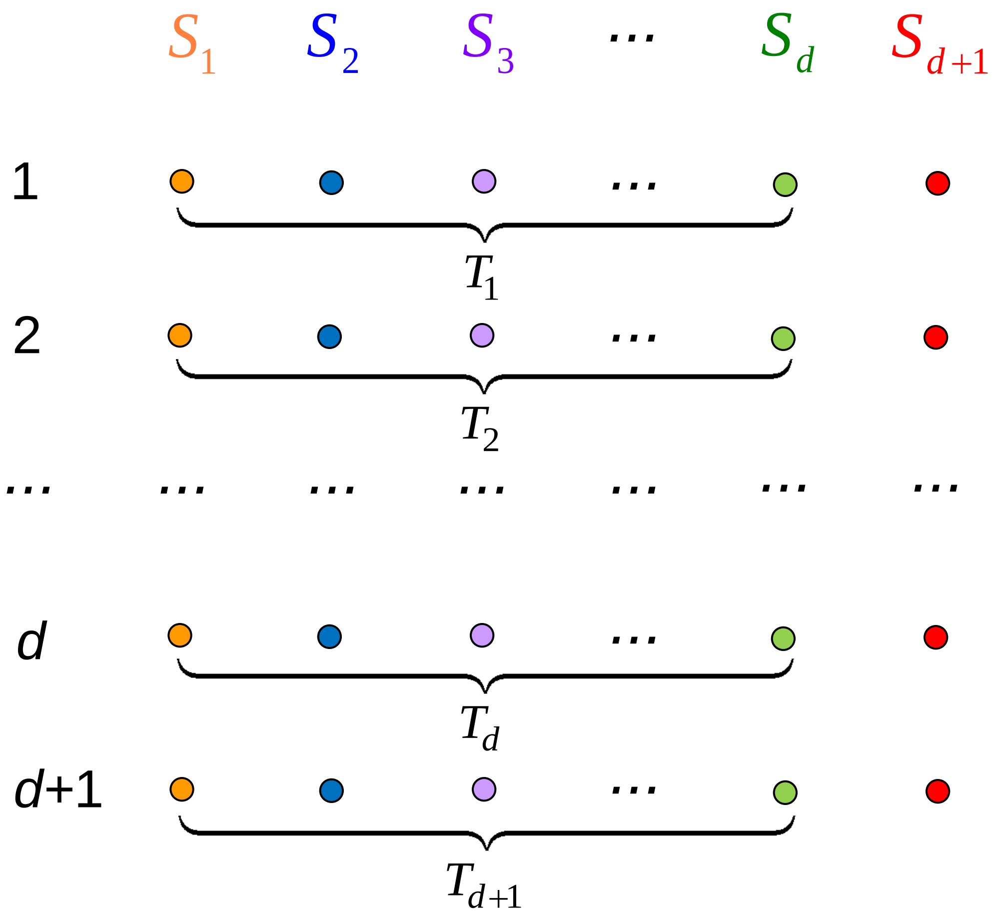
Given 2 disjoint transversals T_1 and T_2, and T_1 spans $\bar{x}$ *(antipode of* x*),*

- either octahedron (T_1,T_2) contains p,
- or there exists a transversal $T \neq T_1$ consisting of points from T_1 and T_2 that spans $\bar{x}$.



