Irreducible contact graphs and Tammes' problem

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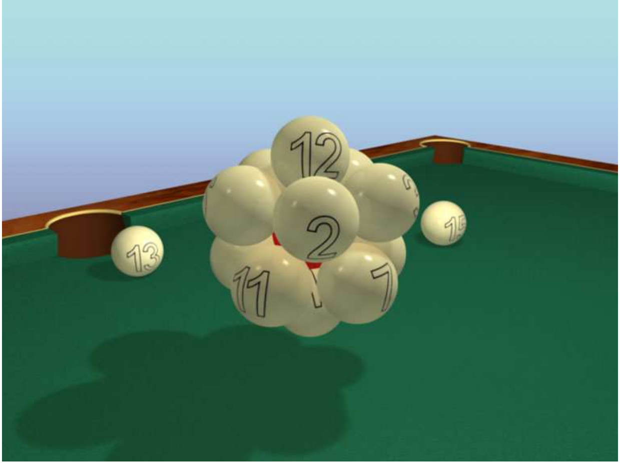
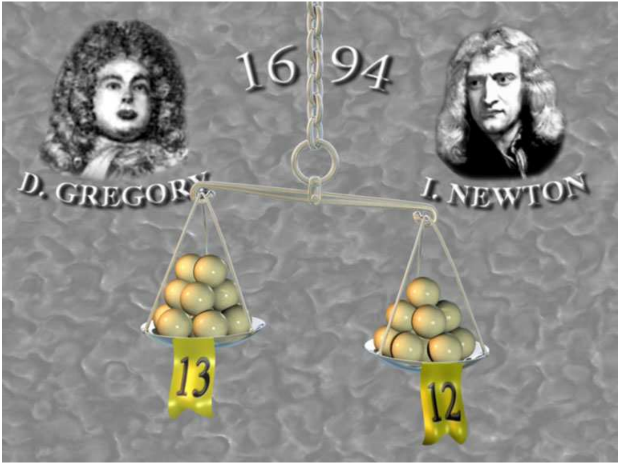
Fields Institute: September 21, 2011

The thirteen spheres problem



The thirteen spheres problem



The thirteen spheres problem: proofs

K. Schütte, and B. L. van der Waerden (1953)

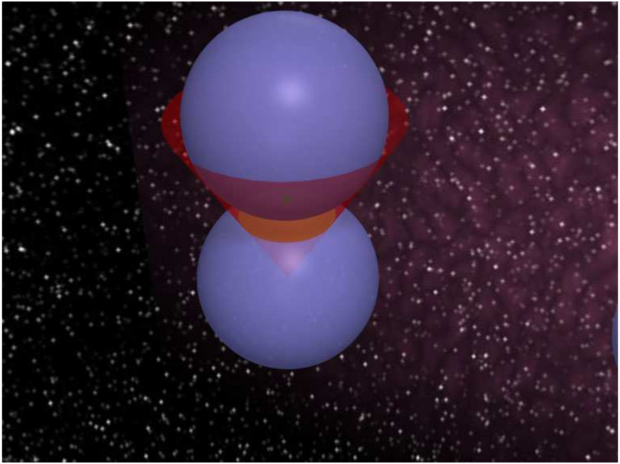
John Leech (1956) : two-page sketch of a proof

. . . It also misses one of the old chapters, about the "problem of the thirteen spheres," whose turned out to need details that we couldn't complete in a way that would make it brief and elegant.

Proofs from THE BOOK, M. Aigner, G. Ziegler, 2nd edition.

W. –Y. Hsiang (2001);
H. Maehara (2001, 2007);
K. Böröczky (2003);
K. Anstreicher (2004);
M. (2006)

The Tammes problem



The Tammes problem

How must N congruent non-overlapping spherical caps be packed on the surface of a unit sphere so that the angular diameter of spherical caps will be as great as possible

Tammes PML (1930). "On the origin of number and arrangement of the places of exit on pollen grains". Diss. Groningen.

The Tammes problem

Let X be a finite subset of $\mathbb{S}^2$. Denote

$$\psi(X) := \min_{x,y \in X} \{\mathrm{dist}(x,y)\}, \text{ where } x \neq y.$$

Then X is a spherical $\psi(X)$-code.

