THE BRICK POLYTOPE

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Fields Institute Toronto

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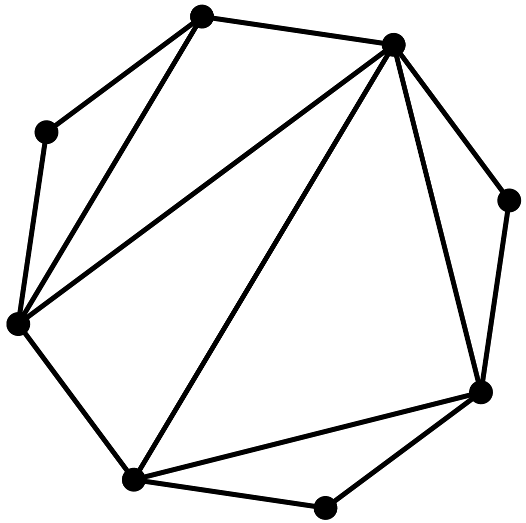
Universidad de Cantabria

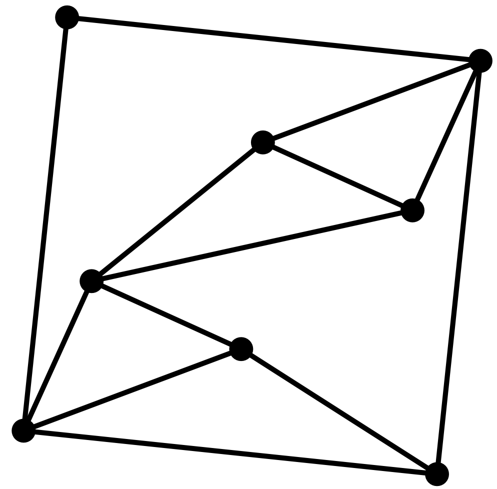
...TRIANGULATIONS

THREE GEOMETRIC STRUCTURES

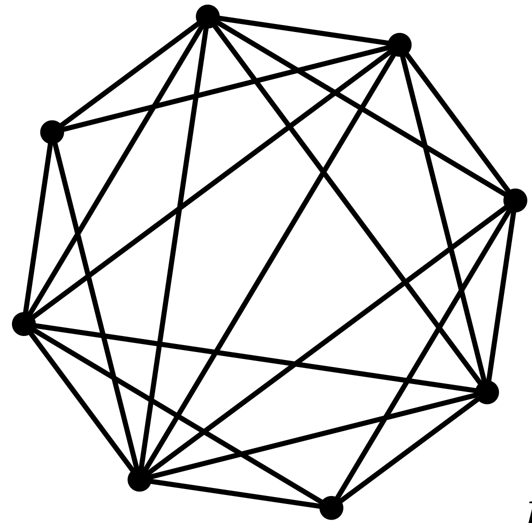
Triangulations



Pseudotriangulations



Multitriangulations



$k = 2$

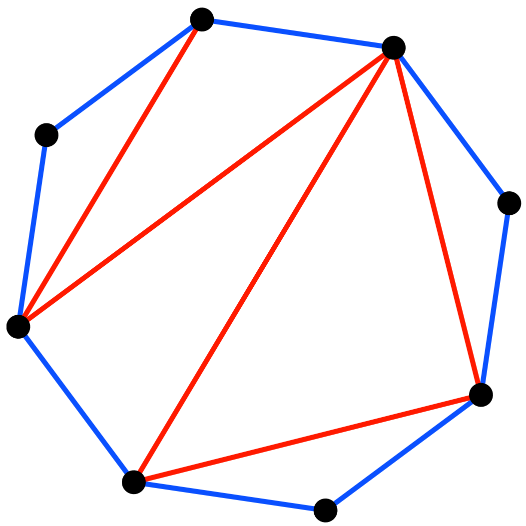
triangulation $=$ maximal crossing-free set of edges,

pseudotriangulation $=$ maximal crossing-free pointed set of edges,

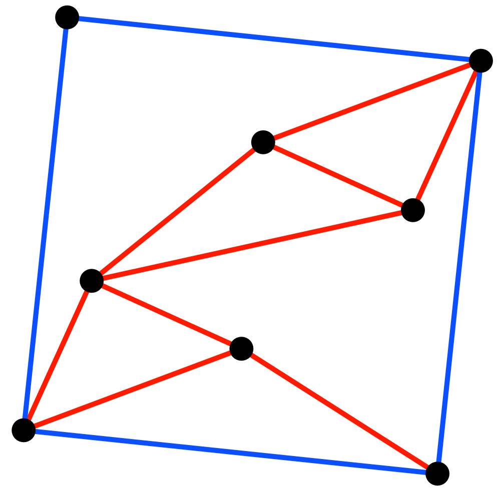
k-triangulation $=$ maximal $(k+1)$-crossing-free set of edges,

THREE GEOMETRIC STRUCTURES

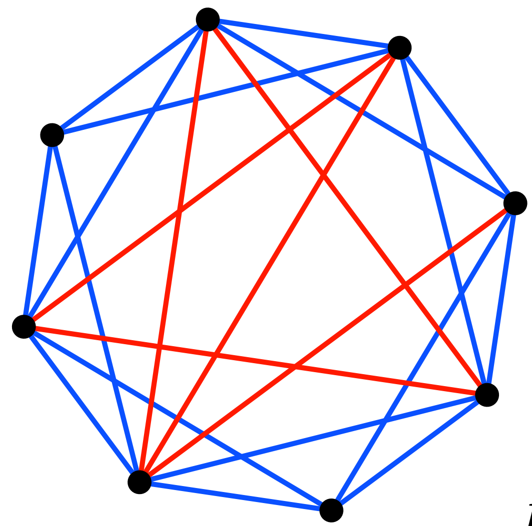
Triangulations



Pseudotriangulations



Multitriangulations



$k = 2$

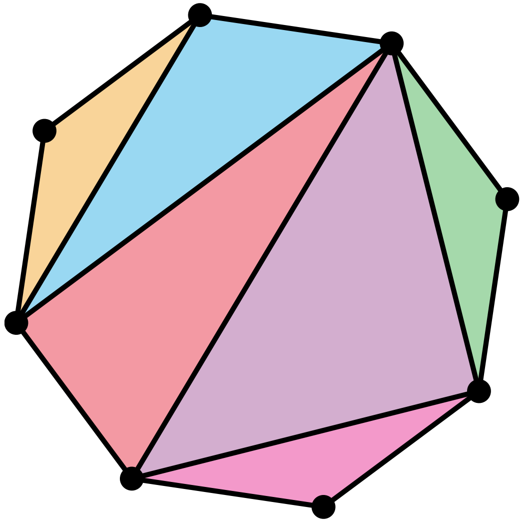
triangulation = maximal crossing-free set of edges,

pseudotriangulation = maximal crossing-free pointed set of edges,

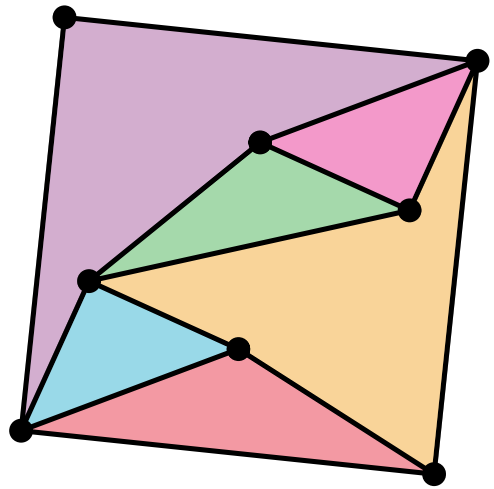
k-triangulation = maximal $(k+1)$-crossing-free set of edges,