Sketch of the Proof

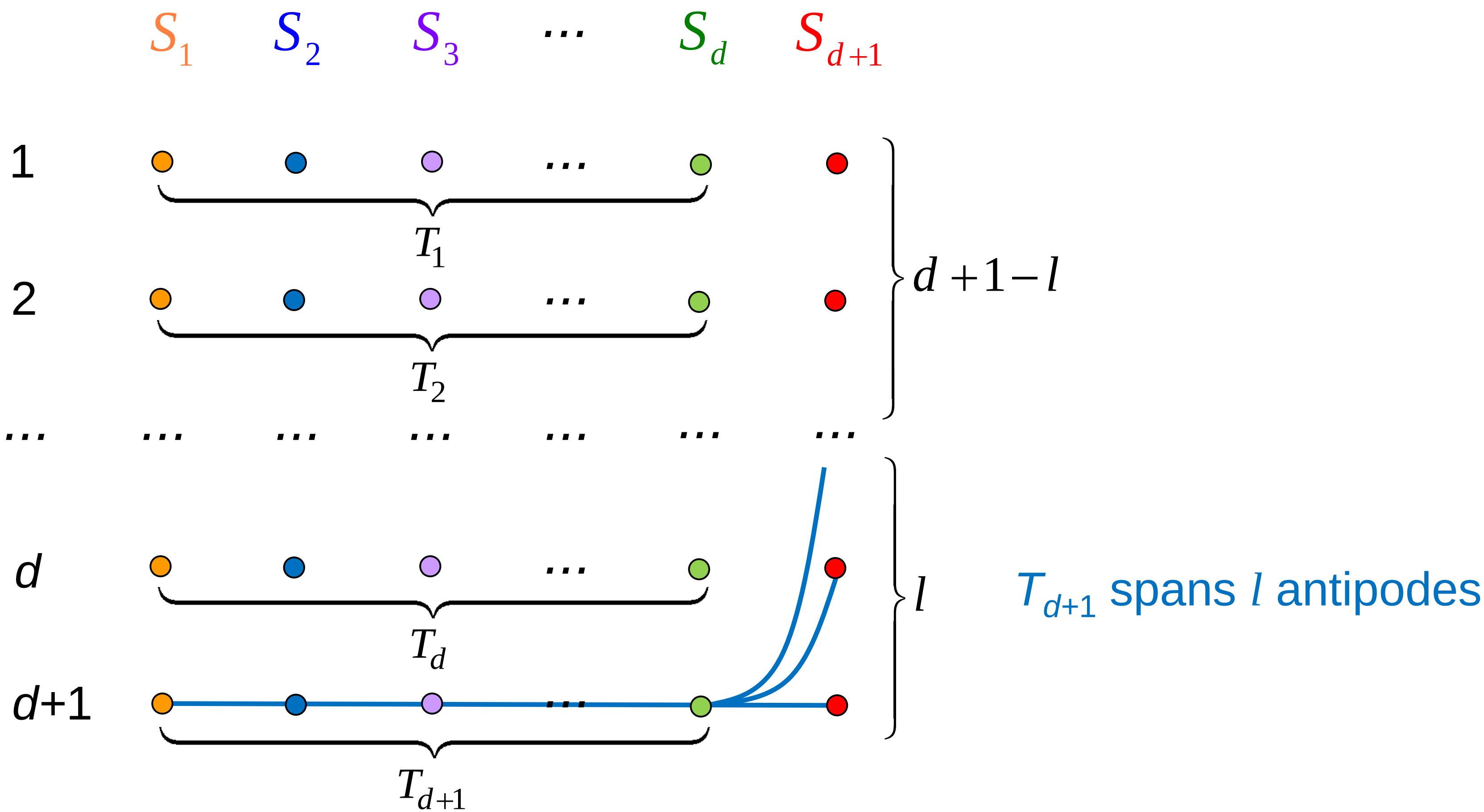
$$\left\lceil \frac{(d+1)^2}{2} \right\rceil \le \mu(d)$$