Denote by d_N the largest angular separation $\psi(X)$ with $|X| = N$ that can be attained in $\mathbb{S}^2$, i.e.

$$d_N := \max_{X \subset \mathbb{S}^2} \{\psi(X)\}, \text{ where } |X| = N.$$

The Tammes problem

L. Fejes Tóth (1943) $N = 3, 4, 6, 12, \infty$

K. Schütte, and B. L. van der Waerden (1951) $N = 5, 7, 8, 9$

L. Danzer (1963) $N = 10, 11$

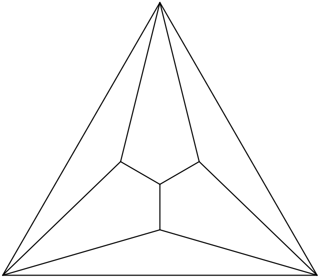
R. M. Robinson (1961) $N = 24$

M. & T. (2010) $N = 13$

d_N

N	d_N
4	109.4712206
5	90.0000000
6	90.0000000
7	77.8695421
8	74.8584922
9	70.5287794
10	66.1468220
11	63.4349488
12	63.4349488
13	57.1367031
....	....
14	55.6705700
15	53.6578501
16	52.2443957
17	51.0903285

G_7



Packing spheres by spheres: Methods

I. *Area inequalities.* L. Fejes Tóth (1943); for $d > 3$ Coxeter (1963) and Böröczky (1978)

II. *Distance and irreducible graphs.* Schütte, and van der Waerden (1951); Danzer (1963); Leech (1956);...

III. *LP and SDP.* Delsarte et al (1977); Kabatiansky and Levenshtein (1978);...

Robert Connelly, Rigidity of packings, European Journal of Combinatorics, 2008

Contact graphs

Let X be a finite set in $\mathbb{S}^2$. The *contact graph* $\mathrm{CG}(X)$ is the graph with vertices in X and edges (x, y), $x, y \in X$ such that

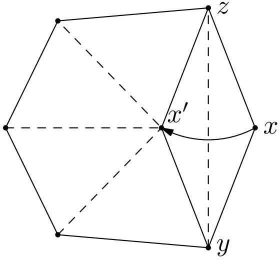
$$\mathrm{dist}(x, y) = \psi(X)$$

Shift of a single vertex

Let X be a finite set in $\mathbb{S}^2$. Let $x \in X$ be a vertex of $\mathrm{CG}(X)$ with $\deg(x) > 0$, i.e. there is $y \in X$ such that $\mathrm{dist}(x, y) = \psi(X)$. We say that there exists a shift of x if x can be slightly shifted to x' such that $\mathrm{dist}(x', X \setminus \{x\}) > \psi(X)$.

Danzer's flip

Danzer [1963] defined the following flip. Let x, y, z be vertices of $\mathrm{CG}(X)$ with $\mathrm{dist}(x, y) = \mathrm{dist}(x, z) = \psi(X)$. We say that x is flipped over yz if x is replaced by its mirror image x' relative to the great circle yz. We say that this flip is *Danzer's flip* if $\mathrm{dist}(x', X \setminus \{x, y, z\}) > \psi(X)$.



Irreducible contact graph

We say that the graph CG(X) is *irreducible* [Schütte - van der Waerden, Fejes Tóth] (or *jammed* [Connelly]) if there are no shift of vertices.

If there are neither Danzer's flips nor shifts of vertices, then we call CG(X) as a *(Danzer's) irreducible graph.*

Maximal graphs G_N

Let X be a subset of $\mathbb{S}^2$ with $|X| = N$. We say that $\mathrm{CG}(X)$ is *maximal* if $\psi(X) = d_N$ and its number of edges is minimum. We denote this graph by G_N.
Actually, this definition does not assume that G_N is unique. We use this designation for some $\mathrm{CG}(X)$ with $\psi(X) = d_N$.

Proposition. Let $\mathrm{CG}(X)$ be a maximal graph G_N. Then for $N \geq 6$ the graph $\mathrm{CG}(X)$ is irreducible.

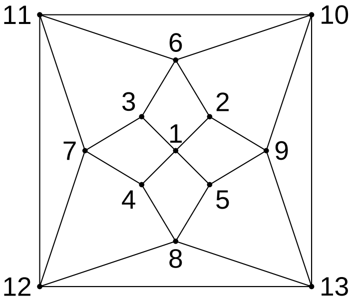
Properties of irreducible graphs

Let the graph $\mathrm{CG}(X)$ be irreducible. Then

1. $\mathrm{CG}(X)$ is a planar graph.
2. Degrees of $\mathrm{CG}(X)$ vertices can take only the values 0 (isolated vertices), 3, 4, or 5.
3. All faces of $\mathrm{CG}(X)$ in $\mathbb{S}^2$ are equilateral convex polygons of sides length $\psi(X)$.
4. All faces of $\mathrm{CG}(X)$ are polygons with at most $\lfloor 2\pi/\psi(X) \rfloor$ vertices.

