Conference on Discrete Geometry and Optimization

Toronto, September 2011

# Exploiting Polyhedral Symmetries
## in Social Choice Theory

Achill Schürmann
(University of Rostock)

arXiv:1109.1545

( based on work "supported" by two Bachelor projects at TU Delft )

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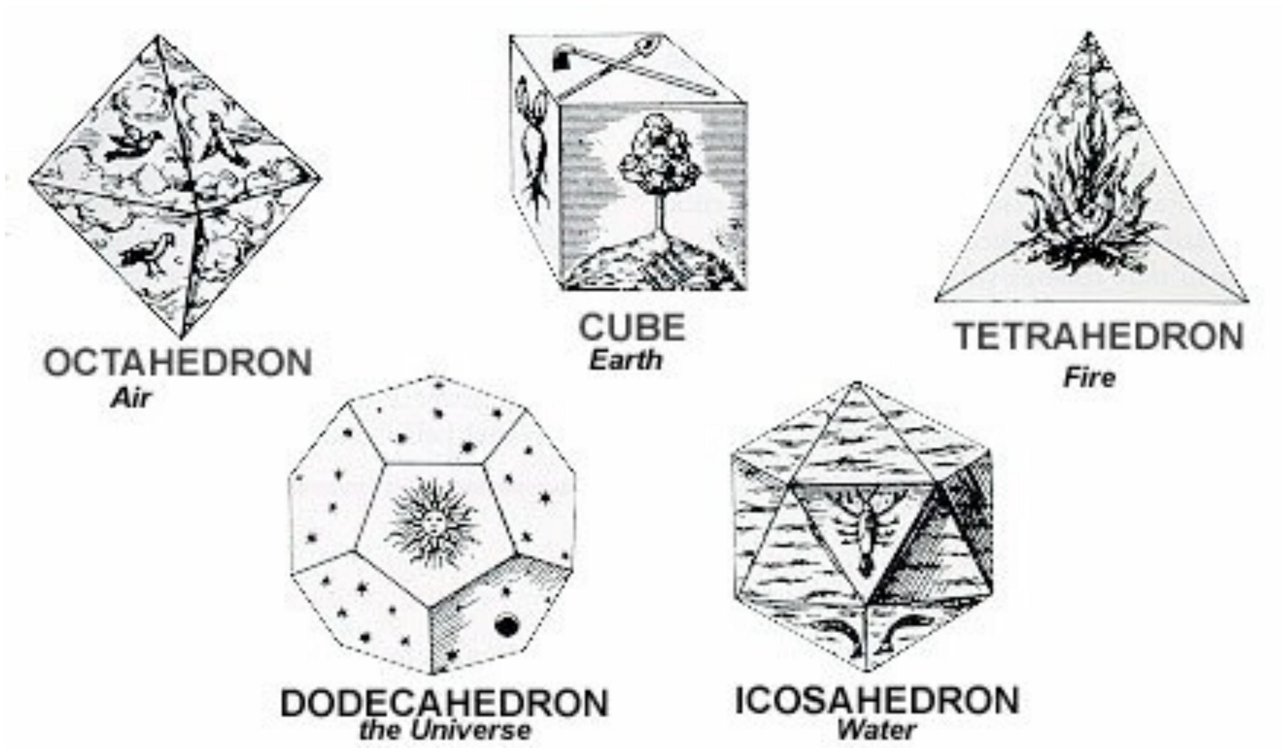

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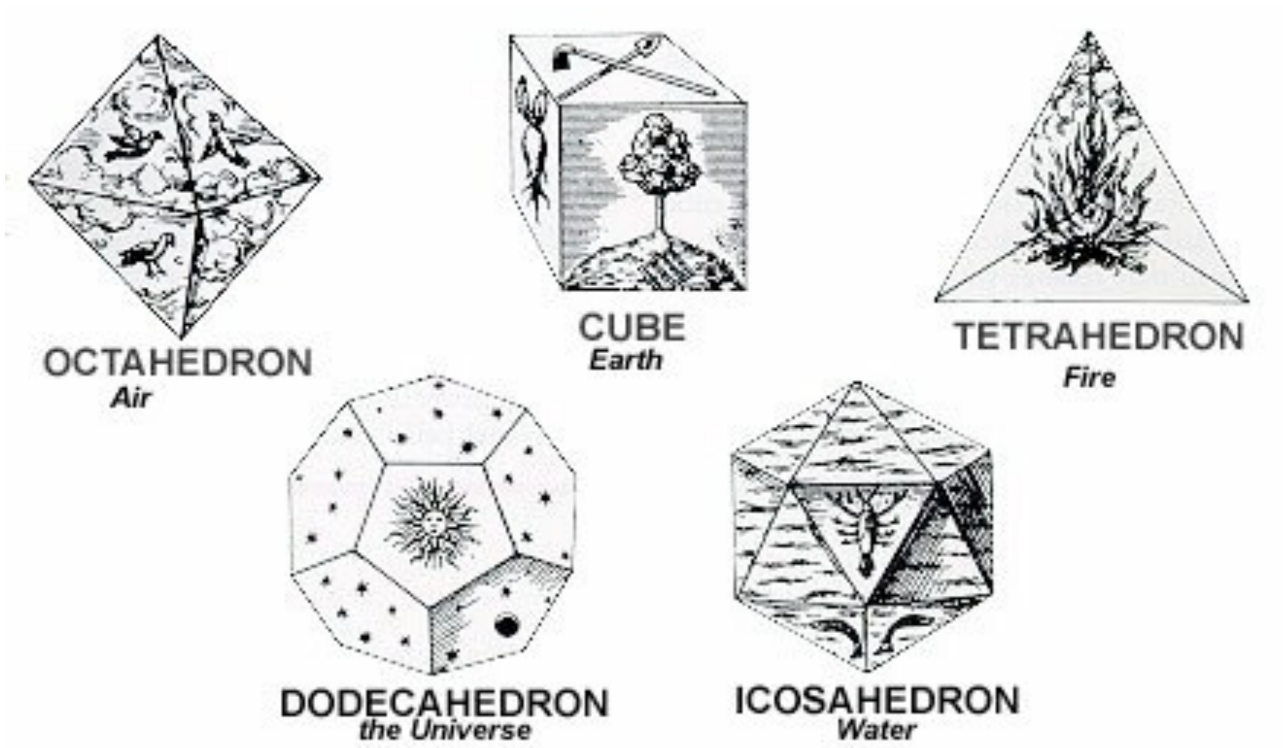


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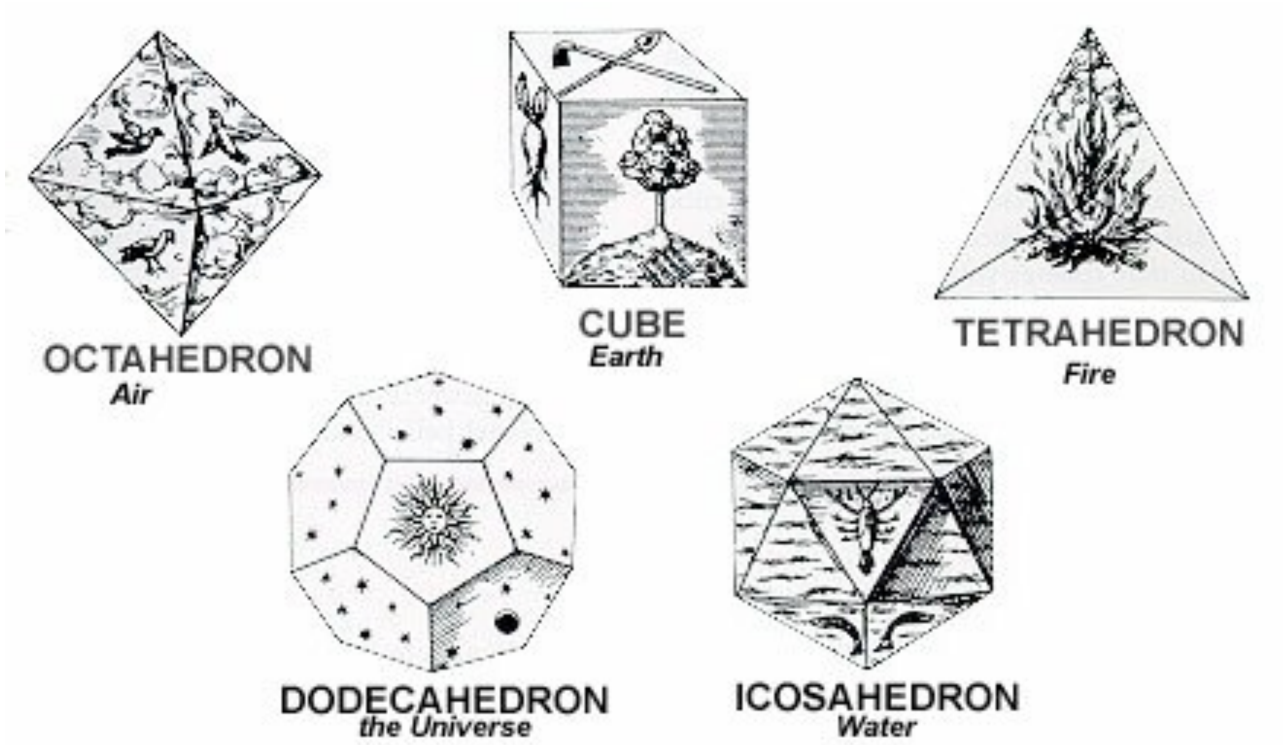