$$\mu(d) \ge ?$$

Sketch of the Proof

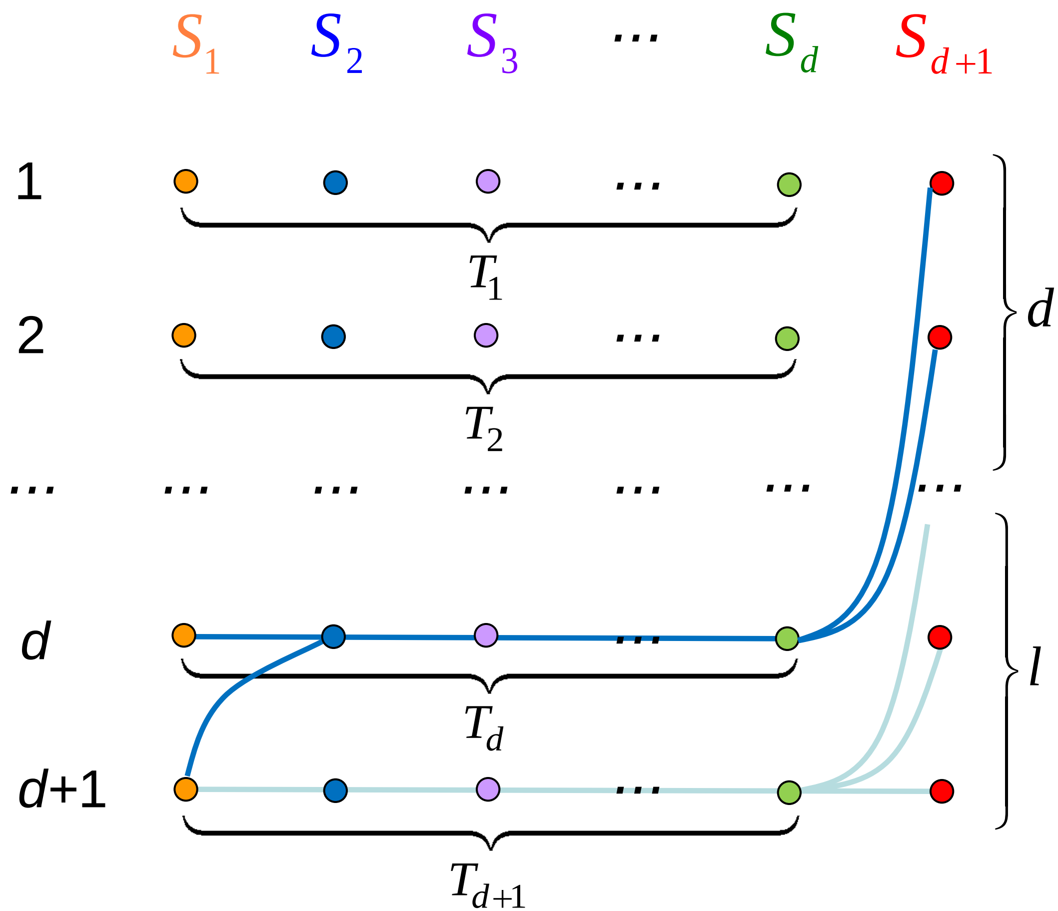
$$\left\lceil \frac{(d+1)^2}{2} \right\rceil \le \mu(d)$$



$$\mu(d) \ge l \ge 1$$

Sketch of the Proof

$$\left\lceil \frac{(d+1)^2}{2} \right\rceil \le \mu(d)$$



consider the d octahedra:

(T_1,T_{d+1}), (T_2,T_{d+1}), ..., (T_d,T_{d+1})

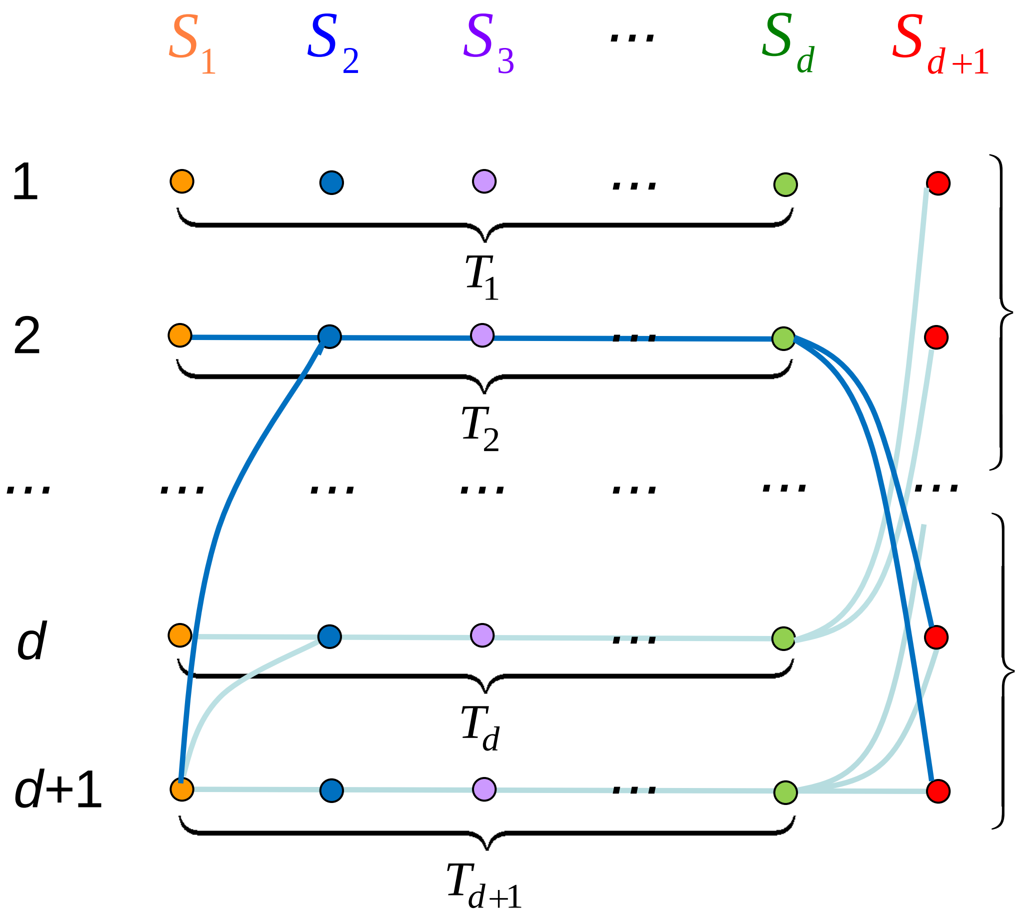
among them:

- b contain p, each generate $d+1-l$ new colourful simplexes.

$$\mu(d) \ge l + b(d+1-l)$$

Sketch of the Proof

$$\left\lceil \frac{(d+1)^2}{2} \right\rceil \le \mu(d)$$



consider the d octahedra:

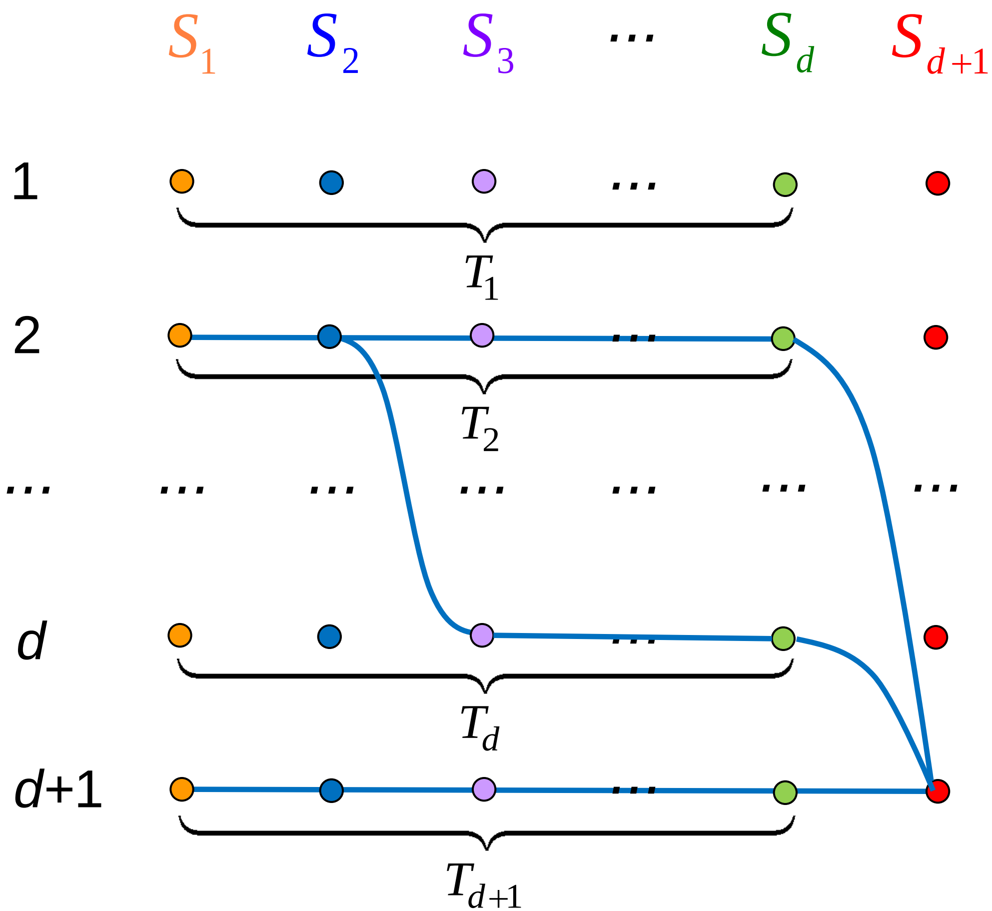
(T_1,T_{d+1}), (T_2,T_{d+1}), ..., (T_d,T_{d+1})

among them:

- b contain p, each generates $d+1-l$ new colourful simplexes.
- $d-b$ do not contain p, each generates l new colourful simplexes (Octahedron Lemma)

$$\mu(d) \ge l + b(d+1-l) + (d-b)l$$

Sketch of the Proof $\left\lceil \frac{(d+1)^2}{2} \right\rceil \le \mu(d)$



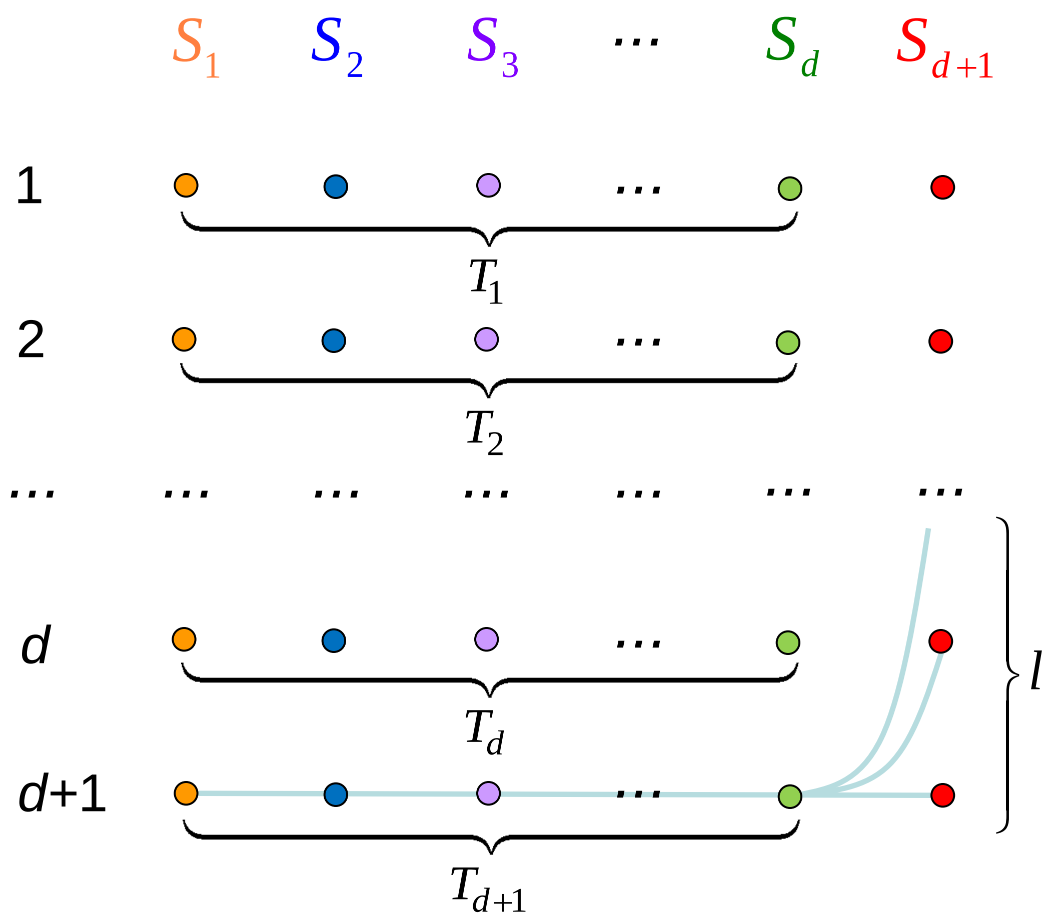
another counting (point-wise):

each antipode from S_{d+1} is spanned by at least j transversals

$$\mu(d) \ge \max\{l + b(d+1-l) + (d-b)l,\ j(d+1)\}$$

Sketch of the Proof

$$\left\lceil \frac{(d+1)^2}{2} \right\rceil \leq \mu(d)$$



Octahedron Lemma:

$$b + j \geq d + 1$$

b : number of transversals containing p

j : minimum number of transversals spanning an antipode of a point in S_{d+1}

$$\mu(d) \geq \max\{l + b(d+1-l) + (d-b)l,\ j(d+1)\} \geq \frac{(d+1)^2}{2} \qquad \text{for } l \leq \frac{d+1}{2}$$

Colourful Research Directions

- Generalize the sufficient condition of Bárány for the existence of a colourful simplex
- Improve lower bound for $\mu(d) = \min_{S,p} depth_S(p)$
- Computational approaches for $\mu(d)$ for small *d*.
- Obtain an efficient algorithm to find a colourful simplex : *Colourful Linear Programming Feasibility* problem