N=13

The contact graph $\Gamma_{13} := \mathrm{CG}(P_{13})$ with $\psi(P_{13}) \approx 57.1367°$



Tammes' problem for $N = 13$

The value $d = \psi(P_{13})$ can be found analytically.

$$2\tan\left(\frac{3\pi}{8} - \frac{a}{4}\right) = \frac{1 - 2\cos a}{\cos^2 a}$$

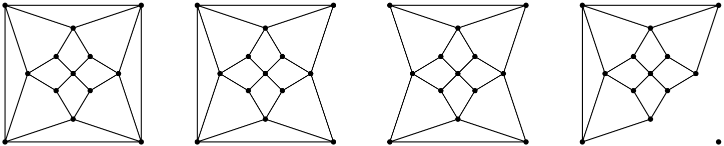
$$d = \cos^{-1}\left(\frac{\cos a}{1 - \cos a}\right).$$

Thus, we have $a \approx 69.4051°$ and $d \approx 57.1367°$.

Tammes' problem for $N = 13$

Theorem. The arrangement of 13 points P_{13} in $\mathbb{S}^2$ is the best possible, the maximal arrangement is unique up to isometry, and $d_{13} = \psi(P_{13})$.

Tammes' problem for $N = 13$: graphs $\Gamma_{13}^{(i)}$



$\Gamma_{13}^{(0)}$ $\Gamma_{13}^{(1)}$ $\Gamma_{13}^{(2)}$ $\Gamma_{13}^{(3)}$

Figure: Graphs $\Gamma_{13}^{(i)}$.

Main lemmas

Lemma 1. *G_{13} is isomorphic to $\Gamma_{13}^{(i)}$ with $i = 0, 1, 2,$ or 3.*

Lemma 2. *G_{13} is isomorphic to $\Gamma_{13}^{(0)}$ and $d_{13} = \psi(P_{13}) \approx 57.1367°$.*

Properties of G_{13}

1. It is a planar graph with 13 vertices.
2. The degree of a vertex is 0,3,4, or 5.
3. All faces are polygons with m=3,4,5, or 6 vertices.
4. If there is an isolated vertex, then it lies in a hexagonal face.
5. No more than one vertex can lie in a hexagonal face.

Proof of Lemma 1

The proof consists of two parts:
(I) Create the list L_{13} of all graphs with 13 vertices that satisfy 1–5;
(II) Using linear approximations and linear programming remove from the list L_{13} all graphs that do not satisfy the geometric properties of G_{13}

Proof of Lemma 1: The list L_{13}

To create L_{13} we use the program *plantri* (Gunnar Brinkmann and Brendan McKay). This program is the isomorph-free generator of planar graphs, including triangulations, quadrangulations, and convex polytopes.

The program plantri generates 94,754,965 graphs in L_{13}. Namely, L_{13} contains 30,829,972 graphs with triangular and quadrilateral faces; 49,665,852 with at least one pentagonal face and with triangular and quadrilaterals; 13,489,261 with at least one hexagonal face which do not contain isolated vertices; 769,375 graphs with one isolated vertex, 505 with two isolated vertices, and no graphs with three or more isolated vertices.

Proof of Lemma 1

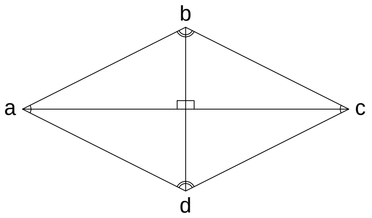
Let G be a graph from the list L_{13} .

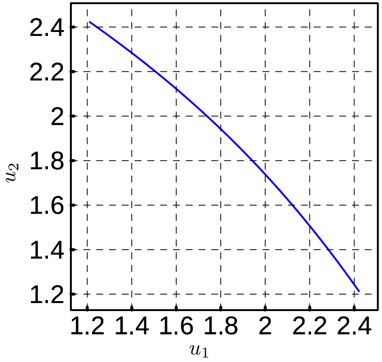
Variables: d (the length of edges), angles of faces.