THREE GEOMETRIC STRUCTURES

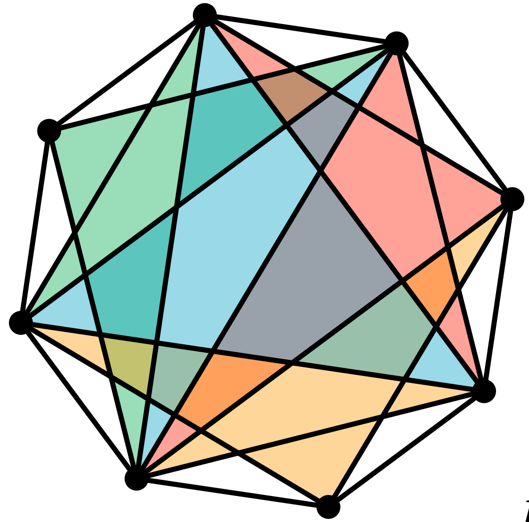
Triangulations



Pseudotriangulations



Multitriangulations

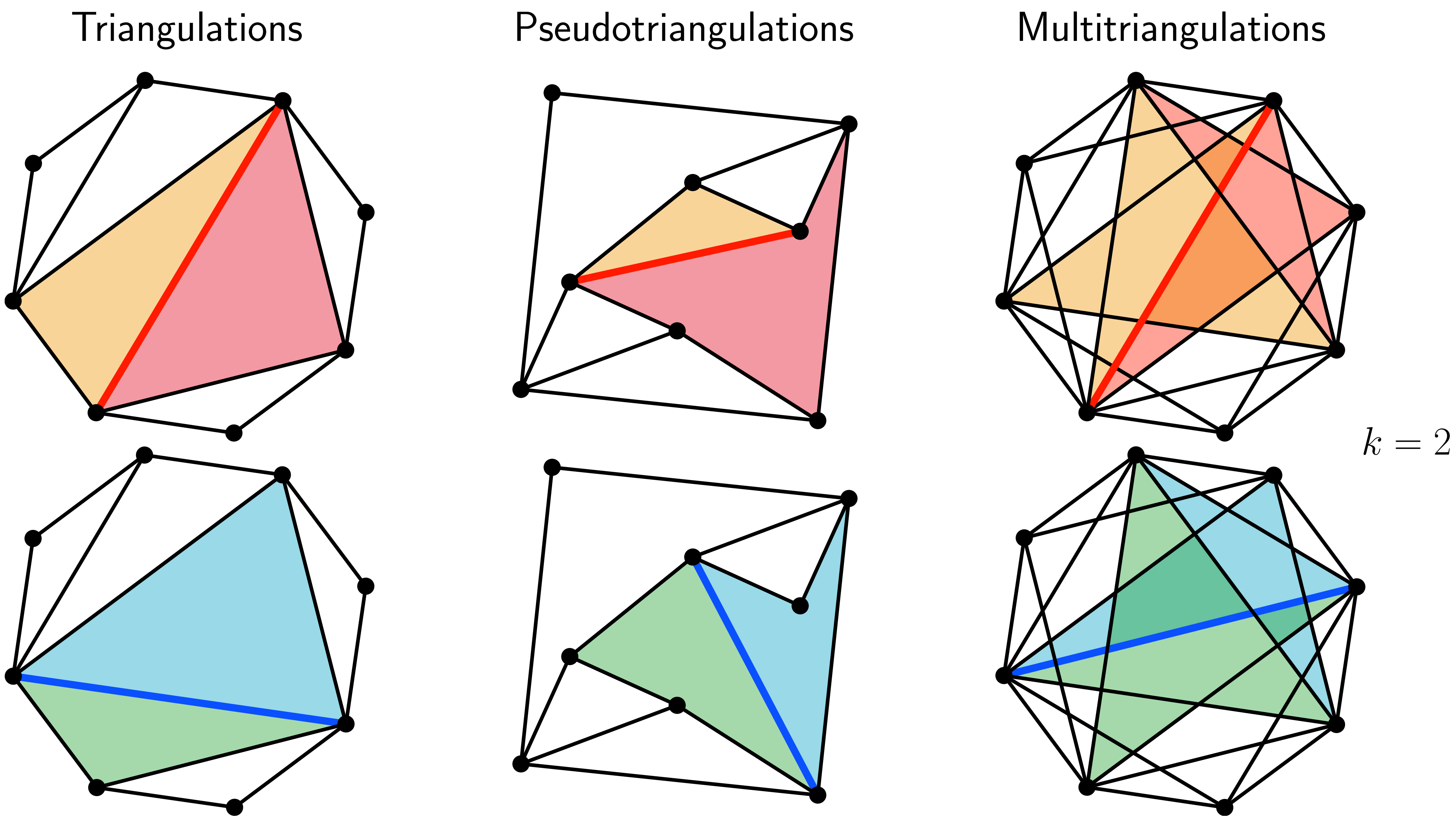


$k = 2$

triangulation	$=$ maximal crossing-free set of edges,
	$=$ decomposition into triangles.
pseudotriangulation	$=$ maximal crossing-free pointed set of edges,
	$=$ decomposition into pseudotriangles.
k-triangulation	$=$ maximal $(k+1)$-crossing-free set of edges,
	$=$ decomposition into k-stars.