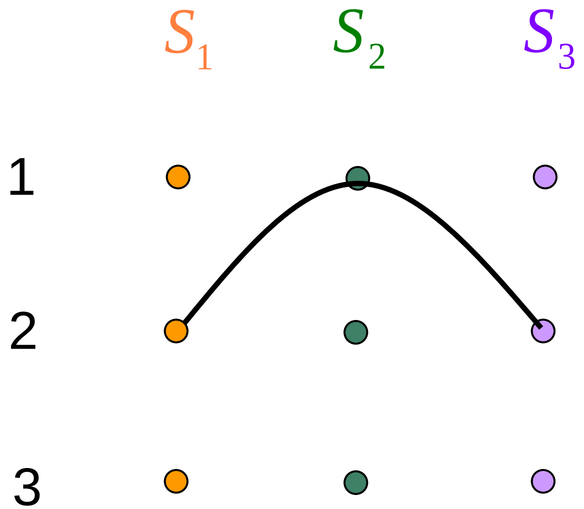
$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

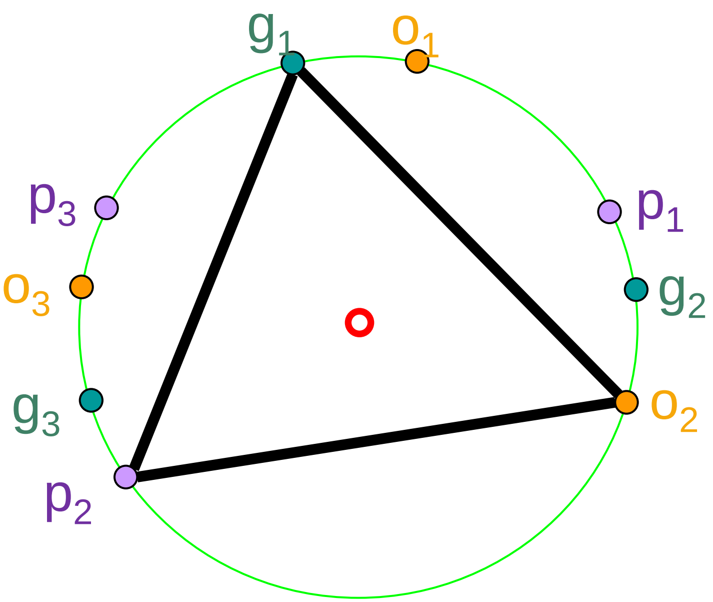
$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \ldots \cap \text{conv}(S_{d+1})$