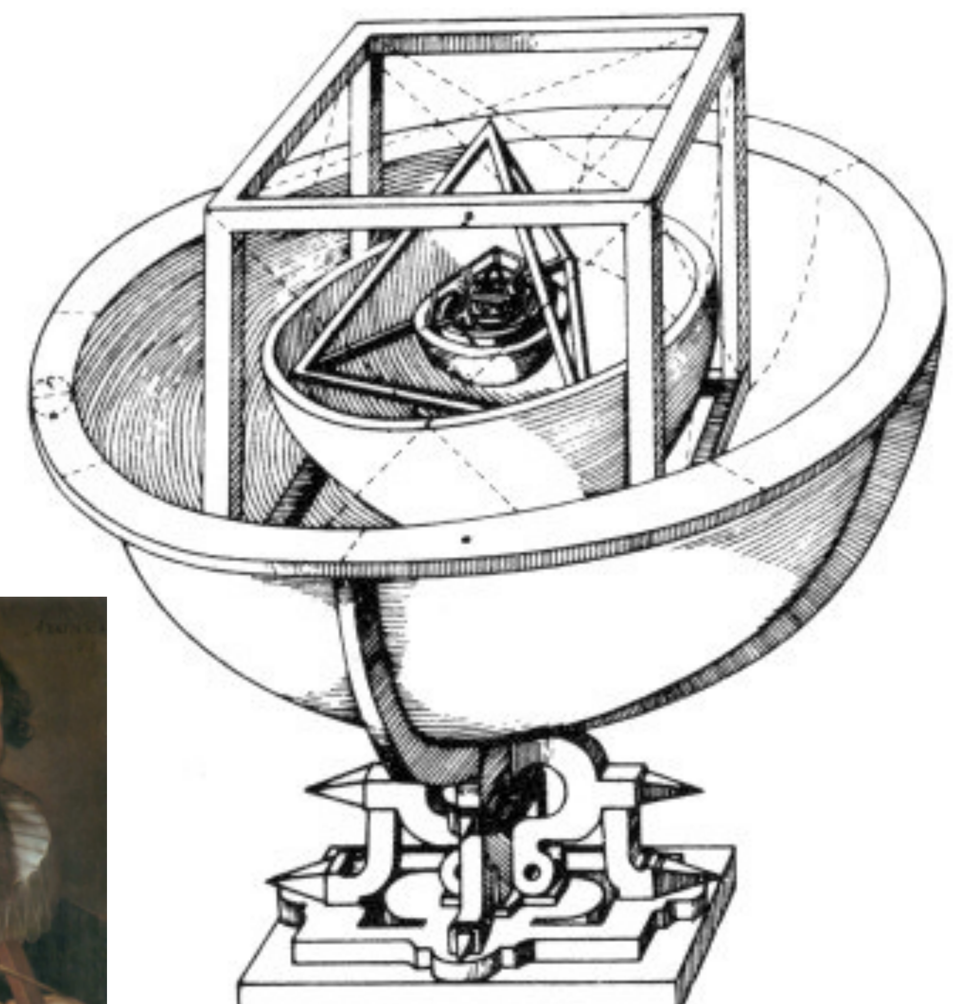

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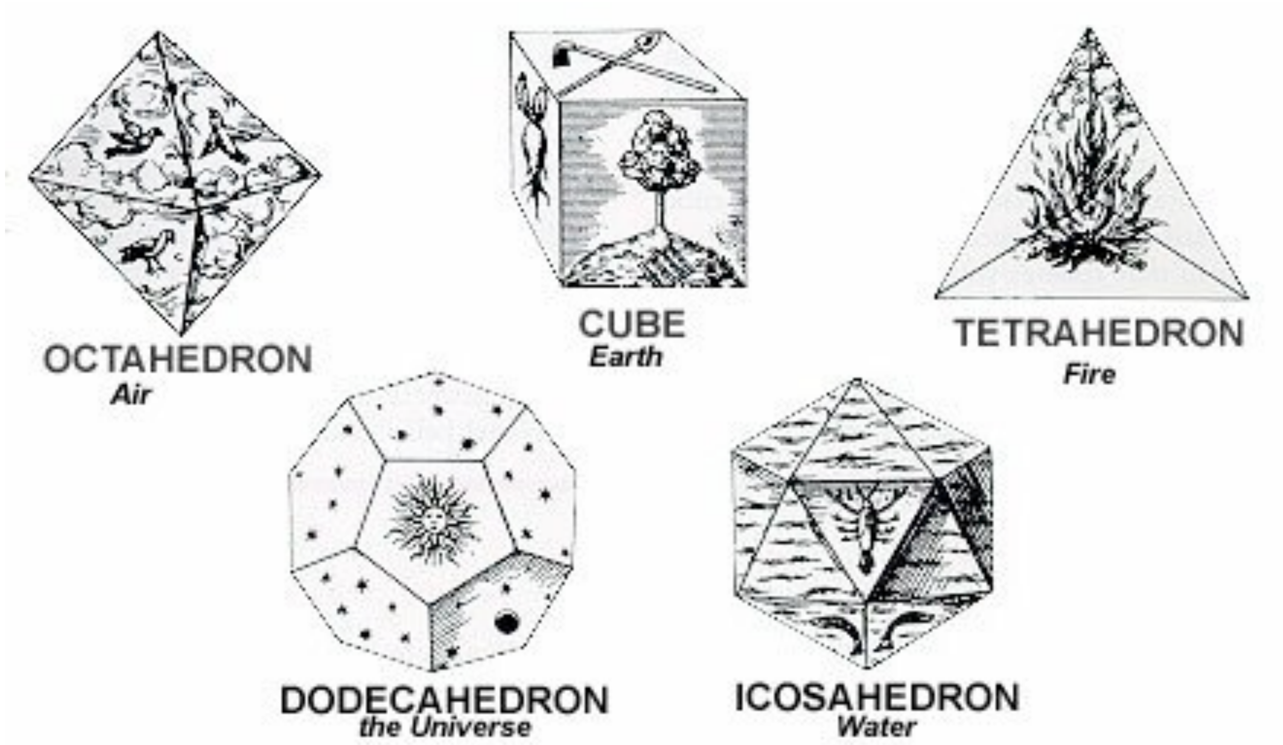


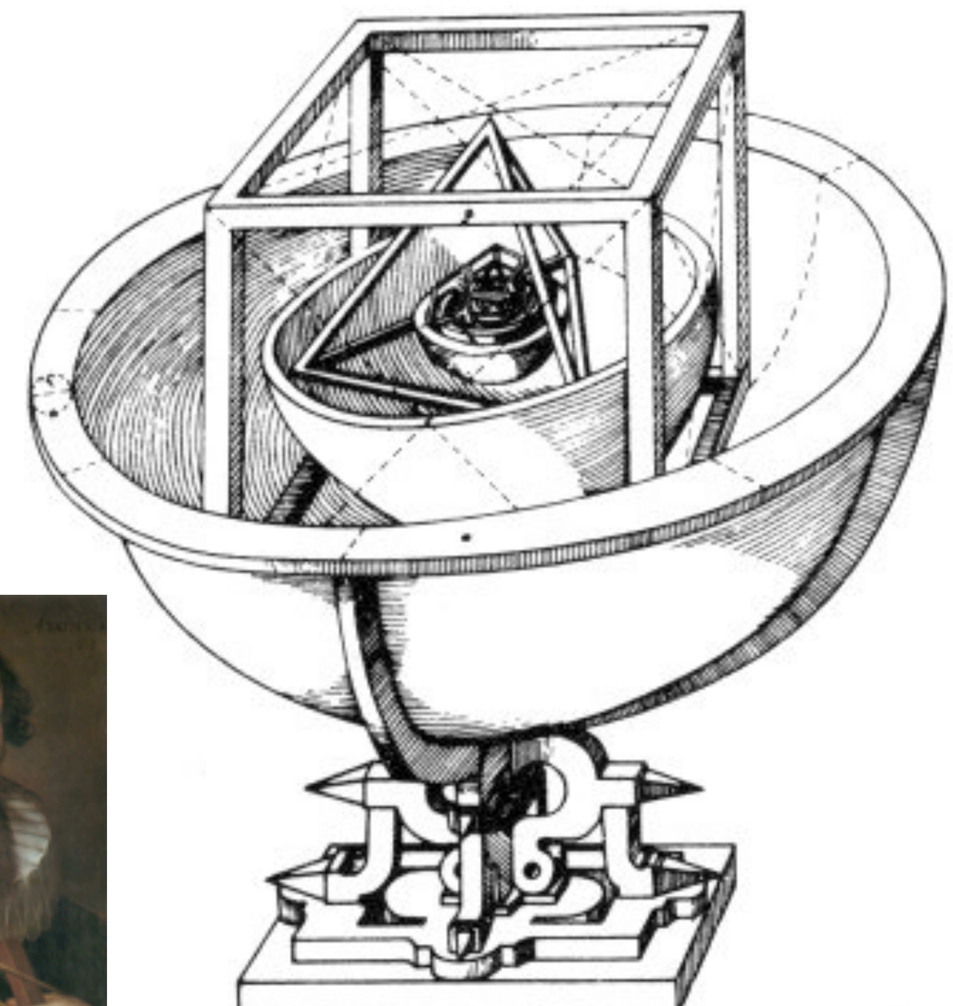

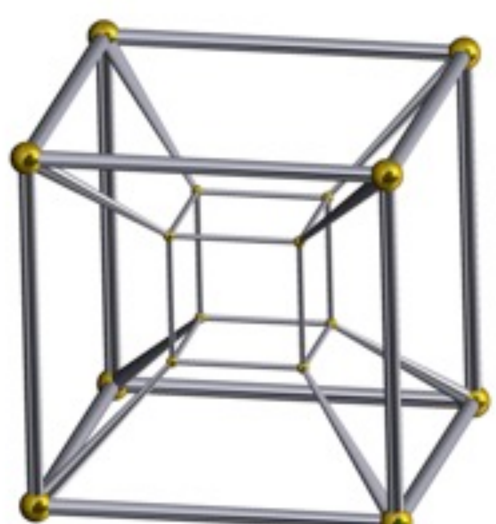

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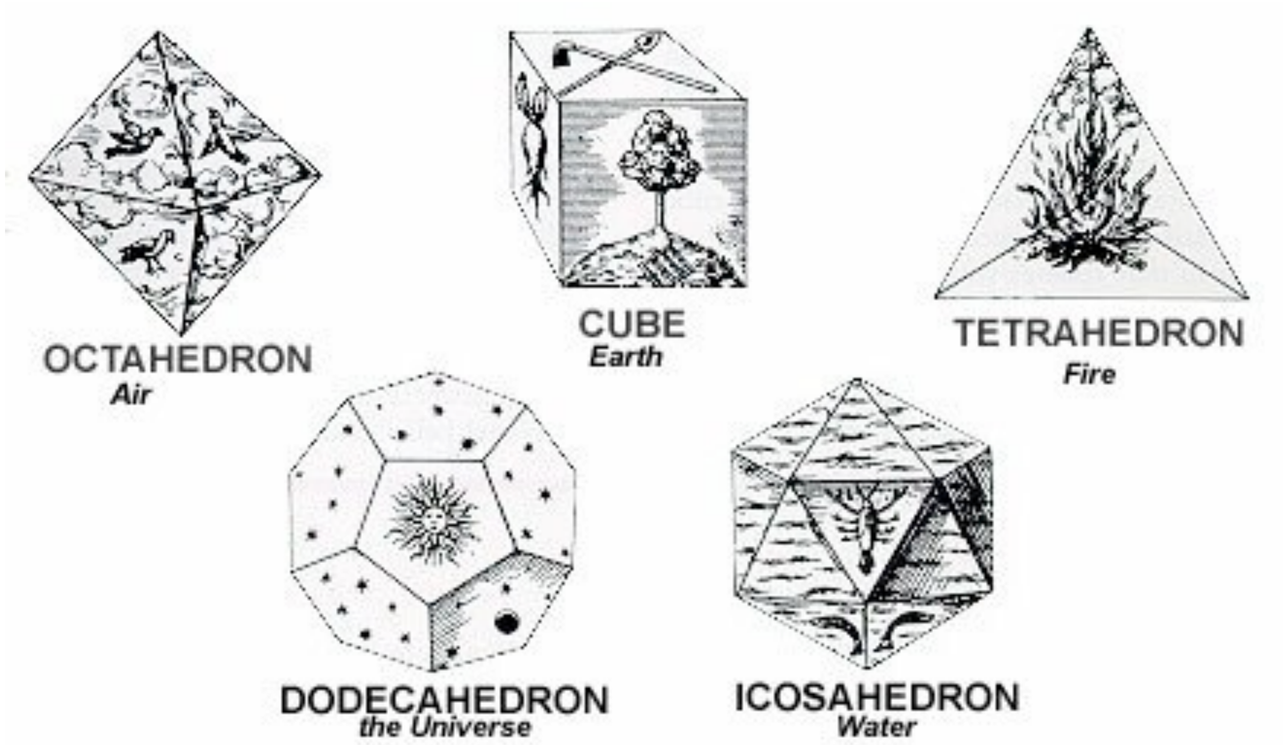