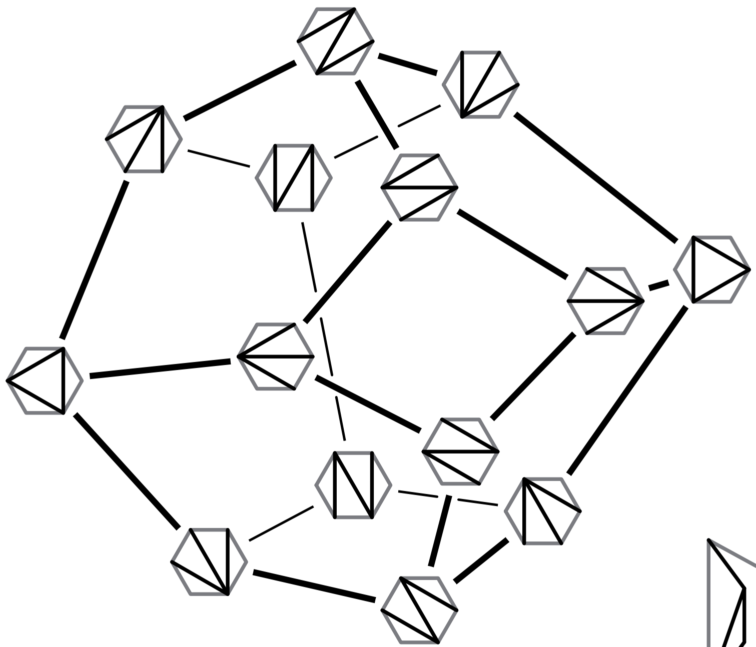
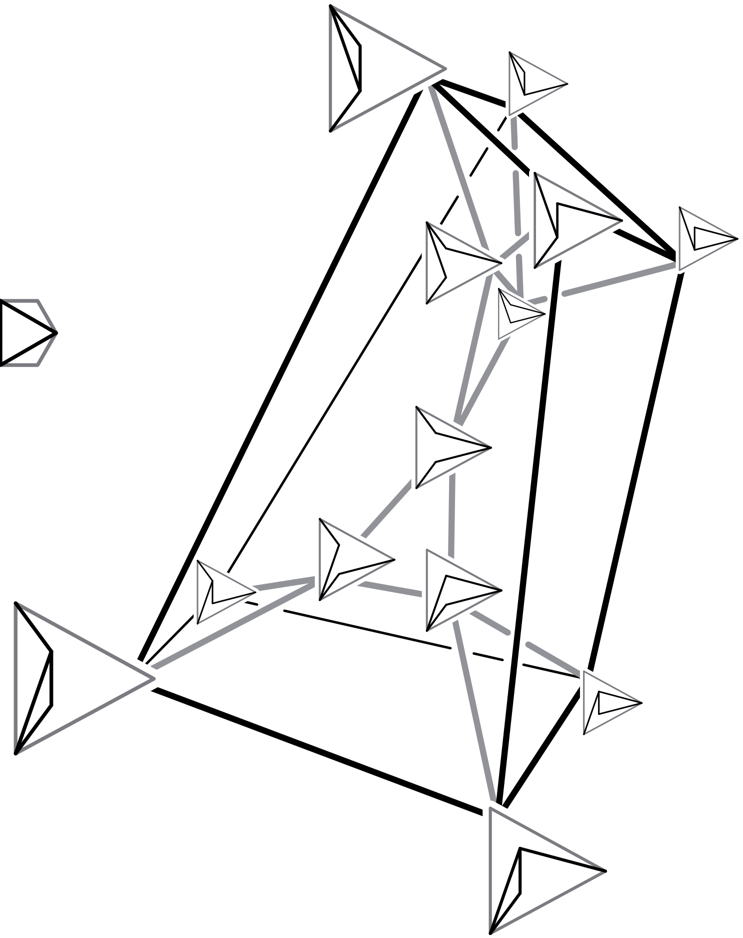
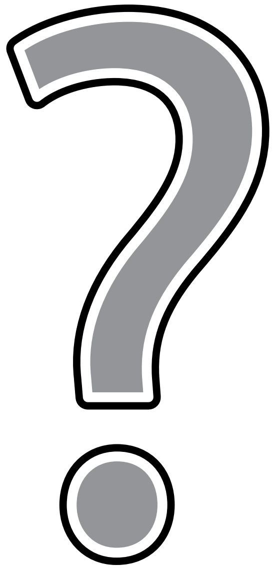
VP & F. Santos, Multitriangulations as complexes of star polygons, 2009.

THREE GEOMETRIC STRUCTURES



flip = exchange an internal edge with the common bisector of the two adjacent cells.

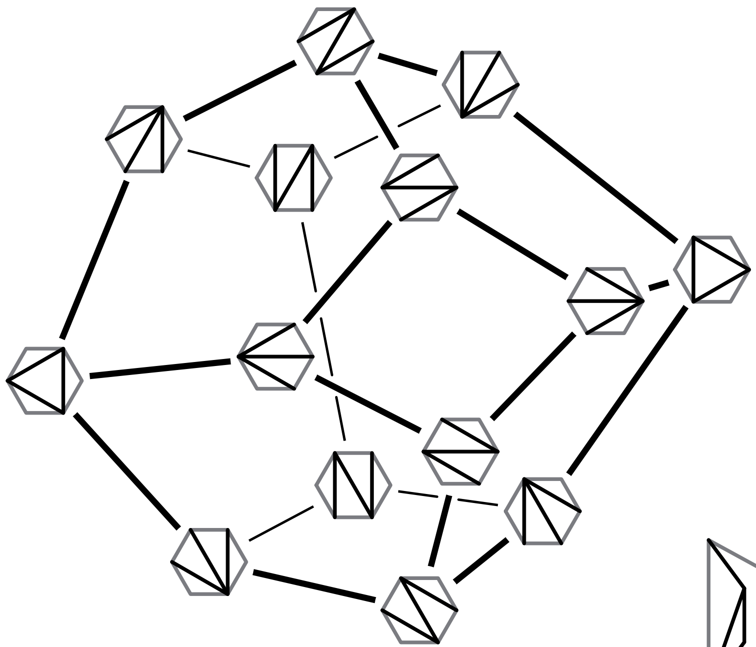
THREE GEOMETRIC STRUCTURES

Triangulations	Pseudotriangulations	Multitriangulations
		

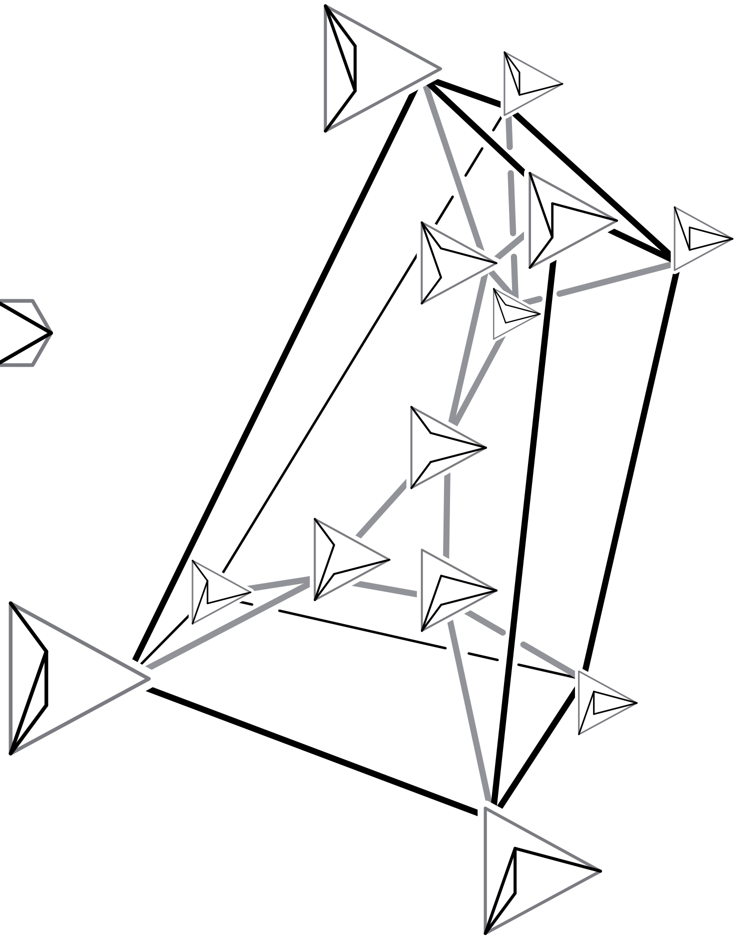
associahedron	$\longleftrightarrow$	crossing-free sets of internal edges.
pseudotriangulations polytope	$\longleftrightarrow$	pointed crossing-free sets of internal edges.
multiassociahedron	$\longleftrightarrow$	$(k+1)$-crossing-free sets of k-internal edges.