Equations and inequalities:

1. $d > 57.1367^0$.
2. For each vertex sum of its angles $= 2\pi$.
3. For a triangle: $u = \arccos\left(\cos d/(1+\cos d)\right)$.
4. For a quadrilateral: an explicit equation.
5. For a pentagon: an approximation by linear inequalities.
6. For an empty hexagon: an approximation by linear inequalities.
7. For a hexagon with an isolated vertex: an approximation by linear inequalities.

Proof of Lemma 1: Quadrilateral

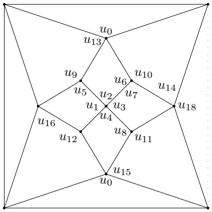


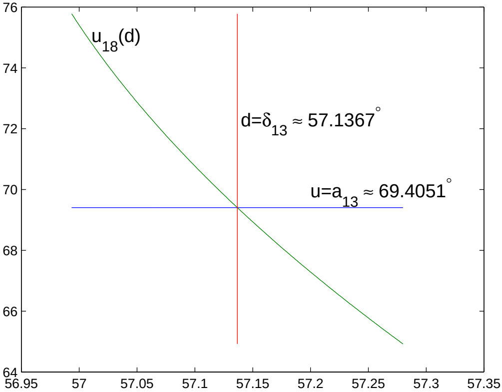


Proof of Lemma 1: Feasible solutions of the system

1. Do linear estimations of equations.
2. Using LP find a convex region containing a possible solution.
3. Using a region do more precise linear approximations and go back to steps 1,2.
4. If a region becomes empty – system is unfeasible.
5. if a region is still not empty split it into two parts, and go back to steps 1–5.
6. If all regions (after splitting) become empty – system if unfeasible.

Proof of Lemma 2



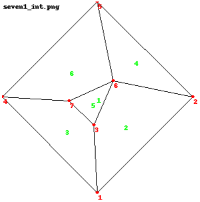


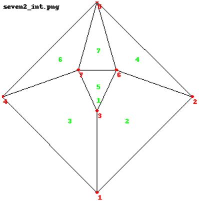
Research directions

1. Danzer (1963) considered (Danzer's) irreducible graphs with $N \leq 10$ vertices. We plan to verify and extend Danzer's classification for N up to 13.
2. We plan to consider maximal and other irreducible graphs for the Tammes problem with $N = 14$ and higher.
3. We will also explore the applicability of the methods discussed here to solve the Tammes problem for $N = 14$ and higher.
4. We plan to extend the concept of irreducible graphs for packing equal circles into two-dimensional manifolds. In according to Daniel Usikov the case of a flat torus is especially interesting for the problem of "super resolution of images".
5. Connelly considered rigidity of circles packings from the point of view of the theory of tensegrity structures. An interesting follow-up project is extending these ideas to combinatorial structure of irreducible graphs.

Irreducible graphs for N=7

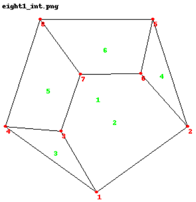
N	d_{min}	d_{max}
$1*$	1.34978	1.35908
$2**$	1.35908	1.35908

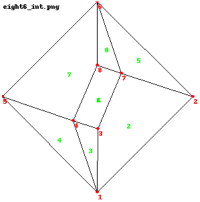


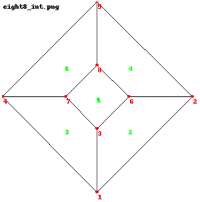


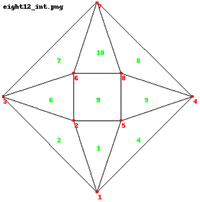
Irreducible graphs for N=8

N	d_{min}	d_{max}
1	1.17711	1.18349
$6*$	1.28619	1.30653
$8*$	1.23096	1.30653
$12**$	1.30653	1.30653







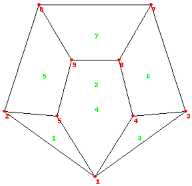


Irreducible graphs for N=9

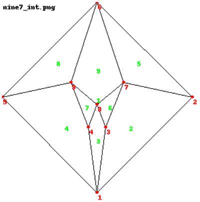
N	d_{min}	d_{max}
4	1.14099	1.14143
7*	1.22308	1.23096
8	1.10525	1.14349
11	1.17906	1.18106
13	1.15448	1.17906
15	1.17906	1.17906
18 **	1.23096	1.23096
20	1.15032	1.18106
21*	1.10715	1.14342

Irreducible graphs for N=9

nine4_int.png



nine7_int.png



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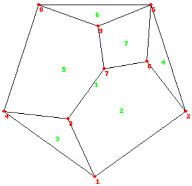