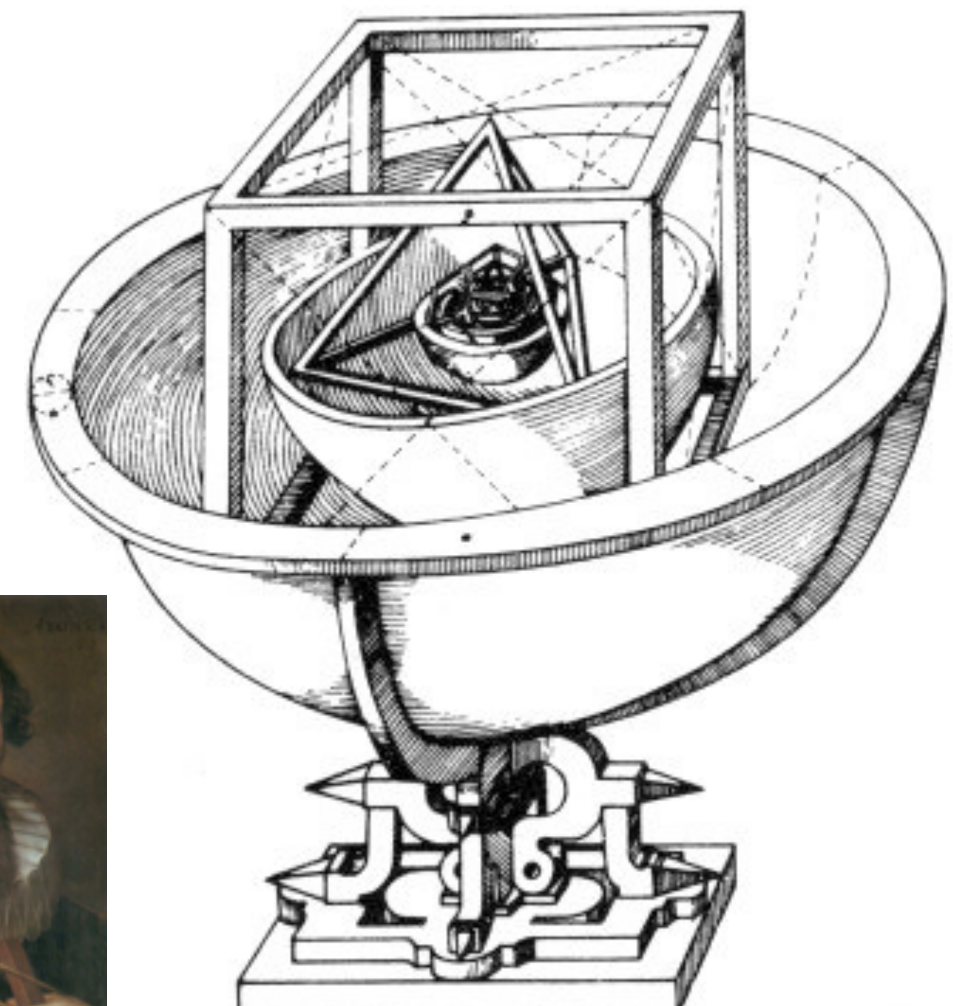

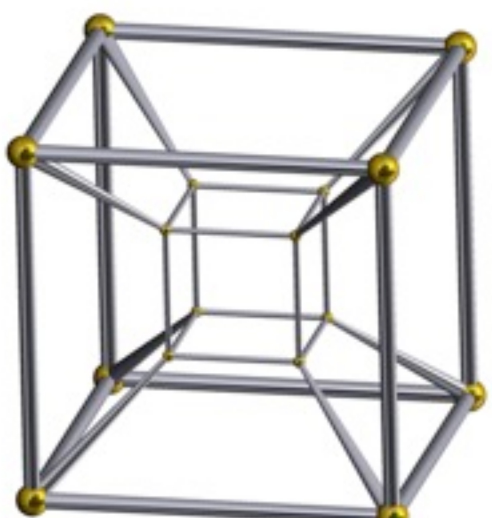

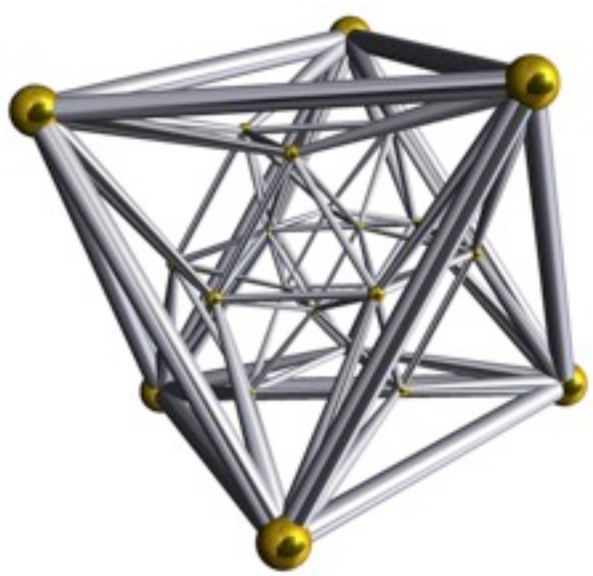

# Social Choice Theory

individual choices ⟶ collective choice

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| | | | | | |
|---|---|---|---|---|---|
| a | b | c | b | a | |
| > | > | > | > | > | |
| b | c | a | c | c | ••• |
| > | > | > | > | > | |
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a

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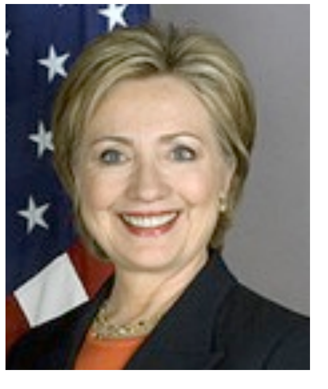

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| > | > | > | > | > | | > |
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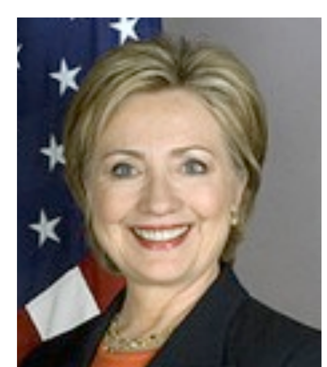
a


b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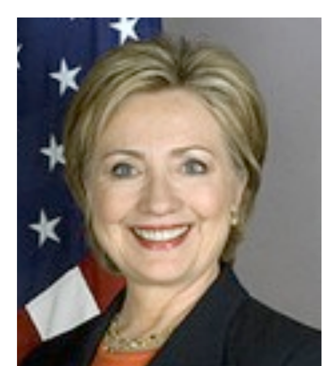
a

a

b



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# Arrows Impossibility Theorem

Kenneth Arrow
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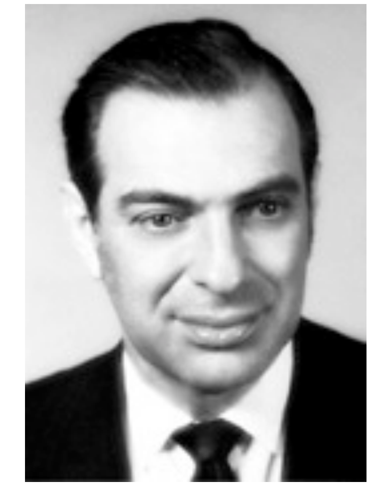

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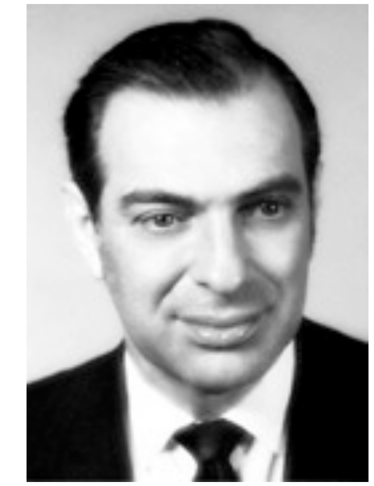

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**THUS:** There may be no "pairwise winner"!
(Condorcet winner)

# Polyhedral Model

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  $(n_{\mathsf{ab}}, n_{\mathsf{ac}}, n_{\mathsf{ba}}, n_{\mathsf{bc}}, n_{\mathsf{ca}}, n_{\mathsf{cb}})$ describes a voting situation

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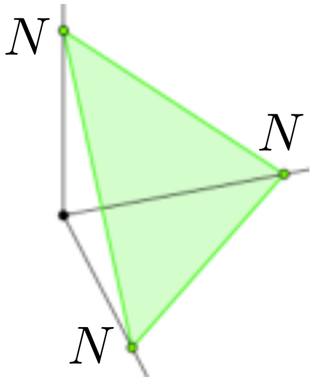

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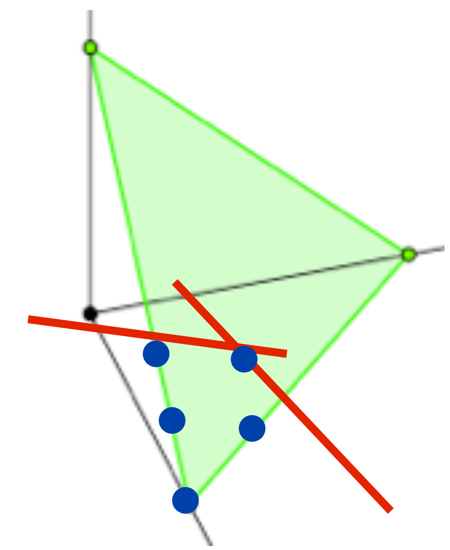
# Counting Lattice Points

- Candidate a is a Condorcet winner if

(1) $n_{\mathsf{ab}} + n_{\mathsf{ac}} + n_{\mathsf{ca}} > n_{\mathsf{ba}} + n_{\mathsf{bc}} + n_{\mathsf{cb}}$ ( a beats b )

(2) and $n_{\mathsf{ab}} + n_{\mathsf{ac}} + n_{\mathsf{ba}} > n_{\mathsf{ca}} + n_{\mathsf{cb}} + n_{\mathsf{bc}}$ ( a beats c )

That is: $(n_{\mathsf{ab}}, n_{\mathsf{ac}}, n_{\mathsf{ba}}, n_{\mathsf{bc}}, n_{\mathsf{ca}}, n_{\mathsf{cb}}) \in \mathbb{Z}^6_{\geq 0}$



is in the polyhedron

$$P_N = \left\{ n \in \mathbb{R}^6 \mid N = \sum_{\mathsf{xy}} n_{xy},\ n_{xy} \geq 0 \text{ and } (1), (2) \right\}$$

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$$\#(P_N \cap \mathbb{Z}^d) = a_{d-1}N^{d-1} + \ldots + a_1 N + a_0$$

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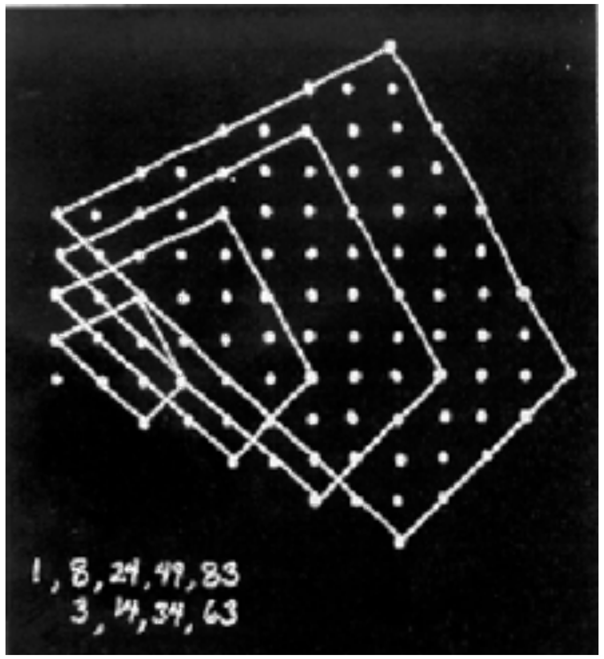

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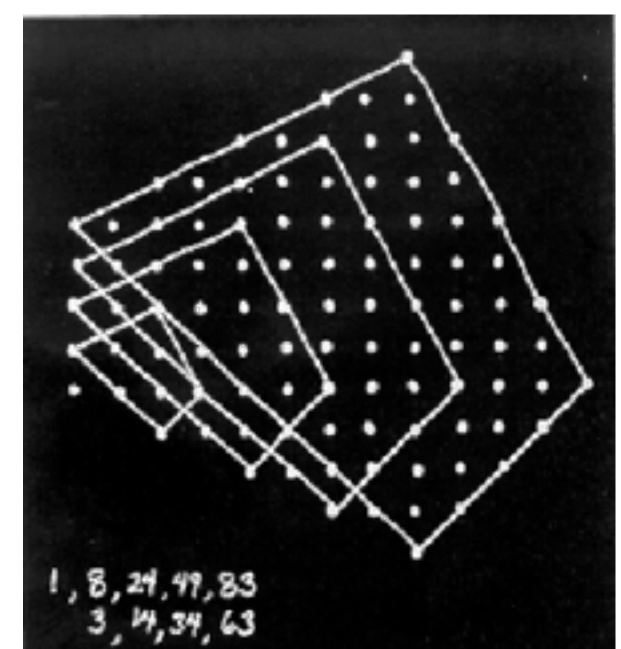

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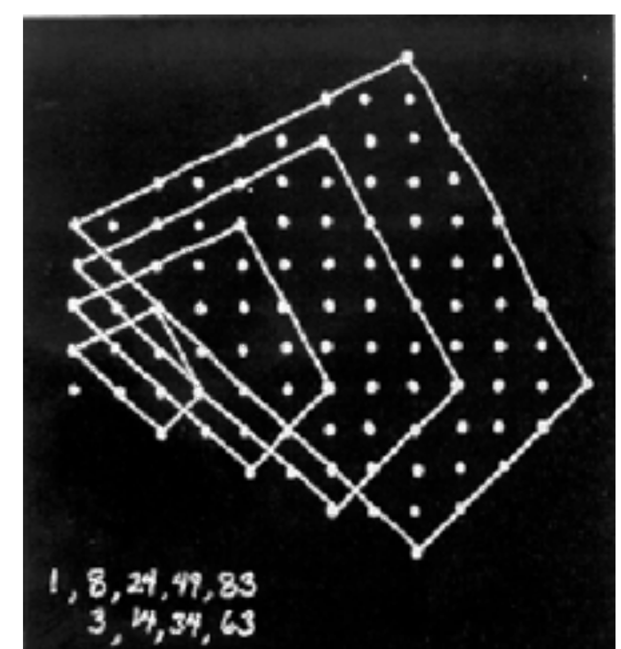