THREE GEOMETRIC STRUCTURES

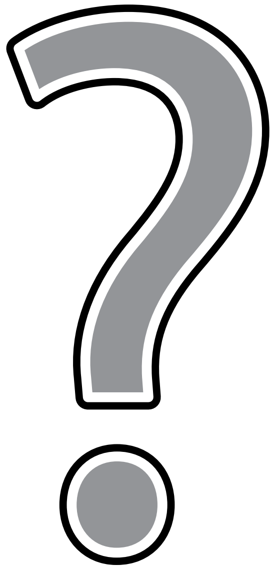
Triangulations



Pseudotriangulations



Multitriangulations



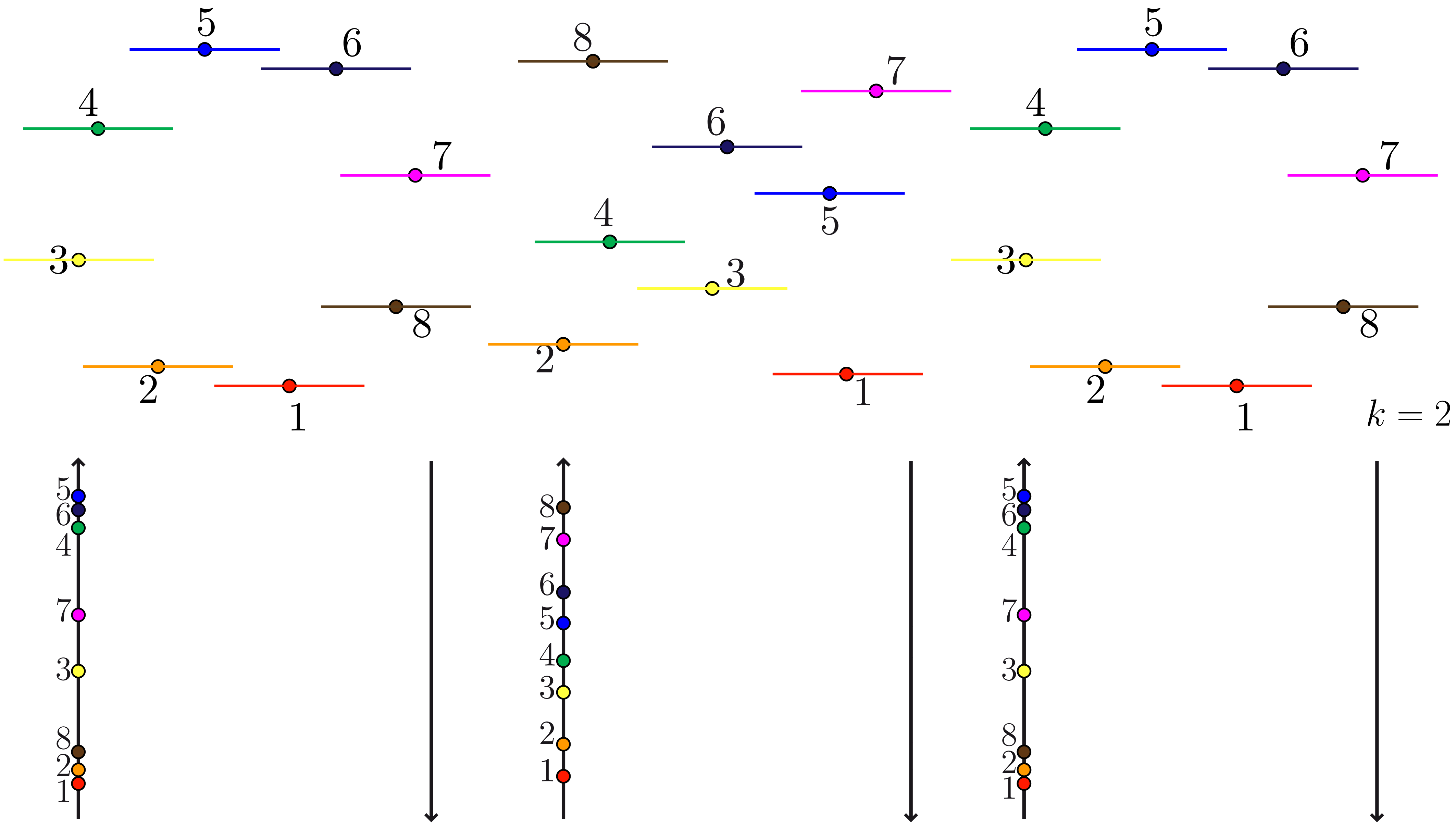
"Our main limits in understanding the combinatorial structure of polytopes still lie in our ability to raise the good questions and in the lack of examples, methods of constructing them, and means of classifying them."

