An Approach to the Dodecahedral Theorem Based on Bounds for Spherical Codes

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Workshop on Optimization
Fields Institute, Toronto
September, 2011

Outline

1. The dodecahedral theorem

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2. Fejes Tóth's proof scheme
3. Relationship to spherical codes

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The Voronoi cell associated with $x_0 = 0$ induced by the points $2\bar{x}_i$, $i = 1, \ldots, m$ is

$$\begin{aligned} V(\bar{x}_1, \ldots, \bar{x}_m) &= \{x \mid \|x\| \leq \|2\bar{x}_i - x\|,\ i = 1, \ldots, m\} \\ &= \{x \mid \bar{x}_i^T x \leq \|\bar{x}_i\|^2,\ i = 1, \ldots, m\}. \end{aligned}$$

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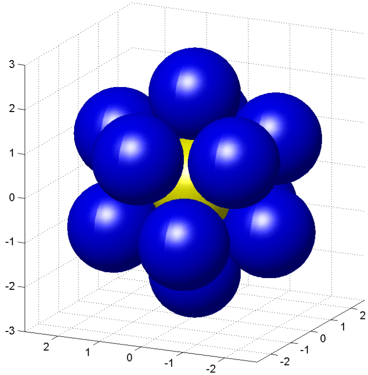

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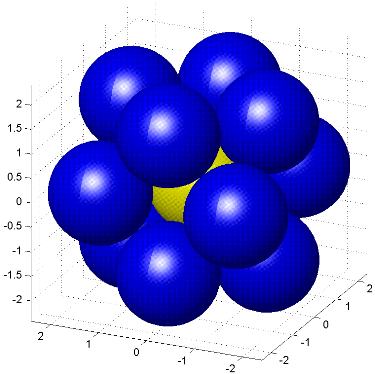

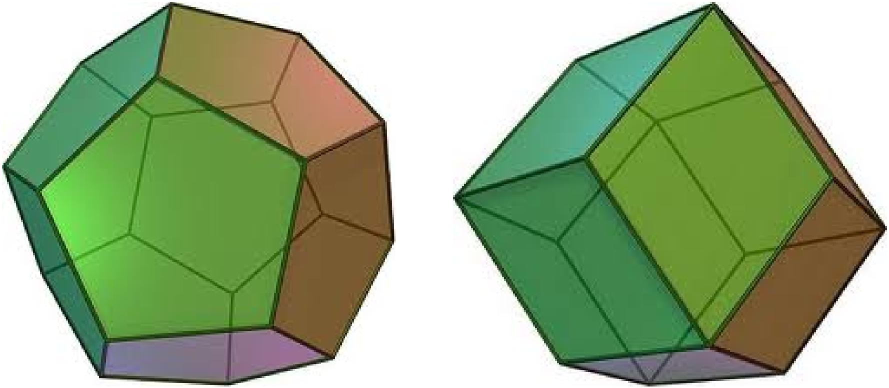

Figure: Regular and rhombic dodecahedra

Outline

1. The dodecahedral theorem
2. Fejes Tóth's proof scheme
3. Relationship to spherical codes
4. Strengthened bounds for spherical codes

Let D denote a regular dodecahedron with inradius one, $R_D = \sqrt{3}\tan 36^\circ \approx 1.2584$ be the radius of a sphere that circumscribes D and $\mathcal{B}_D = \{x \in \Re^3 \mid \|x\| \leq R_D\}$.

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In his 1964 book *Regular Figures*, Fejes Tóth restates the dodecahedral conjecture and describes a scheme that would lead to a complete proof if a key inequality were established.

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Theorem (Fejes Tóth, 1964)

Let $\hat{x}_i$, $i = 1, \ldots, m$ be points in $\Re^3$ with $\|\hat{x}_i\| \geq 1$ for each i. If $m \leq 12$, then $\mathrm{Vol}(V(\hat{x}_1, \ldots, \hat{x}_m) \cap \mathcal{B}_D) \geq \mathrm{Vol}(D)$.

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Note that in the above theorem it is *not* assumed that the points satisfy $\|\hat{x}_i - \hat{x}_j\| \geq 1$, $i \neq j$. Also, the assumption that $\|\hat{x}_i\| < R_D$ for each i could be added, since if $\|\hat{x}_i\| \geq R_D$ the constraint $\hat{x}_i^T x \leq \|\hat{x}_i\|^2$ does not eliminate any points in $\mathcal{B}_D$.