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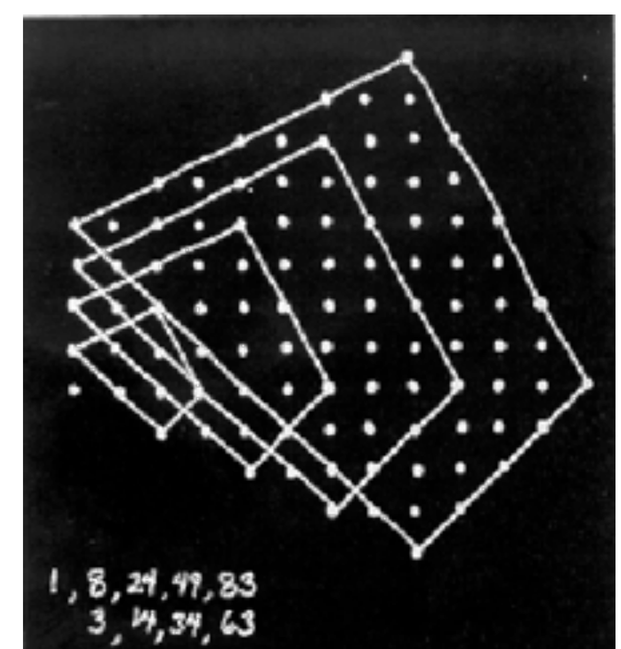

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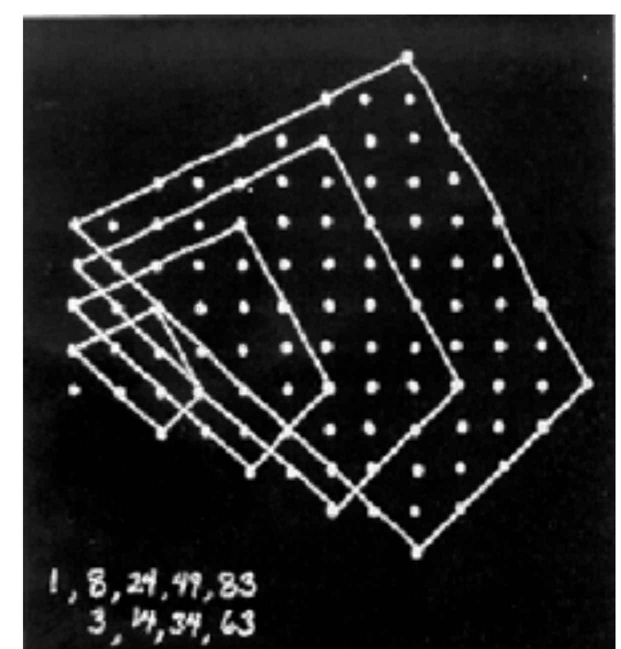

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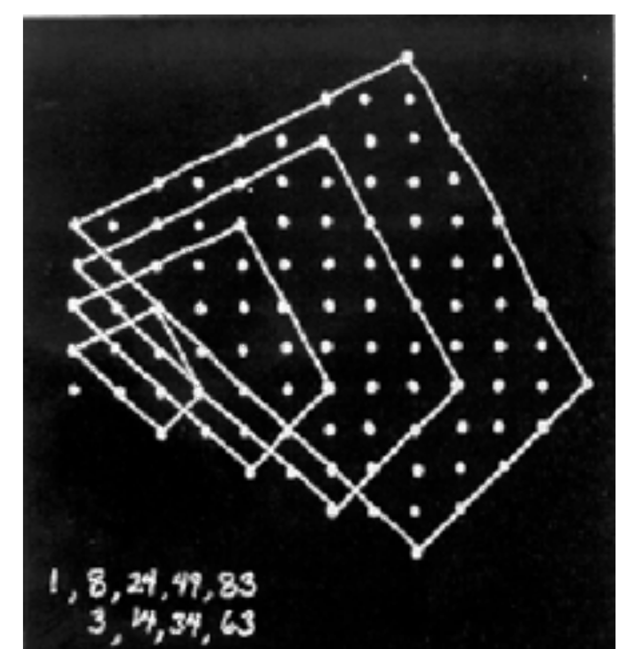
- Parallelity of Approach discovered in 2006 (by Lepelley et al. and Wilson / Pritchard)



# Likeliness of Condorcet paradox

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+ ( -1/64 * {( 1/2 * N + 0 )} + 3/64 ) * N^4
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+ ( -29/32 * {( 1/2 * N + 0 )} + 17/16 ) * N^2
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( Number of voting situations with N voters and candidate a as Condorcet winner )

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Likeliness of Condorcet Paradox $1 - 3\frac{\text{q-poly}}{\binom{N+5}{5}}$

For large elections $(N \to \infty)$:

$$1 - 3\frac{1/384}{1/120} = \frac{1}{16} = 0.0625$$

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- Plurality vs. Plurality Runoff

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$$n_{\text{ab}} + n_{\text{ac}} + n_{\text{ca}} \;>\; n_{\text{ba}} + n_{\text{bc}} + n_{\text{cb}} \qquad \text{( a beats b )}$$

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Likeliness for large elections $(N \to \infty)$: $\dfrac{16}{135} = 0.1185\ldots$

- Plurality vs. Plurality Runoff

$$n_{\text{ab}} + n_{\text{ac}} \;>\; n_{\text{ba}} + n_{\text{bc}} \qquad \text{( a wins plurality over b )}$$

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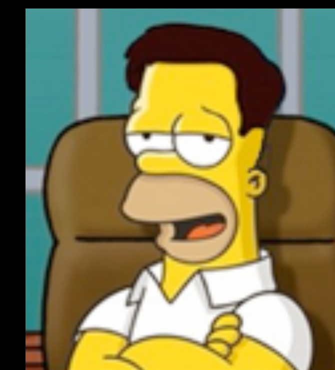

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   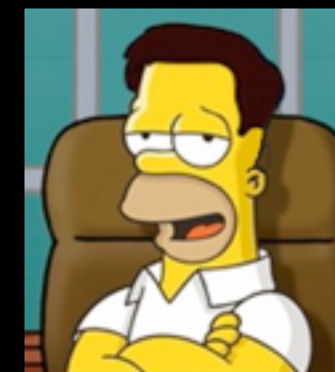

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   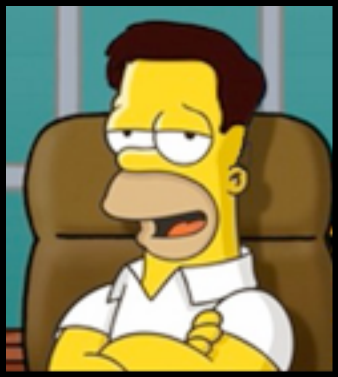

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   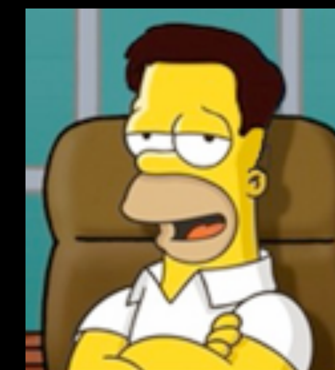

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   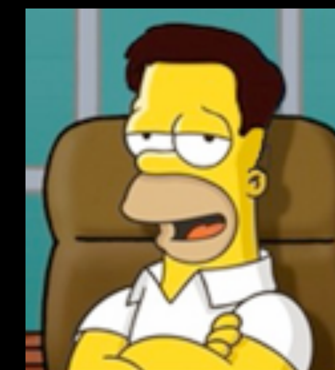

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**IDEA:** Reduce dimension by exploiting symmetry !

# Grouping of variables

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(former lattice points)

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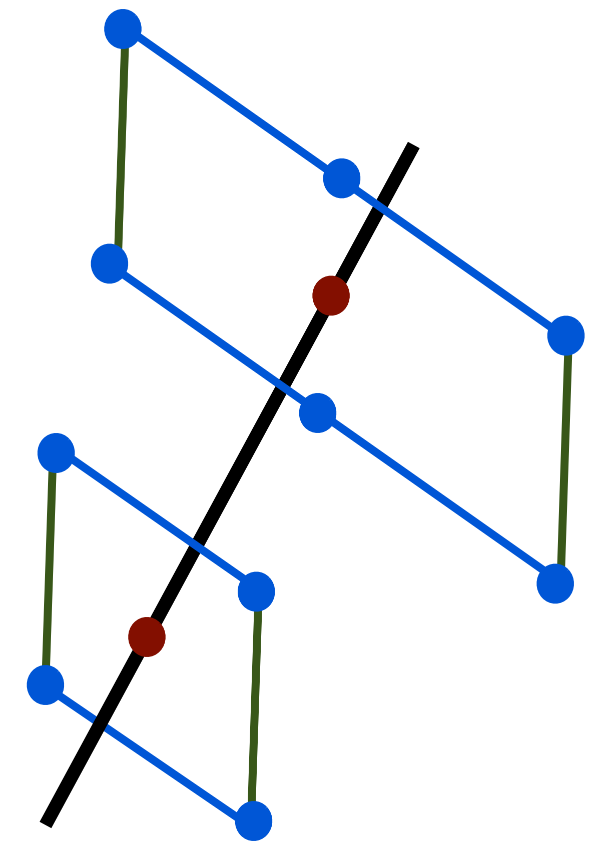

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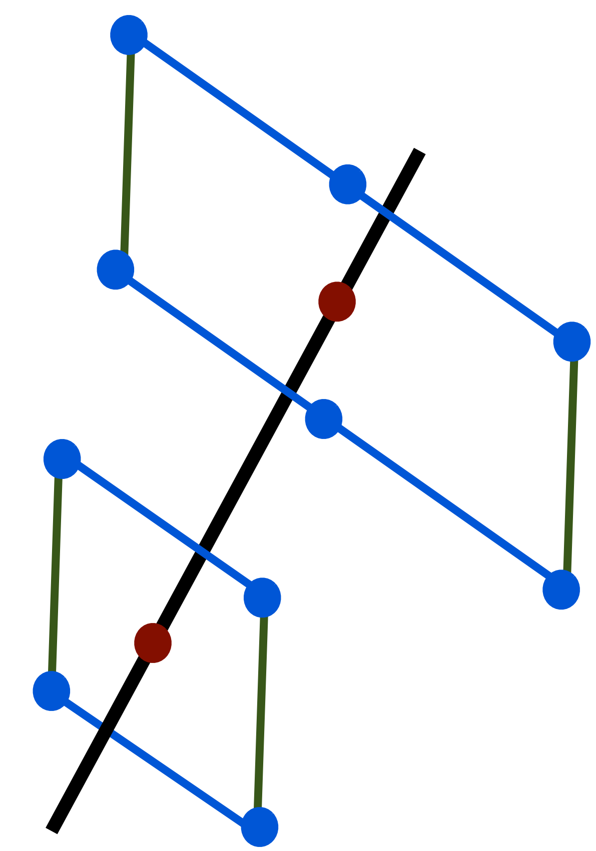

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**William V. Gehrlein**

In an email of Sep. 7th 2011:

Your results particularly got my attention when I finally realized that you had obtained limiting representations for four candidates. This is a significant step forward, and you are not the only person who has been trying to produce such results. However, I believe that you are the first to successfully accomplish this. The only four candidate result that I am aware of is cited in your paper, and I only managed to obtain that by using a trick.