G. Kalai, *Handbook of Discrete & Computational Geometry*.

DUALITY

DUALITY

Triangulations | Pseudotriangulations | Multitriangulations



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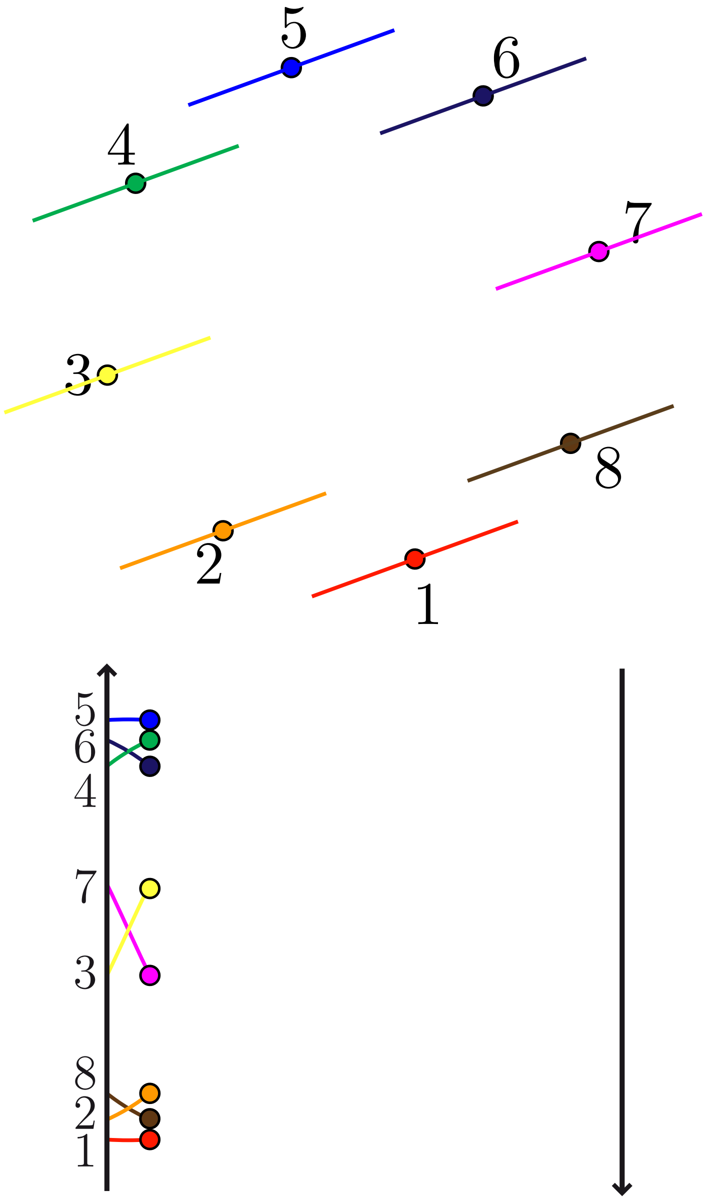
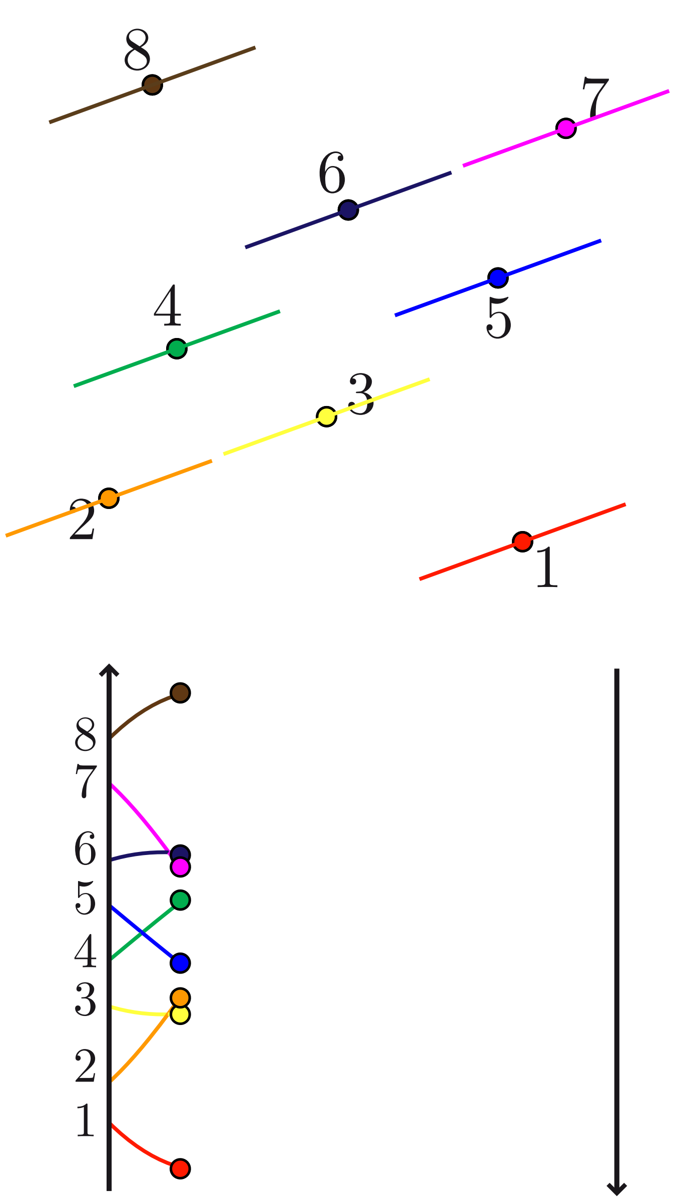
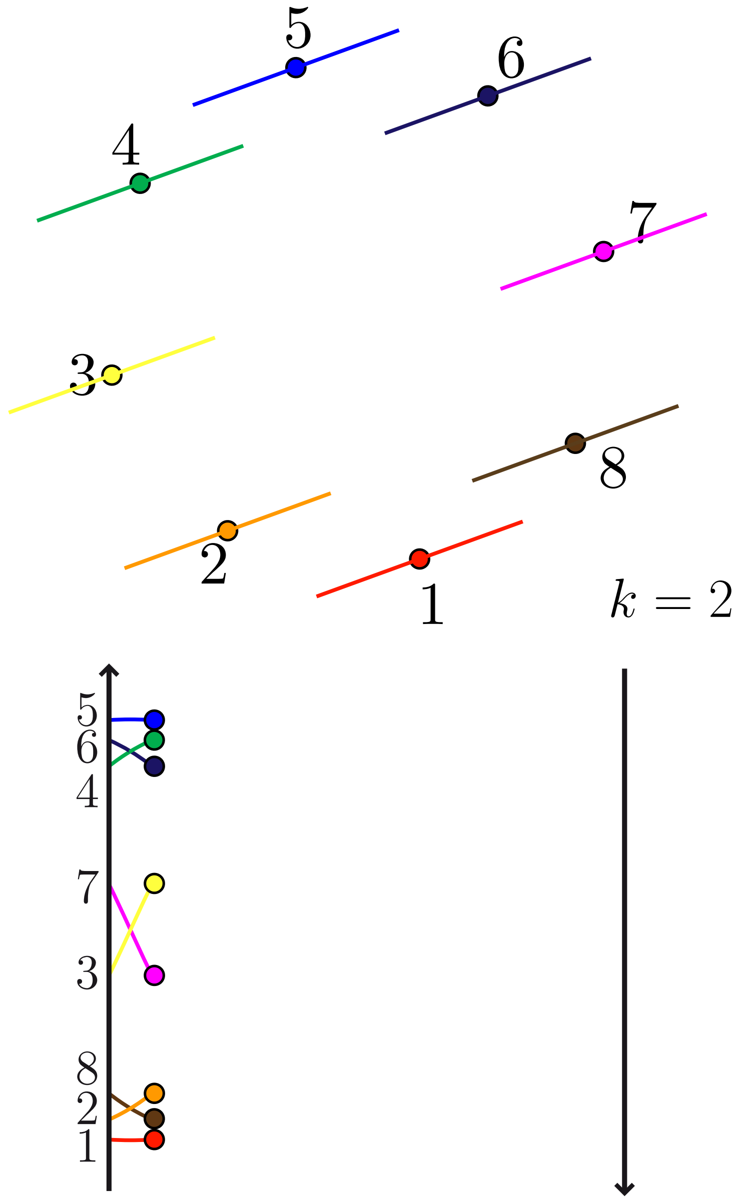
Triangulations