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For the Voronoi cell $V(\hat{x}_1, \ldots, \hat{x}_m)$, let $F_i(\hat{x}_1, \ldots, \hat{x}_m)$ be the face of $V(\hat{x}_1, \ldots, \hat{x}_m) \cap \mathcal{B}_D$ corresponding to the points with $\hat{x}_i^T x = \|\hat{x}_i\|^2$ (it is possible that $F_i(\hat{x}_1, \ldots, \hat{x}_m) = \emptyset$).

Point Adjustment Procedure

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Step 0. *Input* $\bar{x}_i$, $1 \le \|\bar{x}_i\| \le R_D$, $i = 1, \ldots, m$ *with* $m > 12$ *and* $\|\bar{x}_i - \bar{x}_j\| \ge 1$, $i \ne j$. *Let* $\hat{x}_i = \bar{x}_i$, $i = 1, \ldots, m$.

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Step 1. *If* $|\{i \mid 1 < \|\hat{x}_i\| < R_D\}| < 2$ *then go to Step 3. Otherwise choose* $j \ne k$ *such that* $1 < \|\hat{x}_j\| < R_D$, $1 < \|\hat{x}_k\| < R_D$, *and the surface area of* $F_j(\hat{x}_1, \ldots, \hat{x}_m)$ *is less than or equal to that of* $F_k(\hat{x}_1, \ldots, \hat{x}_m)$.

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Step 2. *Let* $\delta = \min\{R_D - \|\hat{x}_j\|, \|\hat{x}_k\| - 1\}$, *and*

$$\hat{x}_j \leftarrow (\|\hat{x}_j\| + \delta)\frac{\hat{x}_j}{\|\hat{x}_j\|}, \quad \hat{x}_k \leftarrow (\|\hat{x}_k\| - \delta)\frac{\hat{x}_k}{\|\hat{x}_k\|}.$$

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Obvious that the adjustment in Step 2 leaves $\sum_{i=1}^{m} \|\hat{x}_i\|$ unchanged, and can be shown that $\mathrm{Vol}(V(\hat{x}_1, \ldots, \hat{x}_m) \cap \mathcal{B}_D)$ is nonincreasing. Note that adjustment in Step 2 is executed at most $m - 1$ times, since each adjustment decreases $|\{i \mid 1 < \|\hat{x}_i\| < R_D\}|$ by at least 1. At termination have at most one i with $1 < \|\hat{x}_i\| < R_D$.

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The previous theorem could then be applied to bound

$$\mathrm{Vol}(V(\bar{x}_1, \ldots, \bar{x}_m)) \geq \mathrm{Vol}(V(\bar{x}_1, \ldots, \bar{x}_m) \cap \mathcal{B}_D) \geq \mathrm{Vol}(V(\hat{x}_1, \ldots, \hat{x}_m) \cap \mathcal{B}_D)$$

if the $\hat{x}_i$ output by the procedure have at most twelve i with $\|\hat{x}_i\| < R_D$. Note that the output points $\hat{x}_i$ may *not* satisfy $\|\hat{x}_i - \hat{x}_j\| \geq 1$, $i \neq j$, but this assumption is not required in the theorem.

This would be the case if the input points $\bar{x}_i$ satisfy the key inequality

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Recall that have at most one i with $1 < \|\hat{x}_i\| < R_D$. Then if $\|\hat{x}_i\| = 1$, $i = 1, \ldots, 12$, key inequality and the fact that $\hat{x}_i \leq R_D$ for each i together imply

$$(m-12)R_D \geq \sum_{i=13}^{m} \|\hat{x}_i\| \geq 12 + (m-12)R_D - 12 = (m-12)R_D,$$

so $\|\hat{x}_i\| = R_D$ for $i = 13, \ldots, m$.

A complete proof of the dodecahedral conjecture thus requires only a proof that the key inequality holds for any $\bar{x}_i$, $i = 1, \ldots, m$ with $1 \leq \|\bar{x}_i\| \leq R_D$ for each i, and $\|x_i - x_j\| \geq 1$ for all $i \neq j$.

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Unfortunately Fejes Tóth was unable to prove the key inequality, even though all evidence suggests that it actually holds with R_D replaced by the larger constant $7/\sqrt{27} \approx 1.347$.

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Note key inequality for $m = 13$ would give immediate proof of "Thirteen Spheres Problem."

Theorem (13 spheres problem; Kissing number in dimension 3)

In a packing of unit spheres in $\Re^3$, at most 12 spheres can simultaneously touch ("kiss") another sphere.