S, *p general position* and $|S_1|, |S_2|, \ldots, |S_{d+1}| \geq d+1$

Computational Approach

hypergraph representation of colourful point configurations

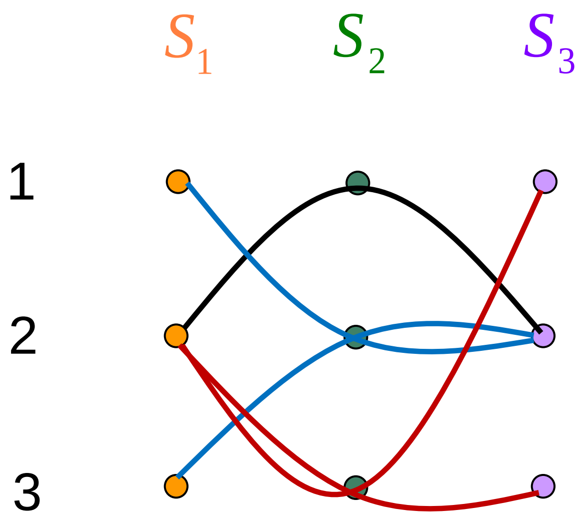


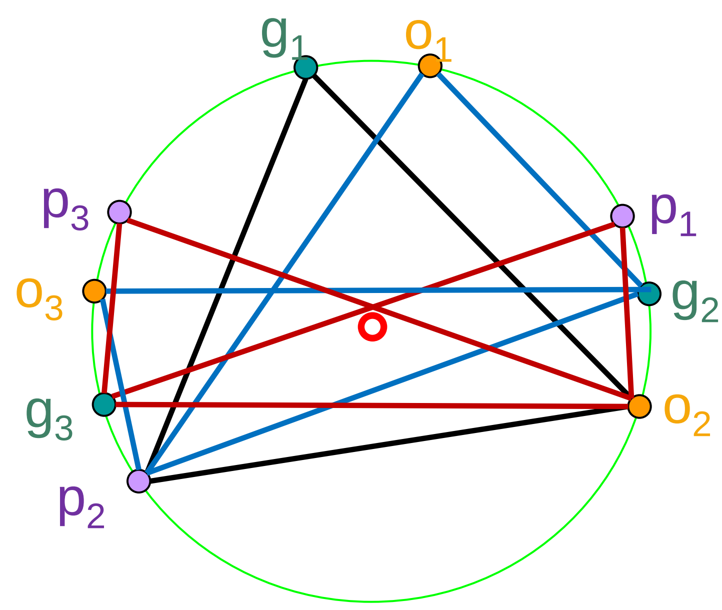


hyper-edge: colourful simplex containing *p*

Computational Approach

hypergraph representation of colourful point configurations

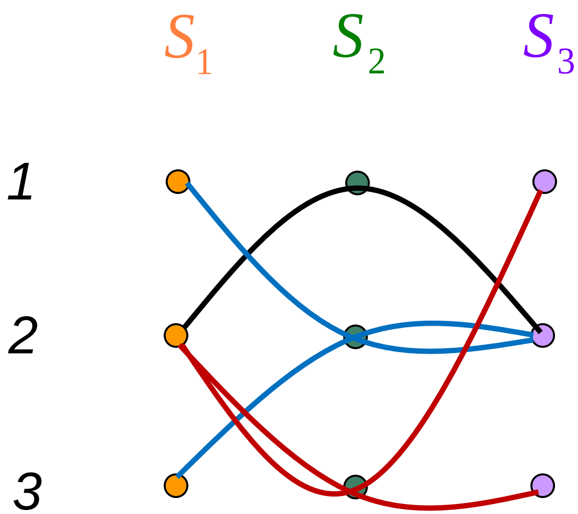


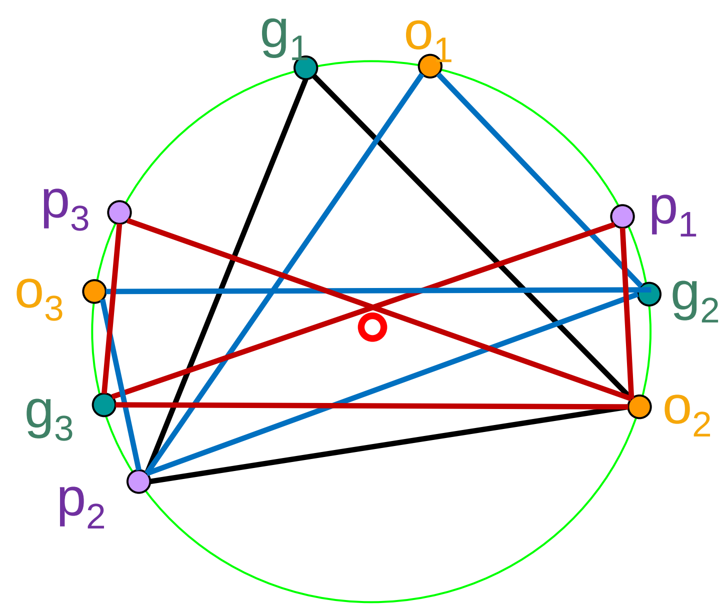


necessary conditions for the hyper-graph:

- every vertex belongs to at least one hyper-edge.
- Octahedron Lemma (parity) holds for any pair of transversals.

Computational Approach





if no hyper-graph with t or less hyper-edges satisfies the 2 necessary conditions, then $\mu(d) > t$

$\Rightarrow$ computational proof that $\mu(3) = 10$

still intractable for $d = 4$ (currently under investigation).

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[Bárány, Onn 1997] and [D., Huang, Stephen, Terlaky 2008]

$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \ldots \cap \text{conv}(S_{d+1})$

S, *p* *general position* and $|S_1|, |S_2|, \ldots, |S_{d+1}| \geq d+1$

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

$$\mu(1) = 2 \qquad \mu(2) = 5 \qquad \mu(3) = 10$$

$$\left\lceil \frac{(d+1)^2}{2} \right\rceil \leq \mu(d) \leq d^2 + 1 \quad \text{for } d \geq 4$$

$$\mu(d) \text{ even for odd } d$$

$$13 \leq \mu(4) \leq 17$$

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✓ ***thank you***