# New results with four candidates

- Condorcet Efficiency of Plurality

$$\lim_{N\to\infty} \text{Prob}(N) = \frac{10658098255011916449318509}{14352135440302080000000000} = 0.74261\ldots$$

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- Plurality vs. Plurality Runoff

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( by integrating polynomial of degree $18$ over a $5$-dimensional polytope )

**WANT**: generalization of Ehrhart theory, counting lattice points with polynomial weights

# The next generation Ehrhart theory

## Counting with polynomial weights

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# Want:

- Methods exploiting general polyhedral symmetry groups

# Exploiting Symmetry in other Polyhedral Computations?

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up to symmetry

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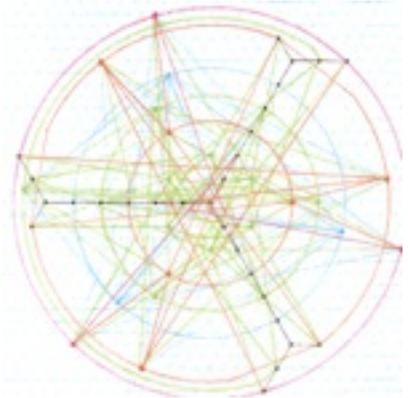
- The contact polytope of the Leech lattice, preprint at arXiv:0906.1427
  - 1 orbit with 196,560 vertices in 24 dimensions
  - 1,197,362,269,604,214,277,200 many facets in 232 orbits



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Sym
Pol

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SymPol

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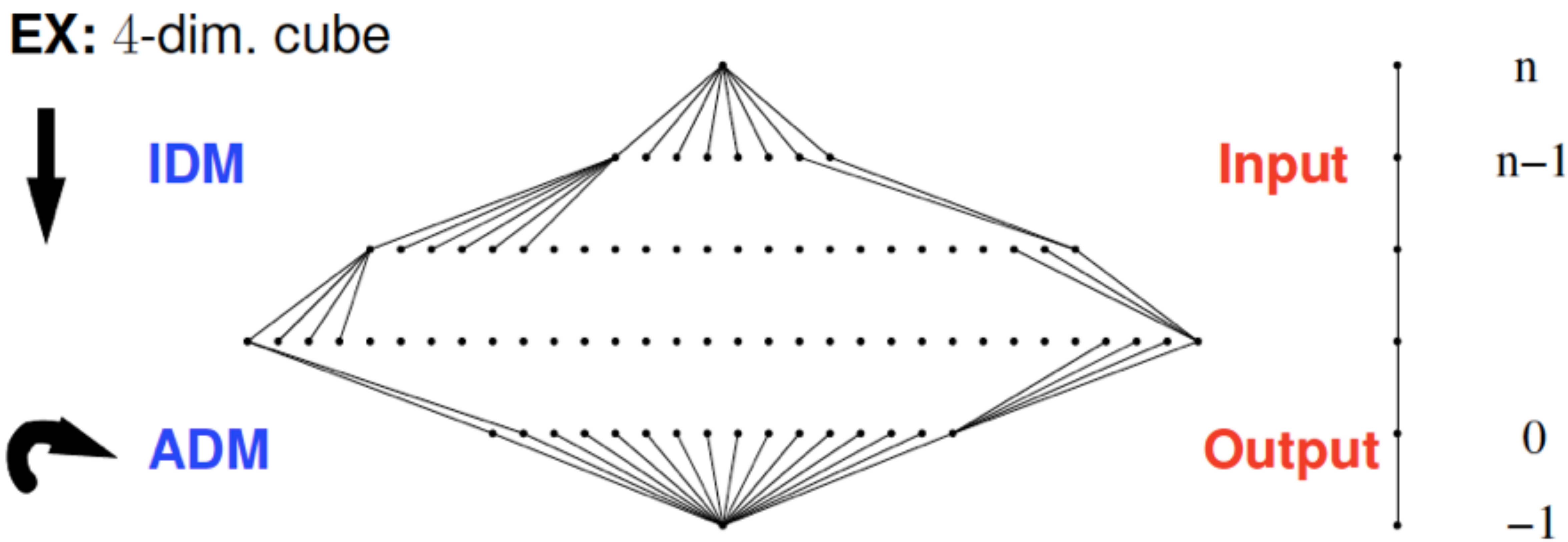

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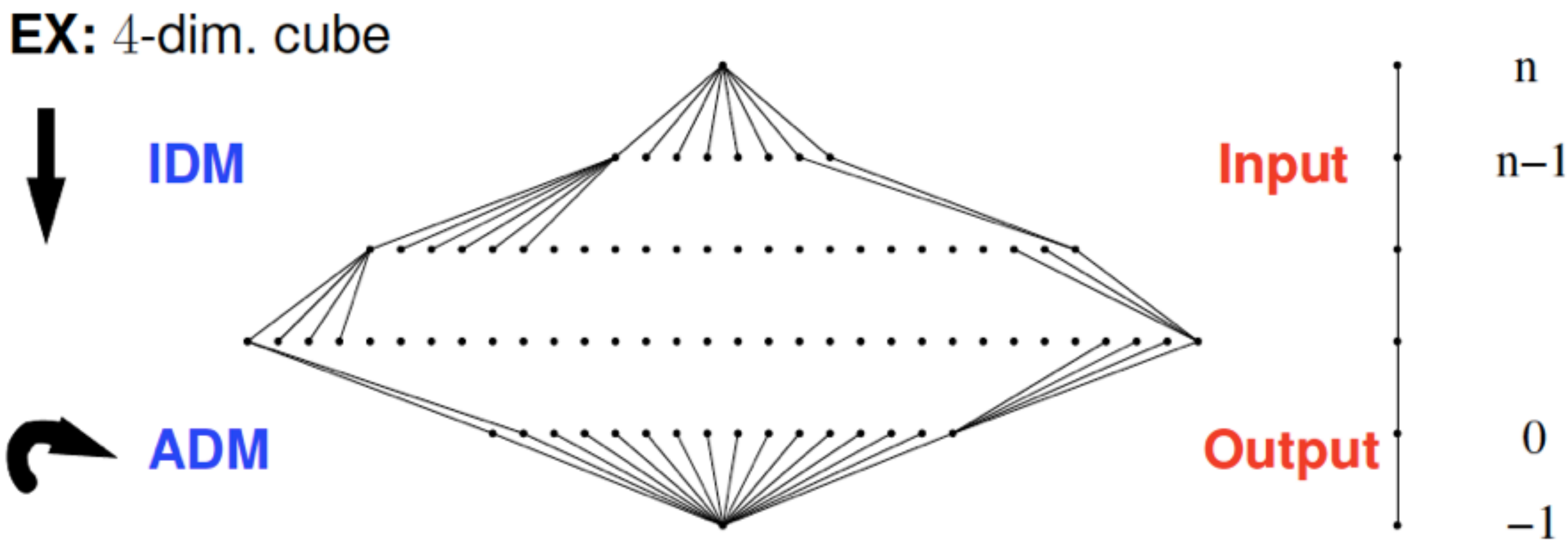

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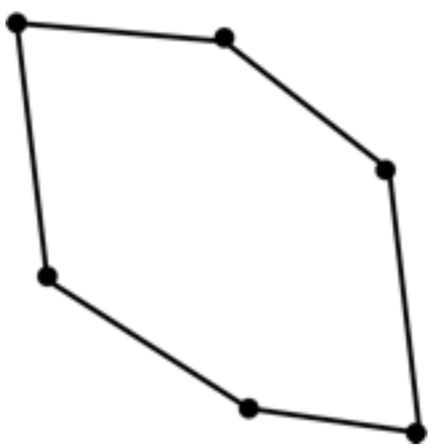
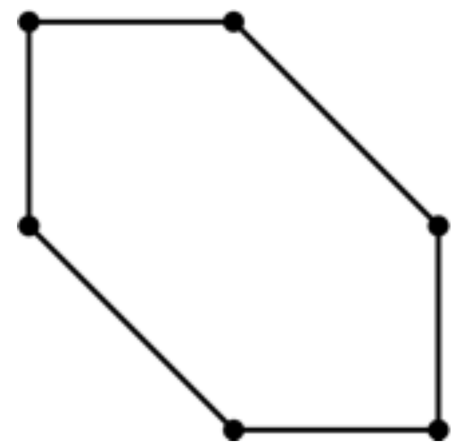


$C_6 \rtimes C_2$
trivial
trivial



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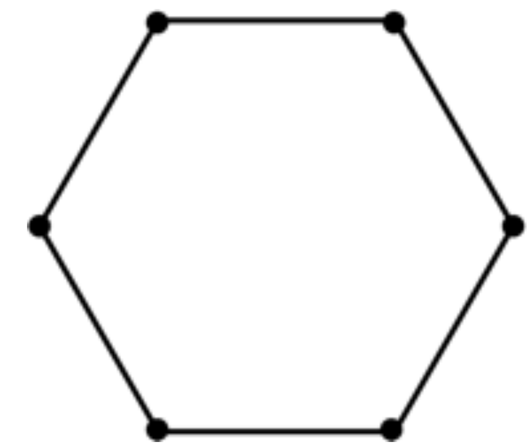

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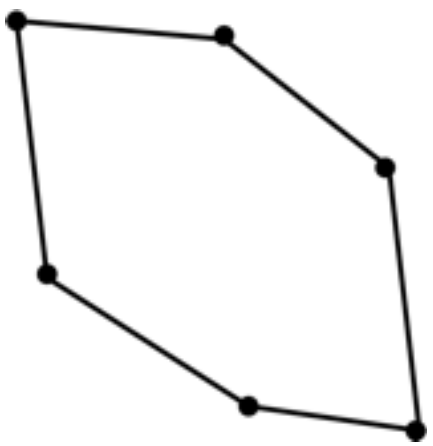

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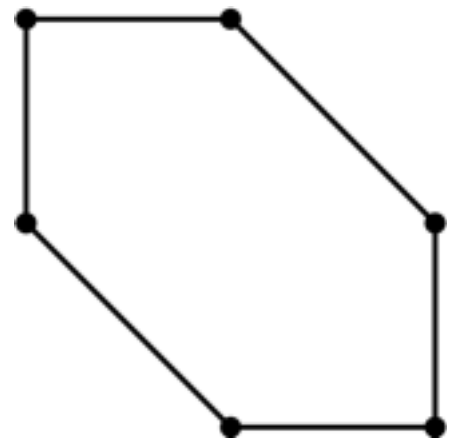

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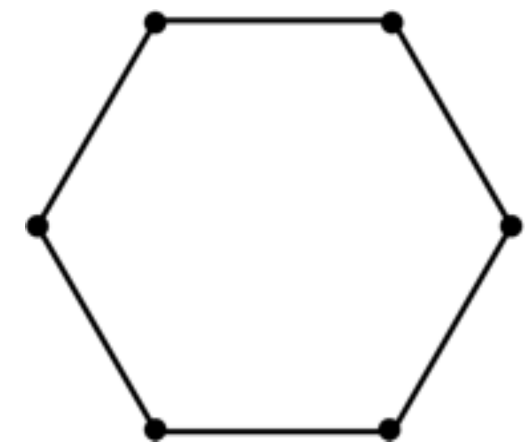

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**DEF:** A linear automorphism of $\{v_1, \ldots, v_m\} \subset \mathbb{R}^n$ is a regular matrix $A \in \mathbb{R}^{n \times n}$ with $Av_i = v_{\sigma(i)}$ for some $\sigma \in S_m$

# Detecting Linear Automorphisms

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**THM:** The group of linear automorphisms is equal to the automorphism group of the complete graph $K_m$ with edge labels $v_i^t Q^{-1} v_j$, where $Q = \sum_{i=1}^{m} v_i v_i^t$

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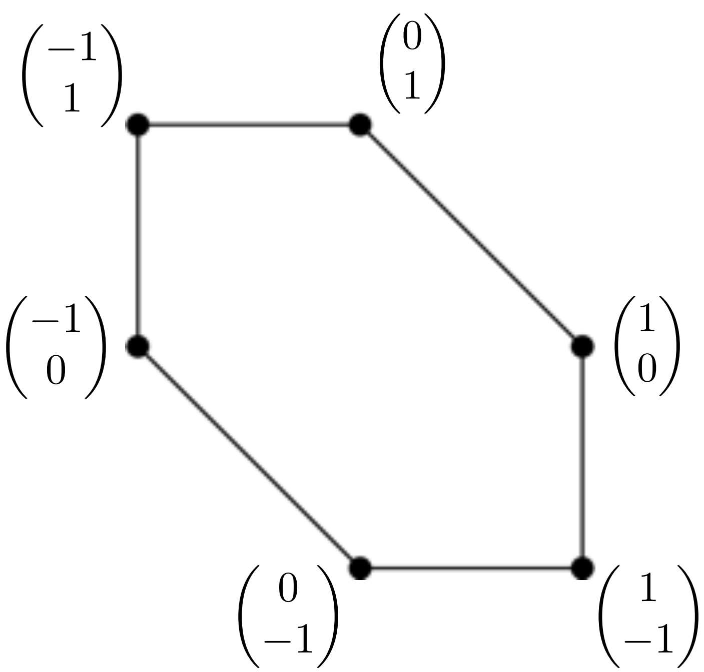
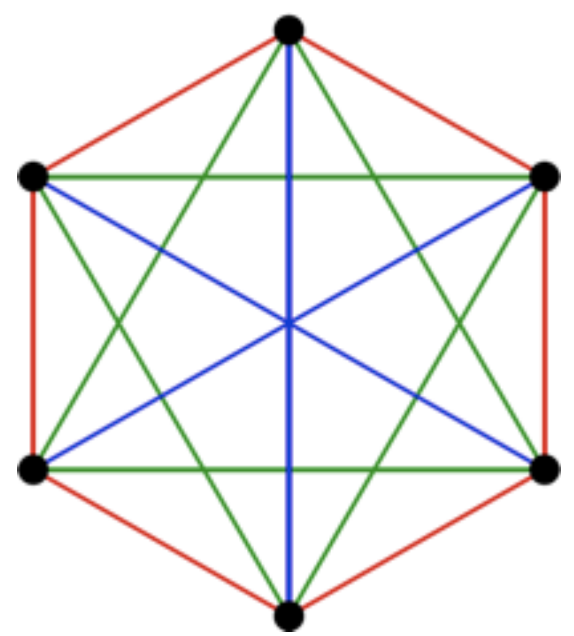