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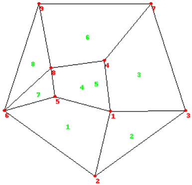

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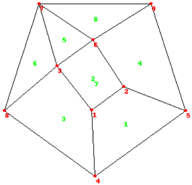

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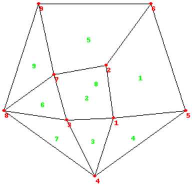

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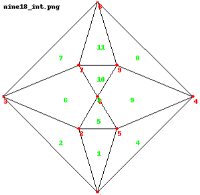

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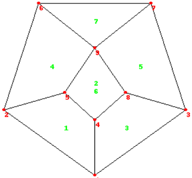

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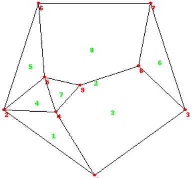
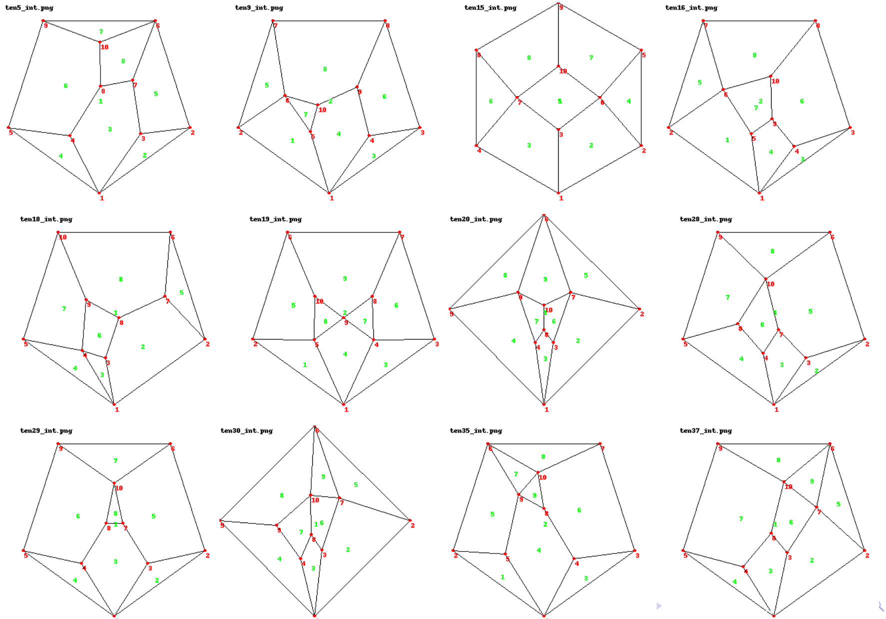


Irreducible graphs for N=10

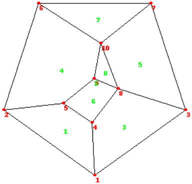
N	d_{min}	d_{max}	N	d_{min}	d_{max}
5	1.0839	1.09751	9	1.08161	1.08439
15	1.03067	1.04695	16	1.10715	1.0988
18	1.07529	1.09431	19	1.09386	1.12285
20∗	1.15278	1.15448	28	1.10012	1.10801
29	1.06344	1.07834	30∗	1.15074	1.15191
35	1.0843	1.08442	37	1.10055	1.10889
44	1.09504	1.10429	45	1.06032	1.09604
48	1.06278	1.1098	60	1.09567	1.10715
64 ∗ ∗	1.15448	1.15448	66	0.998657	1.0467
67	1.0843	1.0844	75	1.08334	1.09547
80∗	1.15341	1.15341	81	1.0988	1.10608
89∗	1.14372	1.15191	91	1.09249	1.1098
92∗	1.15191	1.15245	93	1.09658	1.10977
98∗	1.15191	1.15191	103∗	1.10715	1.10715
104∗	1.10715	1.10715			

Irreducible graphs for N=10



Irreducible graphs for N=10

ten44_int.png



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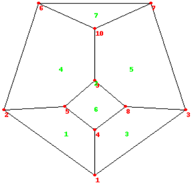

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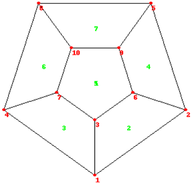

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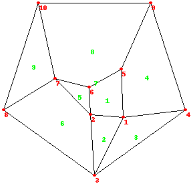