To prove key inequality, need solution (or very good lower bound) for *m*-point norm minimization problem

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m} \|x_i\| \\
\text{s.t.} \quad & \|\bar{x}_i - \bar{x}_j\| \geq 1, \quad i \neq j \\
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Expect that these approaches may have difficulty due to number of variables (40-60), very high degree of symmetry, and need for a relatively tight bound. We will consider another possibility based on the theory of spherical codes.

Outline

A set $\mathcal{C} = \{x_i\}_{i=1}^m \subset \Re^3$ is called a spherical z-code if $\|x_i\| = 1$ for each i, and $x_i^T x_j \le z$ for all $i \neq j$. A packing of unit spheres that all touch a unit sphere centered at the origin generates a spherical $1/2$-code.

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To begin we establish that for R sufficiently small, if $\{\bar{x}_i\}_{i=1}^m$ are points with $1 \le \|\bar{x}_i\| \le R$ for each i and $\|\bar{x}_i - \bar{x}_j\| \ge 1$ for all $i \ne j$, then the normalized points $x_i = \bar{x}/\|\bar{x}_i\|$ form a z-code for a suitable z.

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Lemma (Normalized points form spherical z-code)

Suppose that $1 \leq \|\bar{x}_i\| \leq R$, $i = 1, \ldots, m$, *where* $R \leq \frac{1+\sqrt{5}}{2} \approx 1.618$ *and* $\|\bar{x}_i - \bar{x}_j\| \geq 1$ *for all* $i \neq j$. *Let* $x_i = \bar{x}_i/\|\bar{x}_i\|$, $i = 1, \ldots, m$. *Then* $x_i^T x_j \leq 1 - \frac{1}{2R^2}$ *for all* $i \neq j$.

Next, for $x_i \neq x_j$ with $\|x_i\| = \|x_j\| = 1$, $x_i^T x_j \leq 1 - \frac{1}{2R^2}$, $R \leq \frac{1+\sqrt{5}}{2}$, consider the 2-point norm minimization problem

$$\begin{aligned} \min \quad & \lambda_i + \lambda_j \\ \text{s.t.} \quad & \|\lambda_i x_i - \lambda_j x_j\| \geq 1 \\ & 1 \leq \lambda_i \leq R,\ 1 \leq \lambda_j \leq R. \end{aligned}$$

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Theorem (Solution of 2-point norm minimization problem)

Let $1 \leq R \leq \frac{1+\sqrt{5}}{2}$, $\|x_i\| = \|x_j\| = 1$ and $.5 \leq s = x_i^T x_j \leq 1 - \frac{1}{2R^2}$. Then problem solution has $\lambda_i^* + \lambda_j^* = f(s, R)$, where

$$f(s,R) = \begin{cases} 1 + 2s & \frac{1}{2} \leq s \leq \frac{R}{2}, \\ R(1+s) + \sqrt{1 - R^2(1-s^2)} & \frac{R}{2} \leq s \leq 1 - \frac{1}{2R^2}. \end{cases}$$

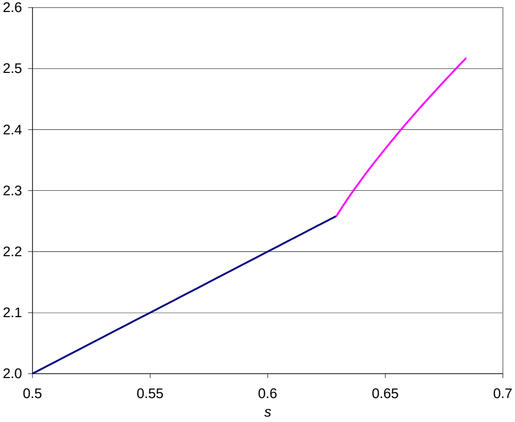

Figure: Solution value in 2-point norm minimization problem for $R = R_D$

Now assume that $m > 12$, $1 \le \|\bar{x}_i\| \le R_D$, $i = 1, \ldots m$, and $\|\bar{x}_i - \bar{x}_j\| \ge 1$ for all $i \ne j$. Let $\lambda_i = \|\bar{x}_i\|$ and $x_i = (1/\lambda_i)\bar{x}_i$. Goal is to prove the key inequality, which can be written

$$\sum_{i=1}^{m}(\lambda_i - 1) \ge 12 + (m-12)R_D - m = (m-12)(R_D - 1).$$

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Define $N_i = |\{j \ne i \mid x_i^T x_j \ge .5\}|$ to be the number of “close neighbors” of x_i, and $\mathcal{N} = \{(i,j),\ i \ne j \mid x_i^T x_j \ge .5\}$. Then $|\mathcal{N}| = \sum_{i=1}^{m} N_i$, and

$$\sum_{(i,j)\in\mathcal{N}} (\lambda_i + \lambda_j - 2) = \sum_{(i,j)\in\mathcal{N}} (\lambda_i - 1) + (\lambda_j - 1) = 2\sum_{i=1}^{m} N_i(\lambda_i - 1).$$