$$Q = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$$



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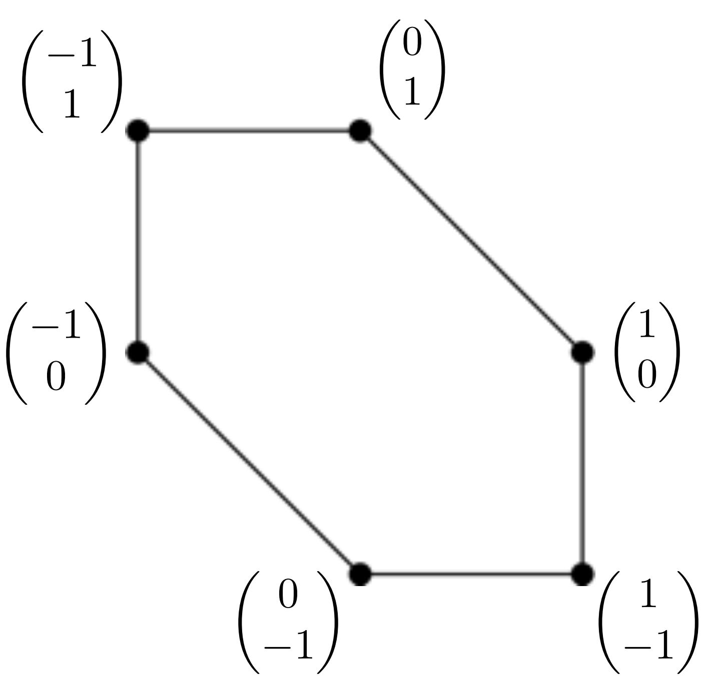


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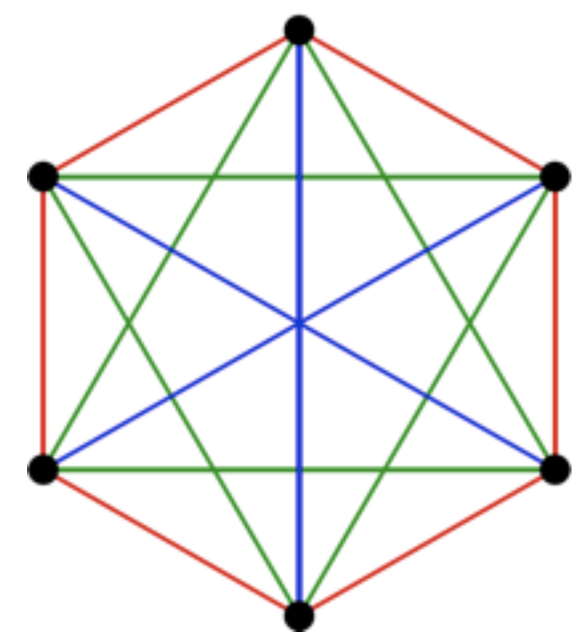

=> use NAUTY by Brendan McKay

# Adjacency Decomposition Method

(for vertex enumeration)

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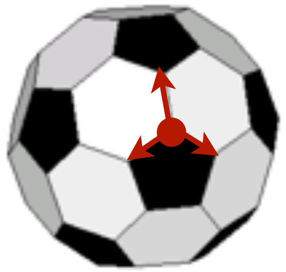

- For each new orbit representative
  - enumerate neighboring vertices

# Adjacency Decomposition Method

(for vertex enumeration)

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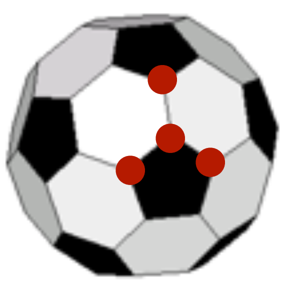

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# Adjacency Decomposition Method

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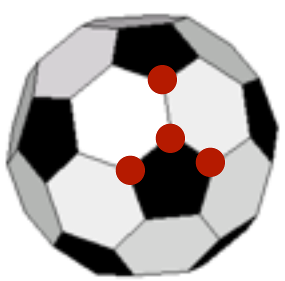

Representation conversion problem

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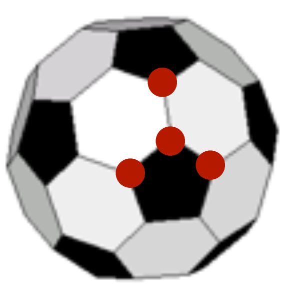

Representation conversion problem

**BOTTLENECK:** Stabilizer and In-Orbit computations

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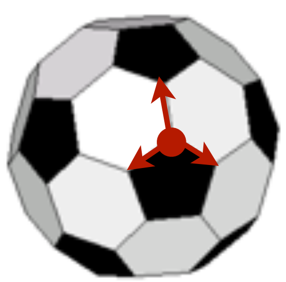

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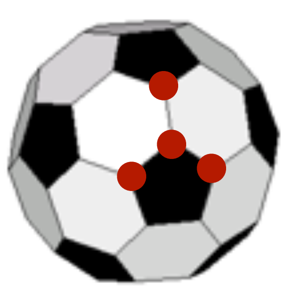

Representation conversion problem

**BOTTLENECK:** Stabilizer and In-Orbit computations

=> Need of efficient data structures and algorithms for permutation groups: BSGS, (partition) backtracking

# Ingredient I:
## Permutation Group Algorithms

- BSGS and (partition) backtrack
  could be provided by GAP, MAGMA or SAGE

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## Permutation Group Algorithms

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Lib

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- We use the callable C++ library PermLib
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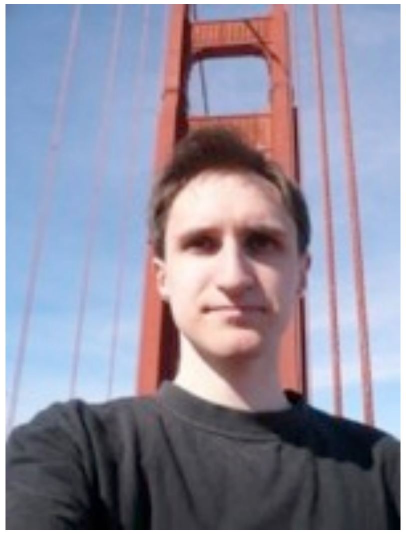
  - with compact API to access core functionality
  - can replace NAUTY



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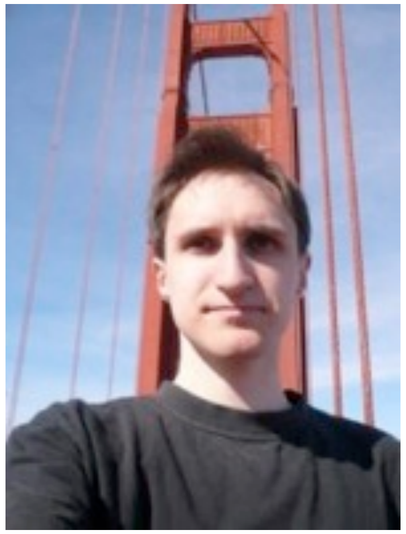

**Vision:**

- Create "integrated algorithms" combining tools of

Polyhedral Combinatorics and Computational Group Theory

# Ingredient II:
## Established Representation Conversion Tools

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  incrementally adding inequalities and recomputing vertices at every step

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WHAT ABOUT Symmetry Exploiting Methods ?

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WHAT ABOUT Symmetry Exploiting Methods ?

- with David Bremner we work(ed) on
  - pivoting methods up to symmetry
  - incremental methods using fundamental domains

# Example 1: Abhinav's Polytope

[Kum11] Abhinav Kumar, *Elliptic fibrations on a generic Jacobian Kummer surface*, arxiv:1105.1715

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H-representation
begin
316 17 integer
0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
...
end
```

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```
permutation group
9
 3 5,7 9,11 14,13 16,19 21,23 25,27 30,2
 ...
4
 33 17 49 308
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...
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```

```
permutation group
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 3 5,7 9,11 14,13 16,19 21,23 25,27 30,29
 ...
4
 33 17 49 308
```

Getting the group:

`sympol --automorphisms-only`

Getting vertices up to symmetry :

`sympol --adm 40 input-file`