Pseudotriangulations



Multitriangulations



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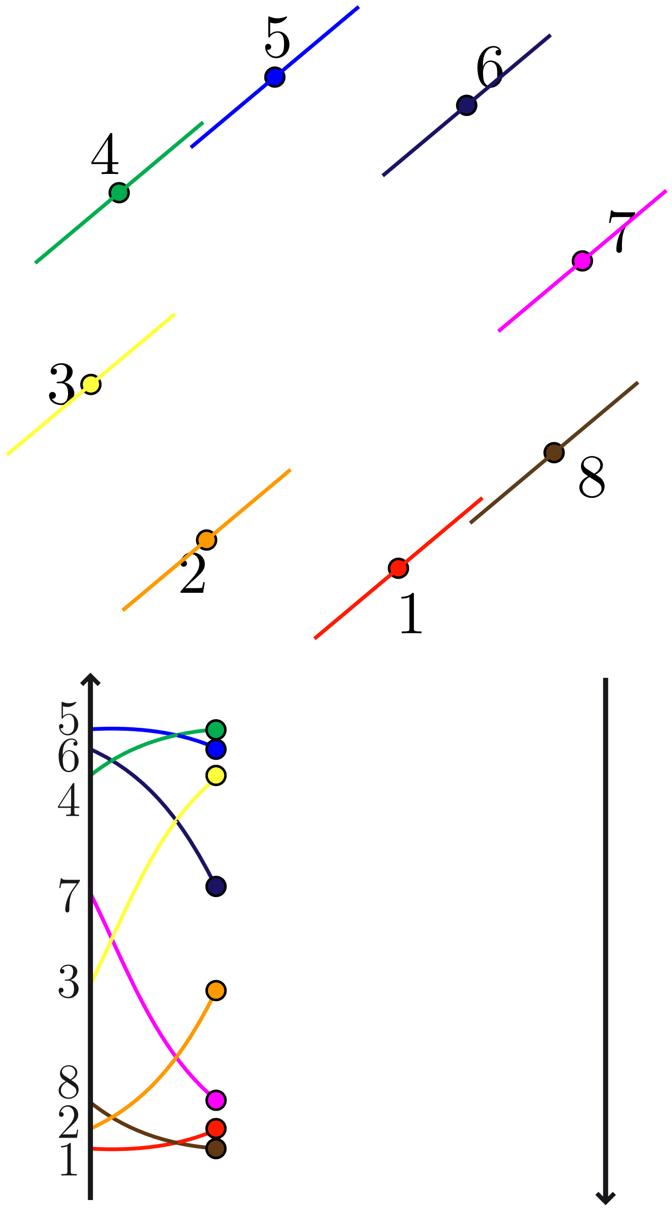
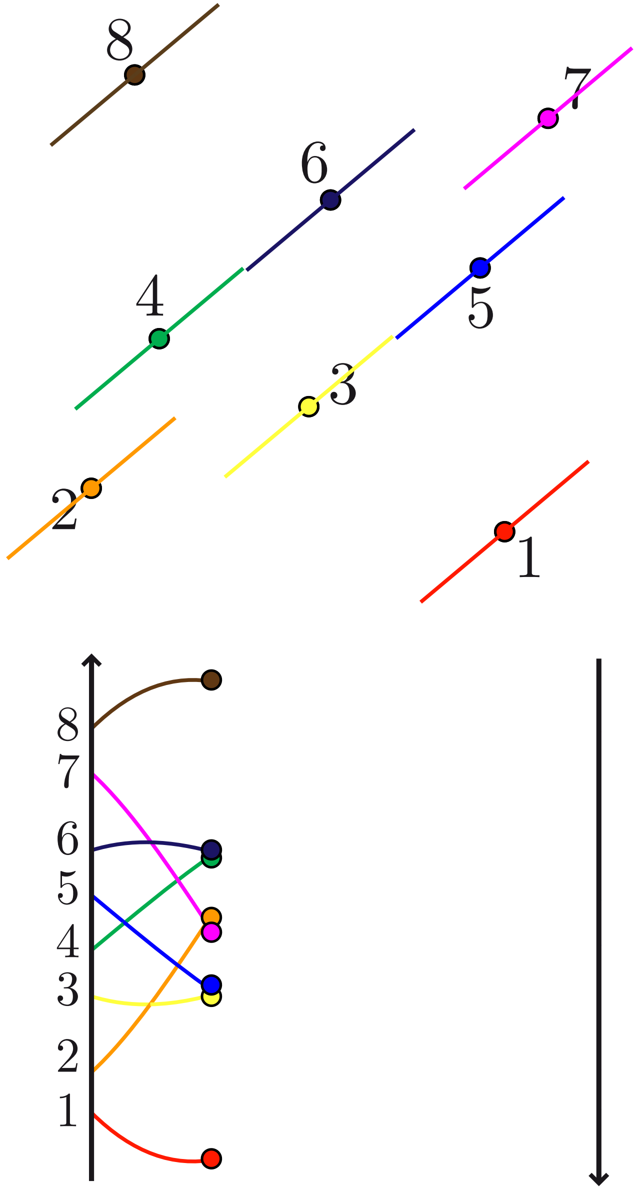
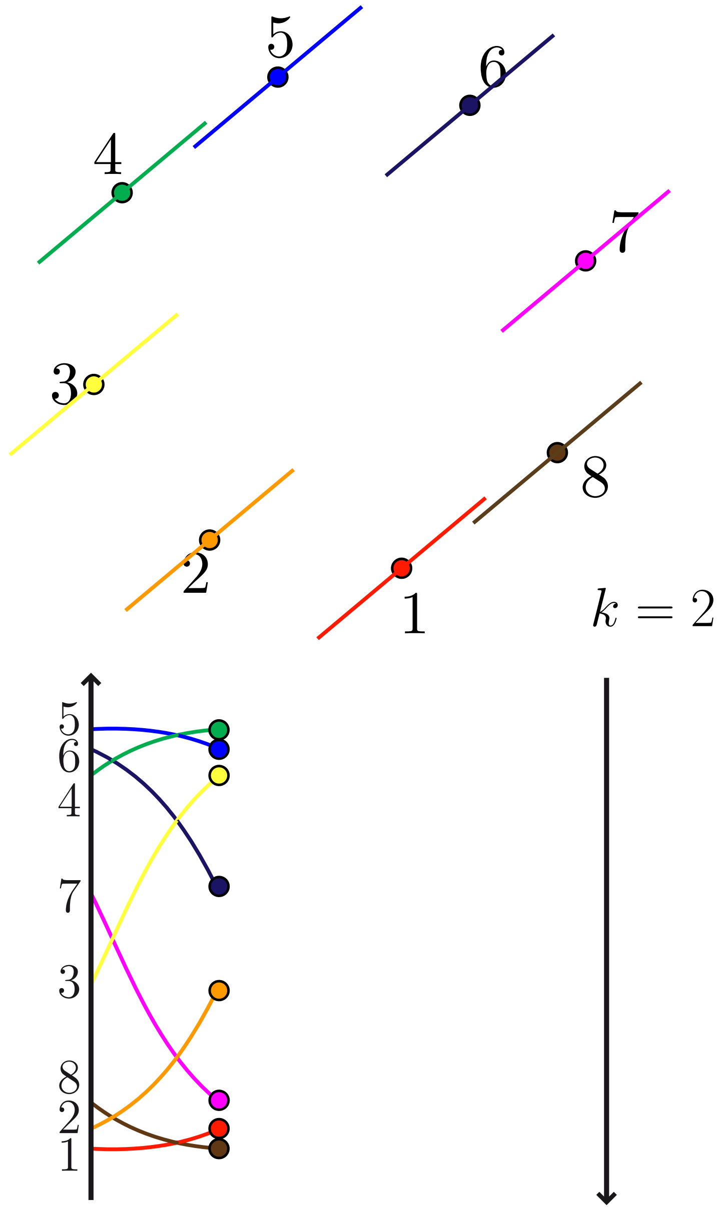
Triangulations