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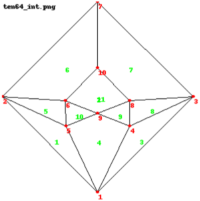

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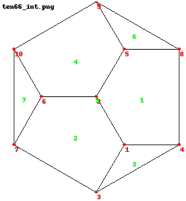

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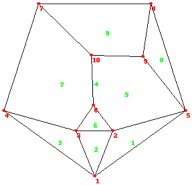

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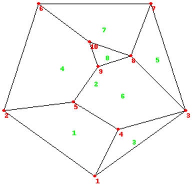

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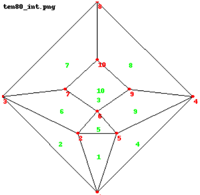

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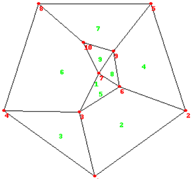

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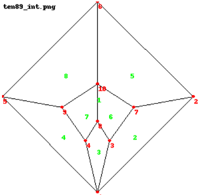

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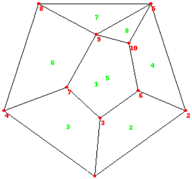
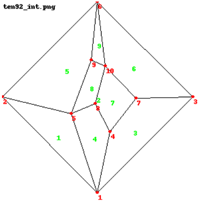
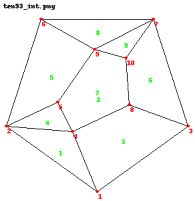
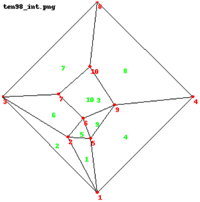
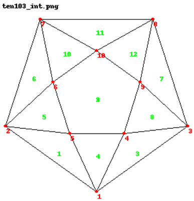


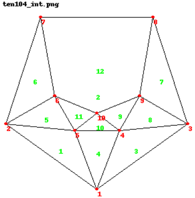
Irreducible graphs for N=10

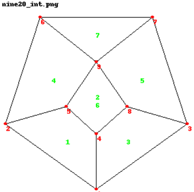




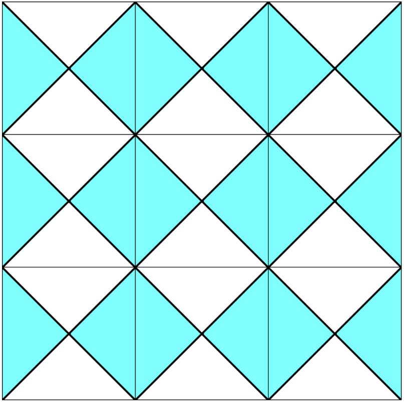


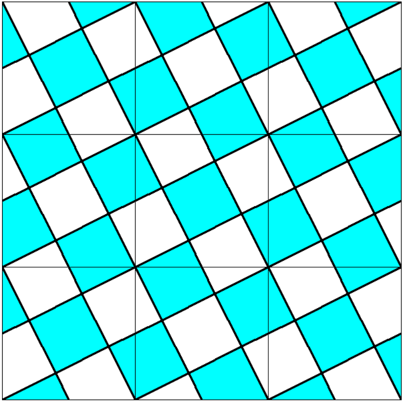




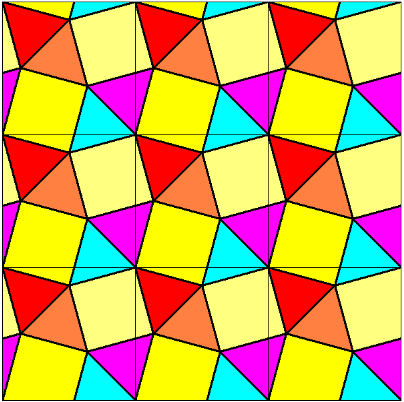


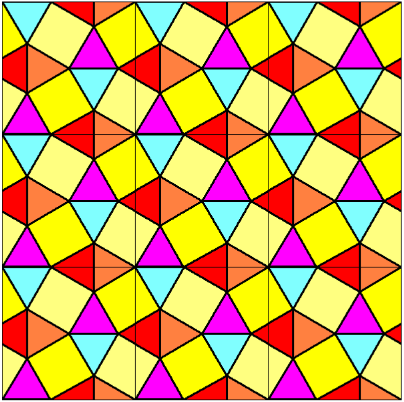
Toric packings: N=2 and N=5





Toric packings: N=4 and N=8





Toric packings: N=6