```
V-representation
* UP TO SYMMETRY
begin
...
end
permutation group
* order 11520
* w.r.t. to the original inequalities/verti
```

# Example II:  Paco’s Prismatoid

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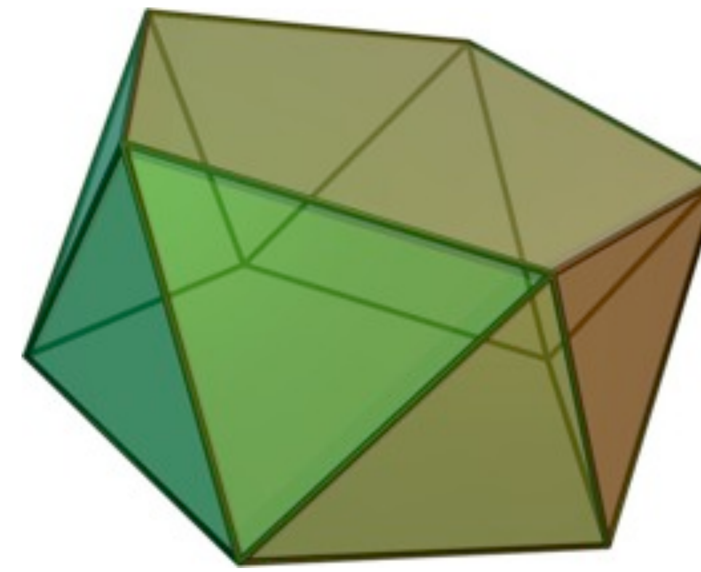
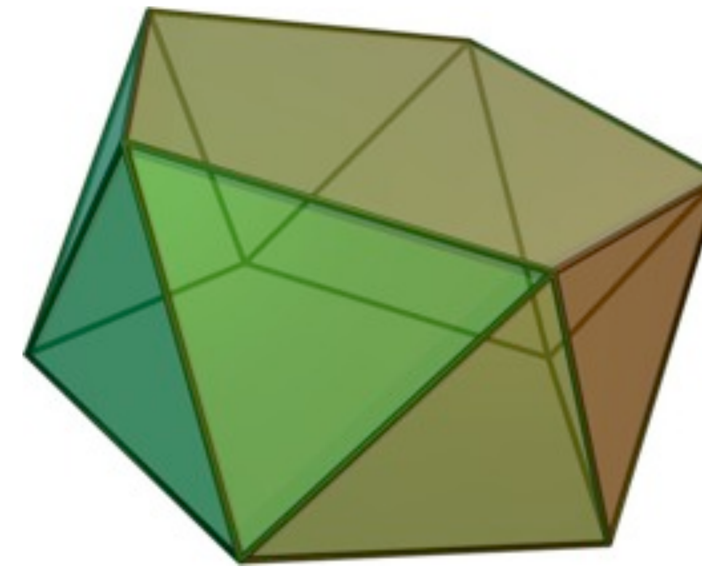
$$
Q := \mathrm{conv}\left\{
\begin{array}{c}
\\ 1^+ \\ 2^+ \\ 3^+ \\ 4^+ \\ 5^+ \\ 6^+ \\ 7^+ \\ 8^+ \\ 9^+ \\ 10^+ \\ 11^+ \\ 12^+ \\ 13^+ \\ 14^+ \\ 15^+ \\ 16^+ \\ 17^+ \\ 18^+ \\ 19^+ \\ 20^+ \\ 21^+ \\ 22^+ \\ 23^+ \\ 24^+
\end{array}
\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\
\left(\begin{array}{ccccc}
18 & 0 & 0 & 0 & 1 \\
-18 & 0 & 0 & 0 & 1 \\
0 & 18 & 0 & 0 & 1 \\
0 & -18 & 0 & 0 & 1 \\
0 & 0 & 45 & 0 & 1 \\
0 & 0 & -45 & 0 & 1 \\
0 & 0 & 0 & 45 & 1 \\
0 & 0 & 0 & -45 & 1 \\
15 & 15 & 0 & 0 & 1 \\
-15 & 15 & 0 & 0 & 1 \\
15 & -15 & 0 & 0 & 1 \\
-15 & -15 & 0 & 0 & 1 \\
0 & 0 & 30 & 30 & 1 \\
0 & 0 & -30 & 30 & 1 \\
0 & 0 & 30 & -30 & 1 \\
0 & 0 & -30 & -30 & 1 \\
0 & 10 & 40 & 0 & 1 \\
0 & -10 & 40 & 0 & 1 \\
0 & 10 & -40 & 0 & 1 \\
0 & -10 & -40 & 0 & 1 \\
10 & 0 & 0 & 40 & 1 \\
-10 & 0 & 0 & 40 & 1 \\
10 & 0 & 0 & -40 & 1 \\
-10 & 0 & 0 & -40 & 1
\end{array}\right)
\end{array}
\quad
\begin{array}{c}
\\ 1^- \\ 2^- \\ 3^- \\ 4^- \\ 5^- \\ 6^- \\ 7^- \\ 8^- \\ 9^- \\ 10^- \\ 11^- \\ 12^- \\ 13^- \\ 14^- \\ 15^- \\ 16^- \\ 17^- \\ 18^- \\ 19^- \\ 20^- \\ 21^- \\ 22^- \\ 23^- \\ 24^-
\end{array}
\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\
\left(\begin{array}{ccccc}
0 & 0 & 0 & 18 & -1 \\
0 & 0 & 0 & -18 & -1 \\
0 & 0 & 18 & 0 & -1 \\
0 & 0 & -18 & 0 & -1 \\
45 & 0 & 0 & 0 & -1 \\
-45 & 0 & 0 & 0 & -1 \\
0 & 45 & 0 & 0 & -1 \\
0 & -45 & 0 & 0 & -1 \\
0 & 0 & 15 & 15 & -1 \\
0 & 0 & 15 & -15 & -1 \\
0 & 0 & -15 & 15 & -1 \\
0 & 0 & -15 & -15 & -1 \\
30 & 30 & 0 & 0 & -1 \\
-30 & 30 & 0 & 0 & -1 \\
30 & -30 & 0 & 0 & -1 \\
-30 & -30 & 0 & 0 & -1 \\
40 & 0 & 10 & 0 & -1 \\
40 & 0 & -10 & 0 & -1 \\
-40 & 0 & 10 & 0 & -1 \\
-40 & 0 & -10 & 0 & -1 \\
0 & 40 & 0 & 10 & -1 \\
0 & 40 & 0 & -10 & -1 \\
0 & -40 & 0 & 10 & -1 \\
0 & -40 & 0 & -10 & -1
\end{array}\right)
\end{array}
\right\}
$$



# Example II: Paco's Prismatoid

$$Q := \operatorname{conv}\left\{ \begin{array}{c} \\ 1^+ \\ 2^+ \\ 3^+ \\ 4^+ \\ 5^+ \\ 6^+ \\ 7^+ \\ 8^+ \\ 9^+ \\ 10^+ \\ 11^+ \\ 12^+ \\ 13^+ \\ 14^+ \\ 15^+ \\ 16^+ \\ 17^+ \\ 18^+ \\ 19^+ \\ 20^+ \\ 21^+ \\ 22^+ \\ 23^+ \\ 24^+ \end{array} \begin{array}{c} x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \\ \begin{pmatrix} 18 & 0 & 0 & 0 & 1 \\ -18 & 0 & 0 & 0 & 1 \\ 0 & 18 & 0 & 0 & 1 \\ 0 & -18 & 0 & 0 & 1 \\ 0 & 0 & 45 & 0 & 1 \\ 0 & 0 & -45 & 0 & 1 \\ 0 & 0 & 0 & 45 & 1 \\ 0 & 0 & 0 & -45 & 1 \\ 15 & 15 & 0 & 0 & 1 \\ -15 & 15 & 0 & 0 & 1 \\ 15 & -15 & 0 & 0 & 1 \\ -15 & -15 & 0 & 0 & 1 \\ 0 & 0 & 30 & 30 & 1 \\ 0 & 0 & -30 & 30 & 1 \\ 0 & 0 & 30 & -30 & 1 \\ 0 & 0 & -30 & -30 & 1 \\ 0 & 10 & 40 & 0 & 1 \\ 0 & -10 & 40 & 0 & 1 \\ 0 & 10 & -40 & 0 & 1 \\ 0 & -10 & -40 & 0 & 1 \\ 10 & 0 & 0 & 40 & 1 \\ -10 & 0 & 0 & 40 & 1 \\ 10 & 0 & 0 & -40 & 1 \\ -10 & 0 & 0 & -40 & 1 \end{pmatrix} \end{array} \quad \begin{array}{c} \\ 1^- \\ 2^- \\ 3^- \\ 4^- \\ 5^- \\ 6^- \\ 7^- \\ 8^- \\ 9^- \\ 10^- \\ 11^- \\ 12^- \\ 13^- \\ 14^- \\ 15^- \\ 16^- \\ 17^- \\ 18^- \\ 19^- \\ 20^- \\ 21^- \\ 22^- \\ 23^- \\ 24^- \end{array} \begin{array}{c} x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \\ \begin{pmatrix} 0 & 0 & 0 & 18 & -1 \\ 0 & 0 & 0 & -18 & -1 \\ 0 & 0 & 18 & 0 & -1 \\ 0 & 0 & -18 & 0 & -1 \\ 45 & 0 & 0 & 0 & -1 \\ -45 & 0 & 0 & 0 & -1 \\ 0 & 45 & 0 & 0 & -1 \\ 0 & -45 & 0 & 0 & -1 \\ 0 & 0 & 15 & 15 & -1 \\ 0 & 0 & 15 & -15 & -1 \\ 0 & 0 & -15 & 15 & -1 \\ 0 & 0 & -15 & -15 & -1 \\ 30 & 30 & 0 & 0 & -1 \\ -30 & 30 & 0 & 0 & -1 \\ 30 & -30 & 0 & 0 & -1 \\ -30 & -30 & 0 & 0 & -1 \\ 40 & 0 & 10 & 0 & -1 \\ 40 & 0 & -10 & 0 & -1 \\ -40 & 0 & 10 & 0 & -1 \\ -40 & 0 & -10 & 0 & -1 \\ 0 & 40 & 0 & 10 & -1 \\ 0 & 40 & 0 & -10 & -1 \\ 0 & -40 & 0 & 10 & -1 \\ 0 & -40 & 0 & -10 & -1 \end{pmatrix} \end{array} \right\}$$

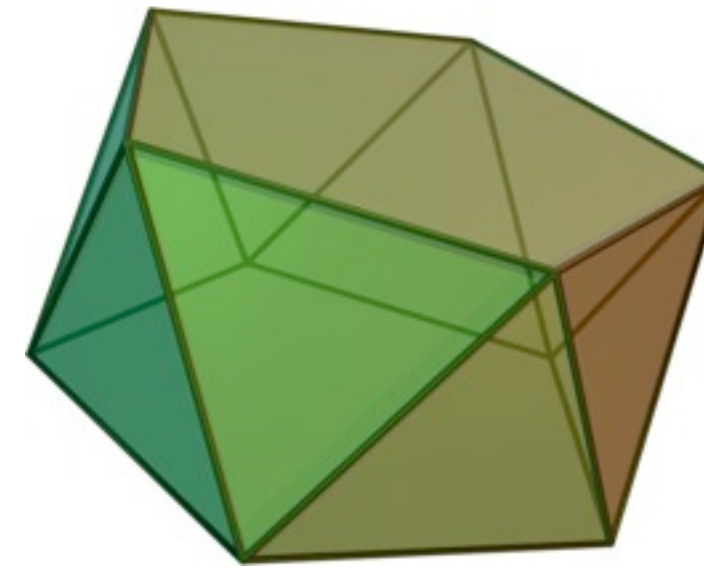


```
sympol --idm-adm-level 0 1 --adjacencies input-file
```

# Example II: Paco's Prismatoid

$Q := \mathrm{conv}$

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $1^+$ | 18 | 0 | 0 | 0 | 1 | $1^-$ | 0 | 0 | 0 | 18 | −1 |
| $2^+$ | −18 | 0 | 0 | 0 | 1 | $2^-$ | 0 | 0 | 0 | −18 | −1 |
| $3^+$ | 0 | 18 | 0 | 0 | 1 | $3^-$ | 0 | 0 | 18 | 0 | −1 |
| $4^+$ | 0 | −18 | 0 | 0 | 1 | $4^-$ | 0 | 0 | −18 | 0 | −1 |
| $5^+$ | 0 | 0 | 45 | 0 | 1 | $5^-$ | 45 | 0 | 0 | 0 | −1 |
| $6^+$ | 0 | 0 | −45 | 0 | 1 | $6^-$ | −45 | 0 | 0 | 0 | −1 |
| $7^+$ | 0 | 0 | 0 | 45 | 1 | $7^-$ | 0 | 45 | 0 | 0 | −1 |
| $8^+$ | 0 | 0 | 0 | −45 | 1 | $8^-$ | 0 | −45 | 0 | 0 | −1 |
| $9^+$ | 15 | 15 | 0 | 0 | 1 | $9^-$ | 0 | 0 | 15 | 15 | −1 |
| $10^+$ | −15 | 15 | 0 | 0 | 1 | $10^-$ | 0 | 0 | 15 | −15 | −1 |
| $11^+$ | 15 | −15 | 0 | 0 | 1 | $11^-$ | 0 | 0 | −15 | 15 | −1 |
| $12^+$ | −15 | −15 | 0 | 0 | 1 | $12^-$ | 0 | 0 | −15 | −15 | −1 |
| $13^+$ | 0 | 0 | 30 | 30 | 1 | $13^-$ | 30 | 30 | 0 | 0 | −1 |
| $14^+$ | 0 | 0 | −30 | 30 | 1 | $14^-$ | −30 | 30 | 0 | 0 | −1 |
| $15^+$ | 0 | 0 | 30 | −30 | 1 | $15^-$ | 30 | −30 | 0 | 0 | −1 |
| $16^+$ | 0 | 0 | −30 | −30 | 1 | $16^-$ | −30 | −30 | 0 | 0 | −1 |
| $17^+$ | 0 | 10 | 40 | 0 | 1 | $17^-$ | 40 | 0 | 10 | 0 | −1 |
| $18^+$ | 0 | −10 | 40 | 0 | 1 | $18^-$ | 40 | 0 | −10 | 0 | −1 |
| $19^+$ | 0 | 10 | −40 | 0 | 1 | $19^-$ | −40 | 0 | 10 | 0 | −1 |
| $20^+$ | 0 | −10 | −40 | 0 | 1 | $20^-$ | −40 | 0 | −10 | 0 | −1 |
| $21^+$ | 10 | 0 | 0 | 40 | 1 | $21^-$ | 0 | 40 | 0 | 10 | −1 |
| $22^+$ | −10 | 0 | 0 | 40 | 1 | $22^-$ | 0 | 40 | 0 | −10 | −1 |
| $23^+$ | 10 | 0 | 0 | −40 | 1 | $23^-$ | 0 | −40 | 0 | 10 | −1 |
| $24^+$ | −10 | 0 | 0 | −40 | 1 | $24^-$ | 0 | −40 | 0 | −10 | −1 |