Applying the solution of the 2-point norm minimization problem, get

$$
\begin{aligned}
2\sum_{i=1}^{m} N_i(\lambda_i - 1) &\geq \sum_{(i,j)\in\mathcal{N}} [f(x_i^T x_j, R_D) - 2] \\
\sum_{i=1}^{m}(\lambda_i - 1) &\geq \frac{1}{2N_{\max}} \sum_{(i,j)\in\mathcal{N}} [f(x_i^T x_j, R_D) - 2],
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where $N_{\max} := \max\{N_i\}_{i=1}^{m}$.

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\frac{1}{2N_{\max}} \sum_{(i,j)\in\mathcal{N}} [f(x_i^T x_j, R_D) - 2] \geq (m - 12)(R_D - 1).
$$

Using results from spherical trigonometry, can prove

Lemma (Maximum number of close neighbors)

$N_{\max} \leq 6$. *Moreover, for* $m = 13$, *if* $N_{\max} = 6$ *then key inequality holds.*

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To get lower bound for

$$\frac{1}{2N_{\max}} \sum_{(i,j)\in\mathcal{N}} [f(x_i^T x_j, R_D) - 2]$$

can apply Delsarte bound for spherical codes. Recall $\mathcal{C} = \{x_i\}_{i=1}^m$ is a spherical z-code in $\Re^3$, with $z = 1 - 1/(2R_D^2) \approx .6843$.

For $\alpha \in [-1, 1]$, define the distance distribution of the code to be

$$\alpha(s) = \frac{|\{(i,j) \mid x_i^T x_j = s\}|}{m}.$$

Then $\alpha(\cdot) \geq 0$, and $\sum_{-1 \leq s \leq z} \alpha(s) = m - 1$.

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Let $\Phi_k(\cdot)$, $k = 0, 1, \ldots$ denote the Gegenbauer, or ultraspherical, polynomials $\Phi_k(t) = P_k^{(0,0)}(t)$ where $P_k^{(0,0)}$ is a normalized Jacobi polynomial. It can be shown that

$$1 + \sum_{-1\leq s\leq z} \alpha(s)\Phi_k(s) \geq 0, \quad k = 1, 2, \ldots.$$

Then $\sum_{(i,j)\in\mathcal{N}}[f(x_i^T x_j, R_D) - 2] \geq v^*(m)$, where $v^*(m)$ is solution value in the semi-infinite LP problem

$$
\begin{aligned}
\mathrm{LP}(m): \qquad & \min \quad m \sum_{.5\leq s\leq z} [f(s, R_D) - 2]\alpha(s) \\
& \text{s.t.} \quad \sum_{-1\leq s\leq z} \alpha(s)\Phi_k(s) \geq -1, \quad k = 1, \ldots, d \\
& \qquad \sum_{-1\leq s\leq z} \alpha(s) = m - 1 \\
& \qquad \alpha(s) \geq 0, \quad -1 \leq s \leq z.
\end{aligned}
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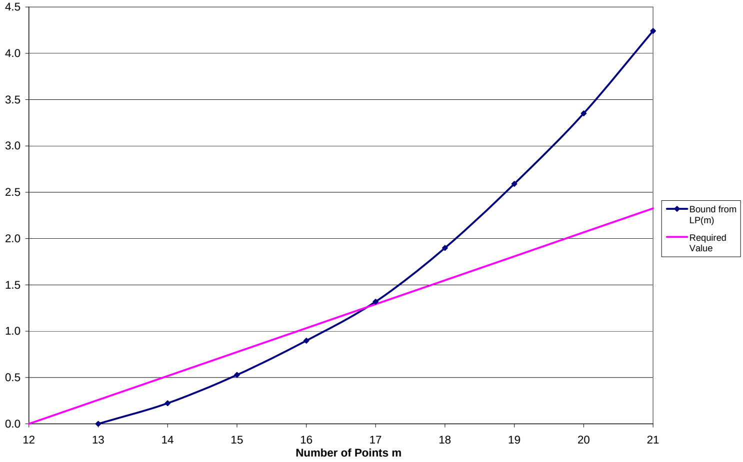
Then $\sum_{(i,j)\in\mathcal{N}}[f(x_i^T x_j, R_D) - 2] \geq v^*(m)$, where $v^*(m)$ is solution value in the semi-infinite LP problem