Pseudotriangulations



Multitriangulations



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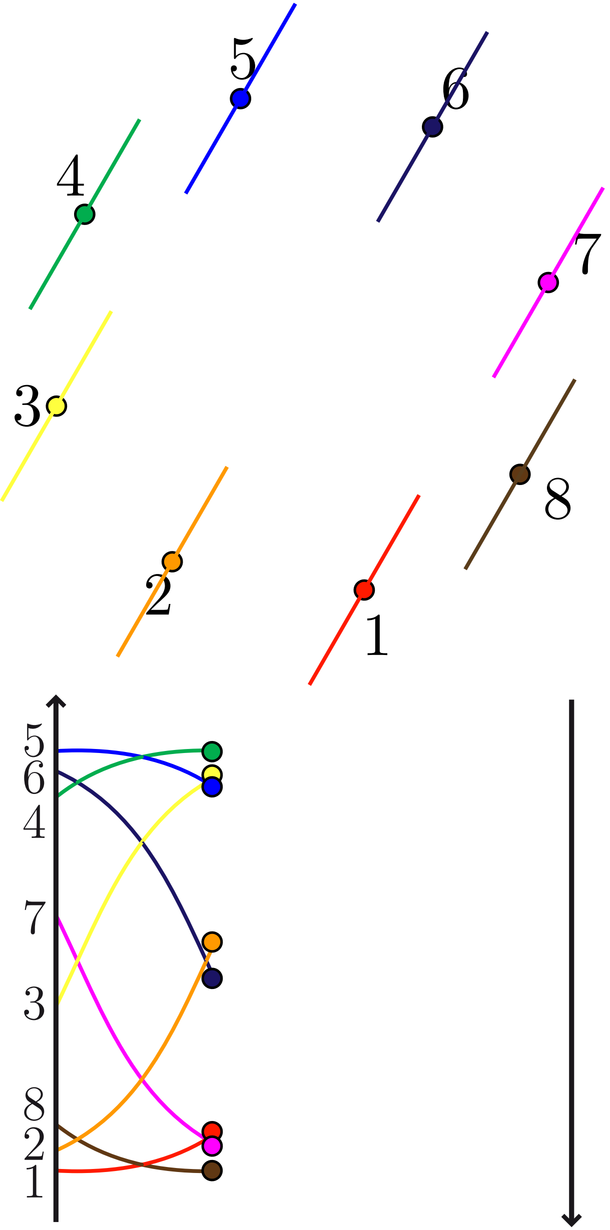
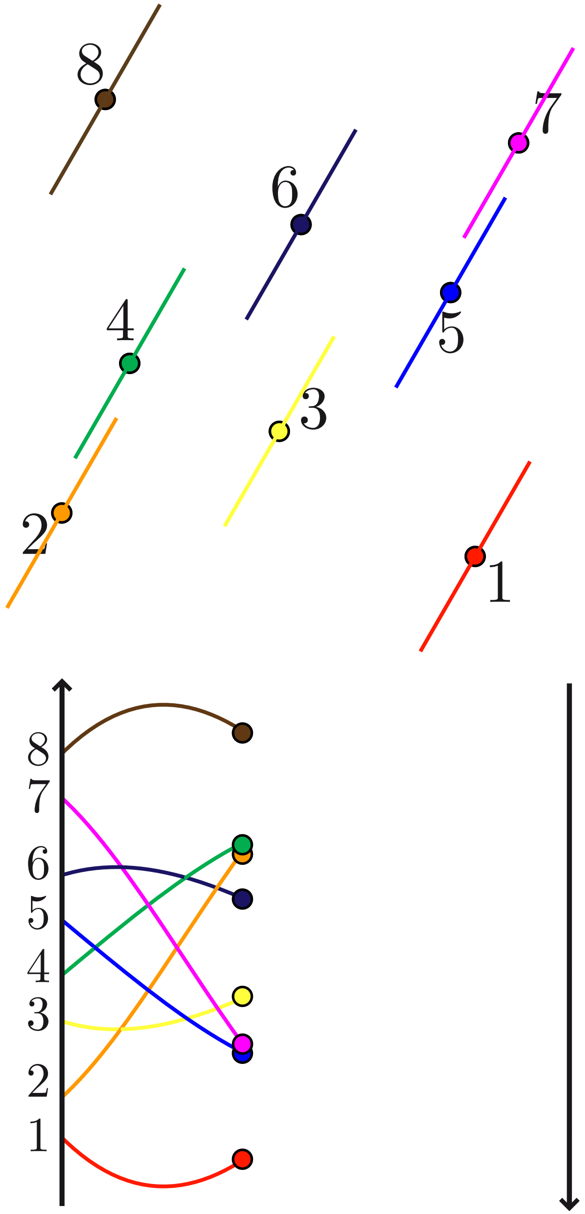
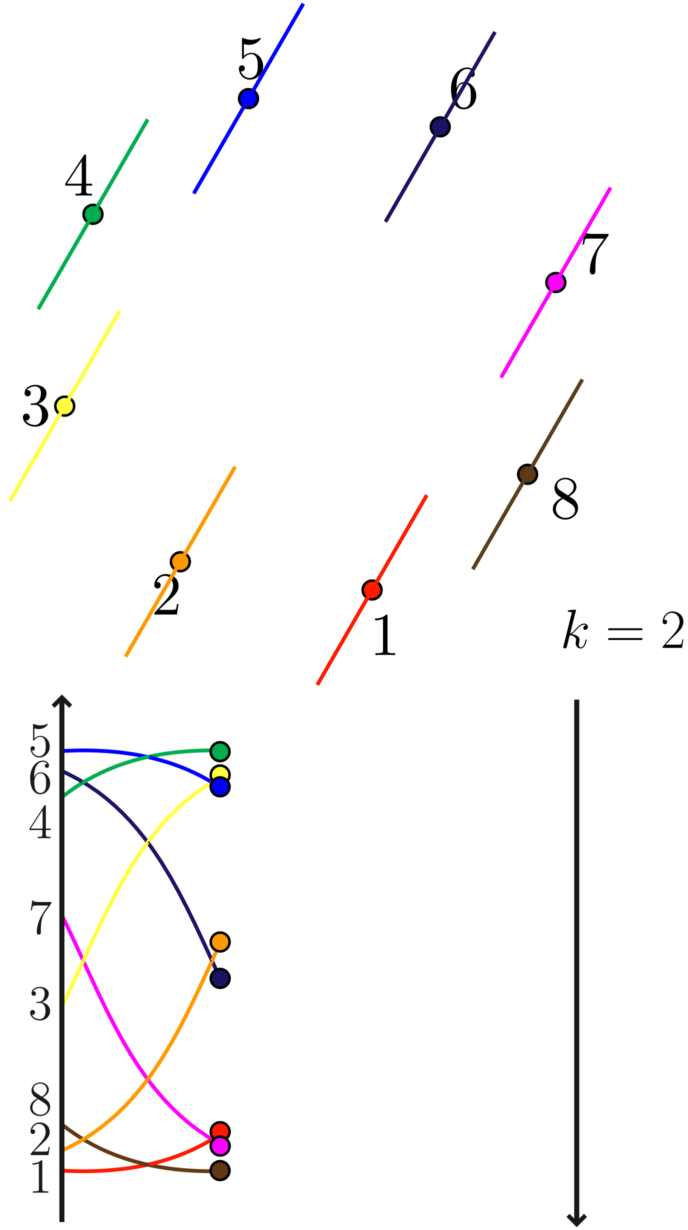


Pseudotriangulations



Multitriangulations



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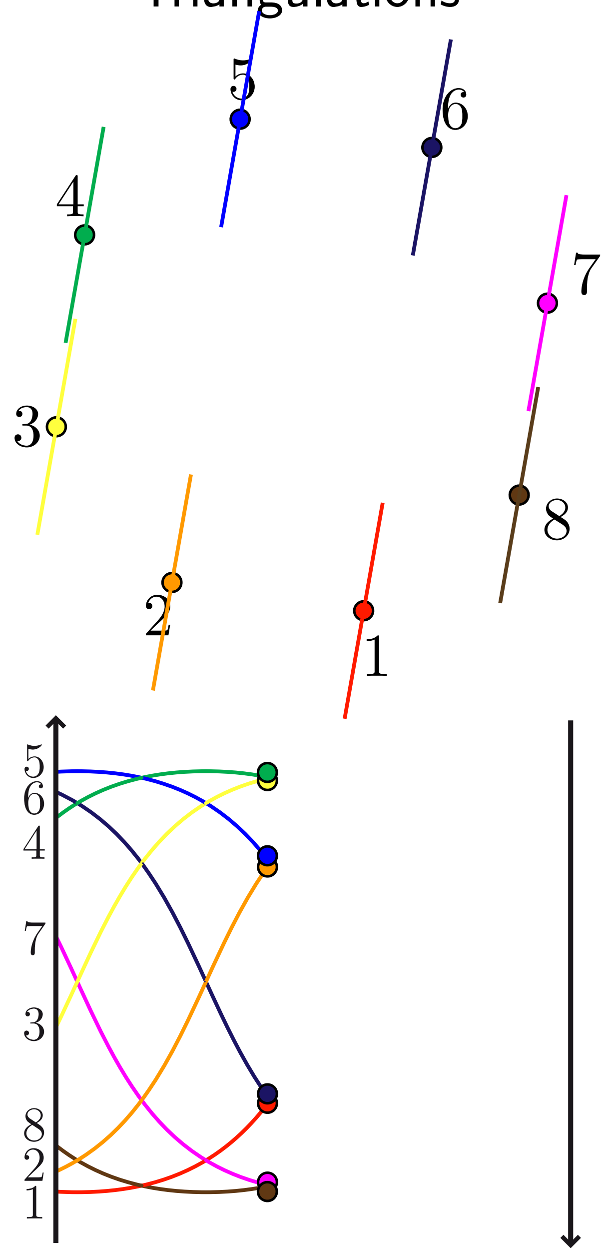