```
sympol --idm-adm-level 0 1 --adjacencies input-file
```

```
graph adjacencies {
  1 -- 2;
  2 -- 4;
  2 -- 3;
  2 -- 2;
  3 -- 10;
  3 -- 3;
  3 -- 6;
  4 -- 5;
  4 -- 4;
  4 -- 6;
  5 -- 5;
  5 -- 7;
  5 -- 6;
  5 -- 8;
  6 -- 6;
  7 -- 7;
  7 -- 9;
  7 -- 8;
  8 -- 8;
  9 -- 12;
```

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Q := \mathrm{conv}\left\{
\begin{array}{c}
\\
1^+\\ 2^+\\ 3^+\\ 4^+\\ 5^+\\ 6^+\\ 7^+\\ 8^+\\ 9^+\\ 10^+\\ 11^+\\ 12^+\\ 13^+\\ 14^+\\ 15^+\\ 16^+\\ 17^+\\ 18^+\\ 19^+\\ 20^+\\ 21^+\\ 22^+\\ 23^+\\ 24^+
\end{array}
\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}\\
\left(\begin{array}{ccccc}
18 & 0 & 0 & 0 & 1\\
-18 & 0 & 0 & 0 & 1\\
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0 & 0 & 45 & 0 & 1\\
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0 & 0 & 0 & -45 & 1\\
15 & 15 & 0 & 0 & 1\\
-15 & 15 & 0 & 0 & 1\\
15 & -15 & 0 & 0 & 1\\
-15 & -15 & 0 & 0 & 1\\
0 & 0 & 30 & 30 & 1\\
0 & 0 & -30 & 30 & 1\\
0 & 0 & 30 & -30 & 1\\
0 & 0 & -30 & -30 & 1\\
0 & 10 & 40 & 0 & 1\\
0 & -10 & 40 & 0 & 1\\
0 & 10 & -40 & 0 & 1\\
0 & -10 & -40 & 0 & 1\\
10 & 0 & 0 & 40 & 1\\
-10 & 0 & 0 & 40 & 1\\
10 & 0 & 0 & -40 & 1\\
-10 & 0 & 0 & -40 & 1
\end{array}\right)
\end{array}
\quad
\begin{array}{c}
\\
1^-\\ 2^-\\ 3^-\\ 4^-\\ 5^-\\ 6^-\\ 7^-\\ 8^-\\ 9^-\\ 10^-\\ 11^-\\ 12^-\\ 13^-\\ 14^-\\ 15^-\\ 16^-\\ 17^-\\ 18^-\\ 19^-\\ 20^-\\ 21^-\\ 22^-\\ 23^-\\ 24^-
\end{array}
\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}\\
\left(\begin{array}{ccccc}
0 & 0 & 0 & 18 & -1\\
0 & 0 & 0 & -18 & -1\\
0 & 0 & 18 & 0 & -1\\
0 & 0 & -18 & 0 & -1\\
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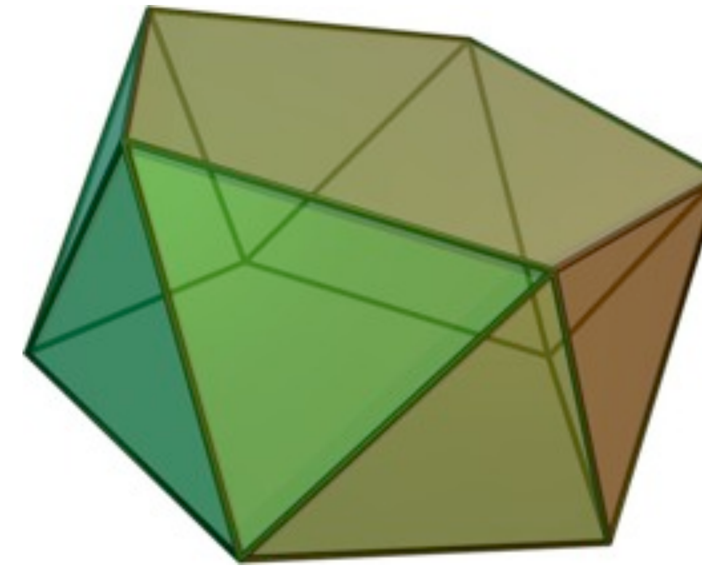
0 & 0 & 15 & -15 & -1\\
0 & 0 & -15 & 15 & -1\\
0 & 0 & -15 & -15 & -1\\
30 & 30 & 0 & 0 & -1\\
-30 & 30 & 0 & 0 & -1\\
30 & -30 & 0 & 0 & -1\\
-30 & -30 & 0 & 0 & -1\\
40 & 0 & 10 & 0 & -1\\
40 & 0 & -10 & 0 & -1\\
-40 & 0 & 10 & 0 & -1\\
-40 & 0 & -10 & 0 & -1\\
0 & 40 & 0 & 10 & -1\\
0 & 40 & 0 & -10 & -1\\
0 & -40 & 0 & 10 & -1\\
0 & -40 & 0 & -10 & -1
\end{array}\right)
\end{array}
\right\}
$$



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graph adjacencies {
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  2 -- 4;
  2 -- 3;
  2 -- 2;
  3 -- 10;
  3 -- 3;
  3 -- 6;
  4 -- 5;
  4 -- 4;
  4 -- 6;
  5 -- 5;
  5 -- 7;
  5 -- 6;
  5 -- 8;
  6 -- 6;
  7 -- 7;
  7 -- 9;
  7 -- 8;
  8 -- 8;
  9 -- 12;
```
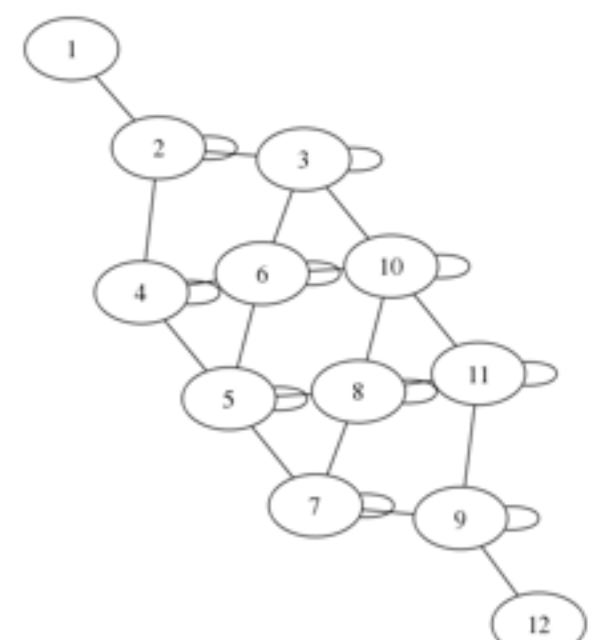

~> neato ~>

(Graphviz)



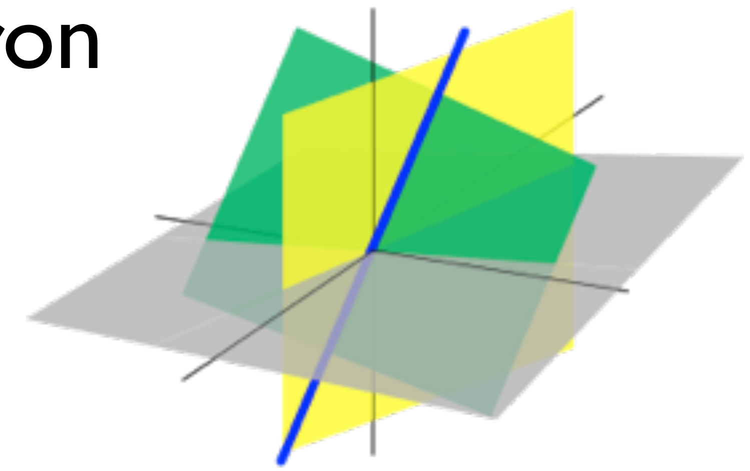
# What else?

# Exploiting Symmetries in LPs and IPs

# Exploiting Symmetries in LPs and IPs

- For LPs one can intersect feasible polyhedron

  with invariant linear subspace

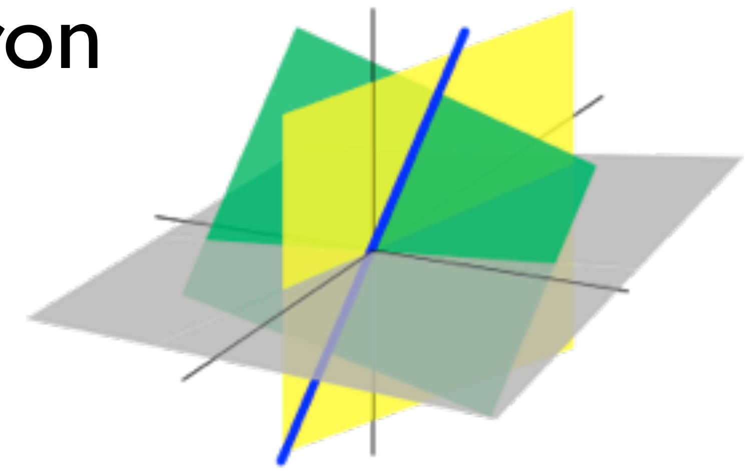


# Exploiting Symmetries in LPs and IPs

- For LPs one can intersect feasible polyhedron

  with invariant linear subspace

  (**not possible for IPs**)



# Exploiting Symmetries in LPs and IPs

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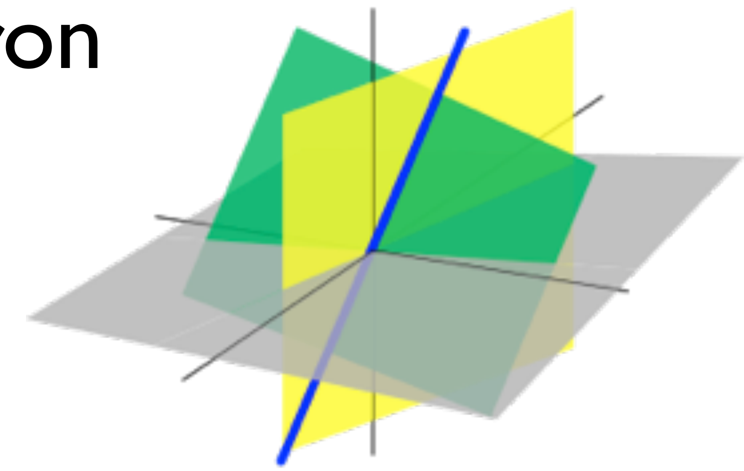
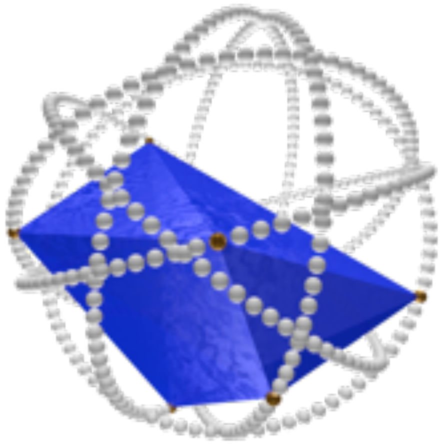


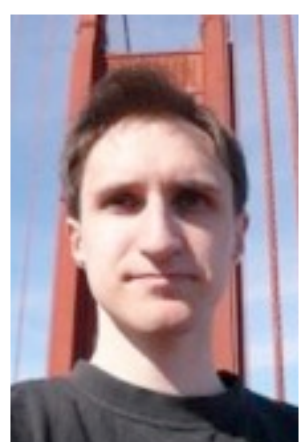
- For IPs several new approaches have been proposed



=> see survey "Symmetry in Integer Linear Programming" by François Margot (2010)


# Exploiting Polyhedral Symmetries in IPs

using invariant linear subspace



# Exploiting Polyhedral Symmetries

# Exploiting Polyhedral Symmetries

- in Lattice Point Counting


# Exploiting Polyhedral Symmetries

- in Lattice Point Counting
- in Polyhedral Representation Conversions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# Exploiting Polyhedral Symmetries

- in Lattice Point Counting
- in Polyhedral Representation Conversions
- in Integer Programming and MILPs


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- in Lattice Point Counting
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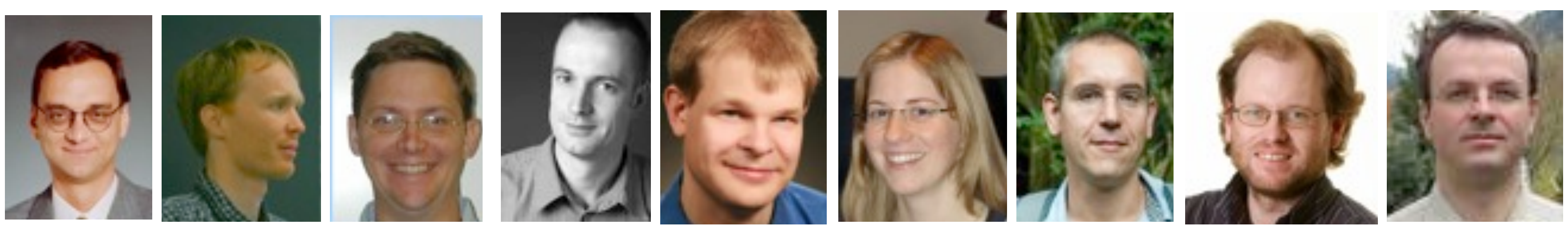

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# ToDo

- Create efficient computational tools / use more math!
- Integrate tools from Computational Group Theory

Thanks!

**http://www.geometrie.uni-rostock.de**