$$\begin{aligned}
\mathrm{LP}(m): \quad & \min \quad m \sum_{.5\leq s\leq z} [f(s, R_D) - 2]\alpha(s) \\
& \text{s.t.} \quad \sum_{-1\leq s\leq z} \alpha(s)\Phi_k(s) \geq -1, \quad k = 1, \ldots, d \\
& \qquad \sum_{-1\leq s\leq z} \alpha(s) = m - 1 \\
& \qquad \alpha(s) \geq 0, \quad -1 \leq s \leq z.
\end{aligned}$$

Constraints of LP(m) are feasible up to $m = 21$. To establish key inequality, need

$$v^*(m)/(2N_{\max}) \geq (m - 12)(R_D - 1), \qquad m = 13, \ldots, 21.$$


4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
12
13
14
15
16
17
18
19
20
21
Number of Points m
Bound from LP(m)
Required Value

Result: Bound from LP(m) sufficient to prove key inequality for $m \geq 17$. Remains to prove inequality for $m = 13, \ldots, 16$.

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Note $v^*(13) = 0$. In fact knew this would be the case ahead of time, since Delsarte bound for kissing number in dimension 3 is 13, not 12. Need to strengthen Delsarte bound to have any chance of proving key inequality for $m = 13$.

Outline

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All three are sufficient to prove that kissing number in dimension 3 is 12. Last approach is most powerful and results in SDP in place of LP.

Resulting problem SDP(m) has form:

$$\begin{aligned} \min \quad & m \sum_{.5 \le s \le z} [f(s, R_D) - 2]\alpha(s) \\ \text{s.t.} \quad & \sum_{s \in Z} \alpha(s)\Phi_k(s) \ge -1, \quad k = 1, \ldots, d \\ & \sum_{s \in Z} \alpha(s) = m - 1, \quad \alpha(s) \ge 0, \quad s \in Z = [-1, z] \end{aligned}$$

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& \sum_{s \in Z} \alpha(s) = m - 1, \quad \alpha(s) \ge 0, \quad s \in Z = [-1, z] \\
& 3\sum_{s \in Z} \alpha(s) S_k(s, s, 1) + \sum_{s,t,u \in Z} \alpha'(s,t,u) S_k(s,t,u) \succeq -S_k(1,1,1), \\
& \qquad k = 1, \ldots, d \\
& \sum_{s,t,u \in Z} \alpha'(s,t,u) = (m-1)(m-2) \\
& \alpha'(s,t,u) \ge 0, \quad s, t, u \in Z.
\end{aligned}
$$

In SDP(m), $\alpha'(\cdot,\cdot,\cdot)$ is the 3-point distance distribution

$$\alpha'(s,t,u) = \frac{|\{(i,j,k) \mid x_i^T x_j = s, x_i^T x_k = t, x_j^T x_k = u\}|}{m},$$

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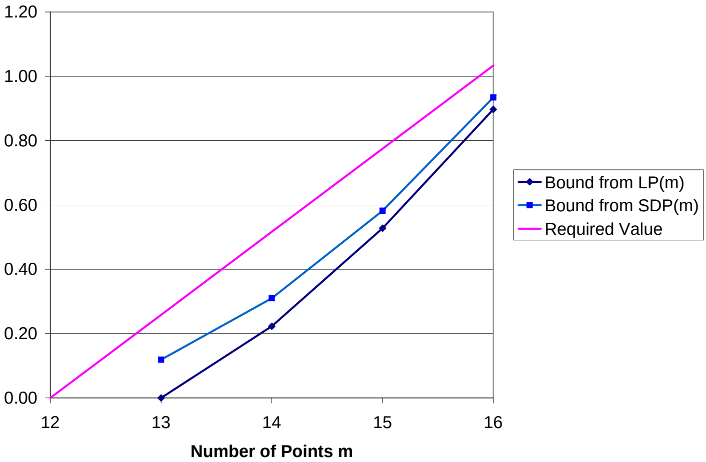
Can also add constraints relating 2-point and 3-point distance distributions and remove original constraints based on $\Phi_k(\cdot)$.


1.20
1.00
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0.60
0.40
0.20
0.00
12
13
14
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Thank You!