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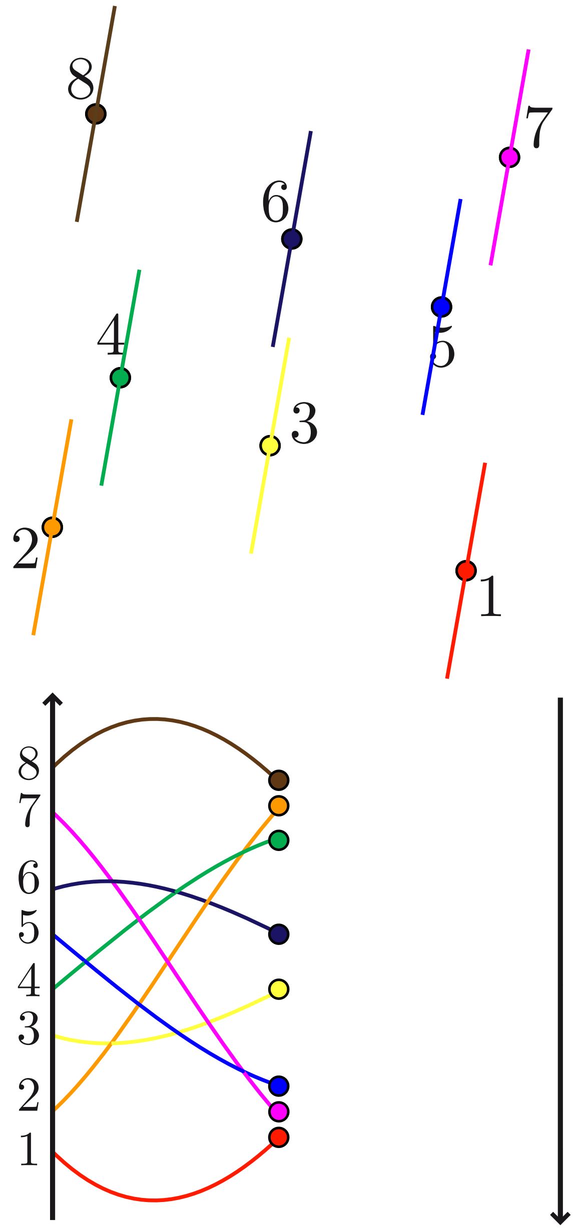
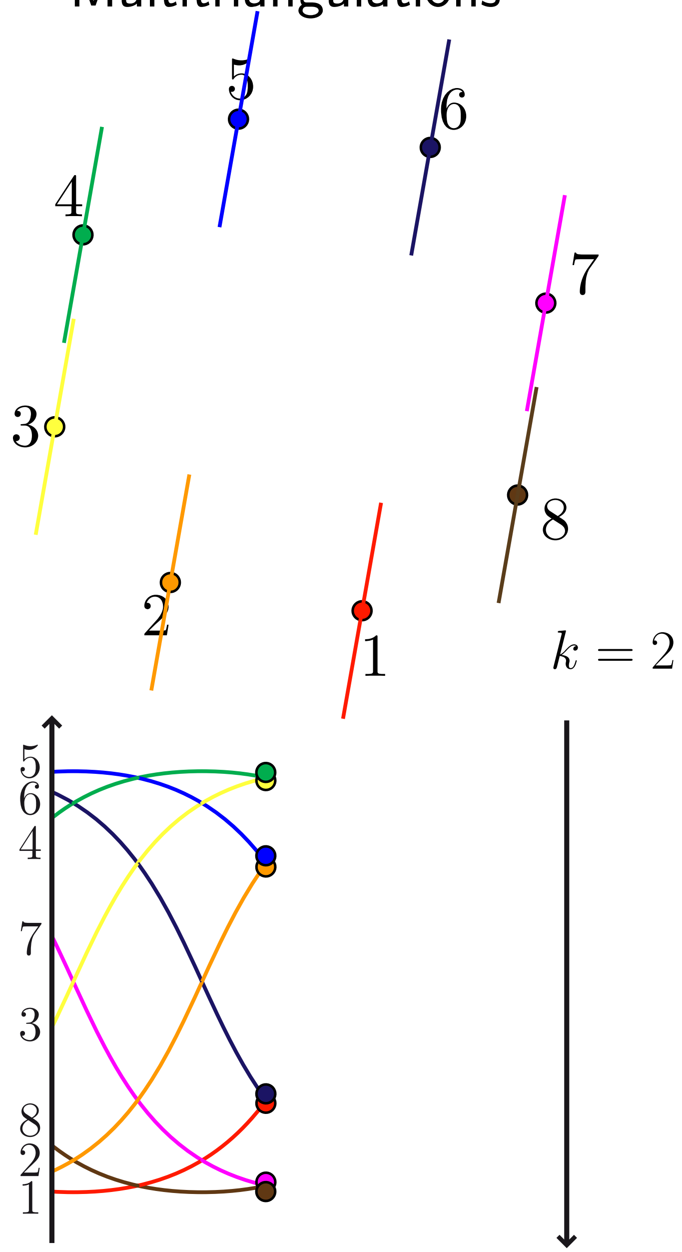


Multitriangulations



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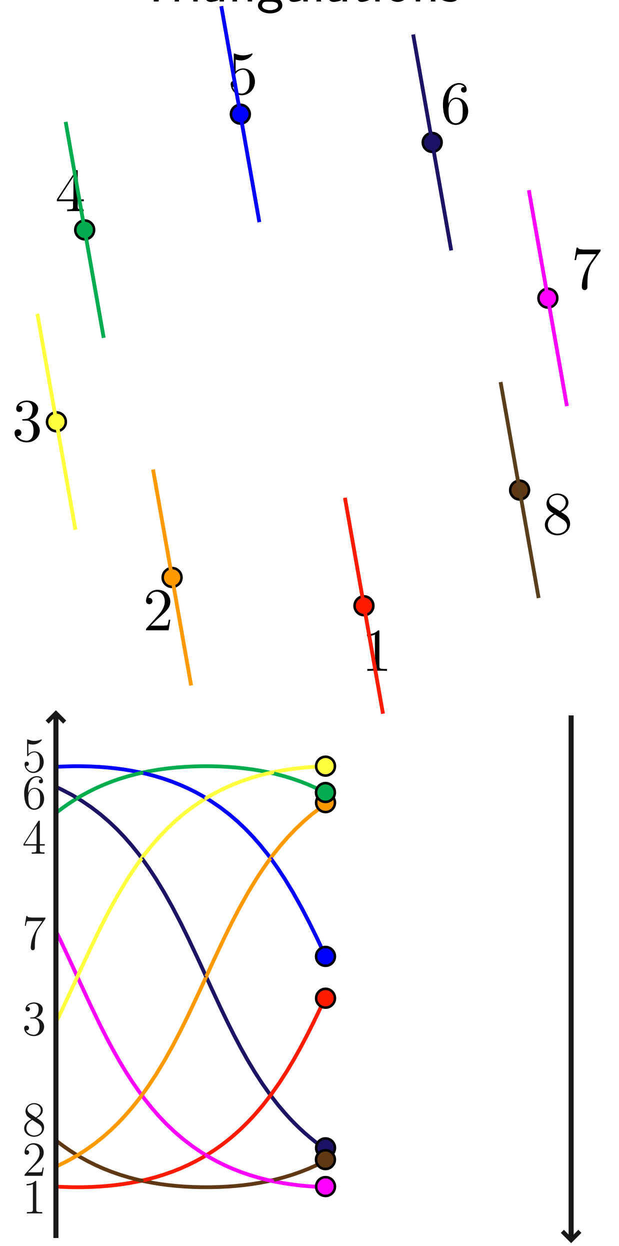

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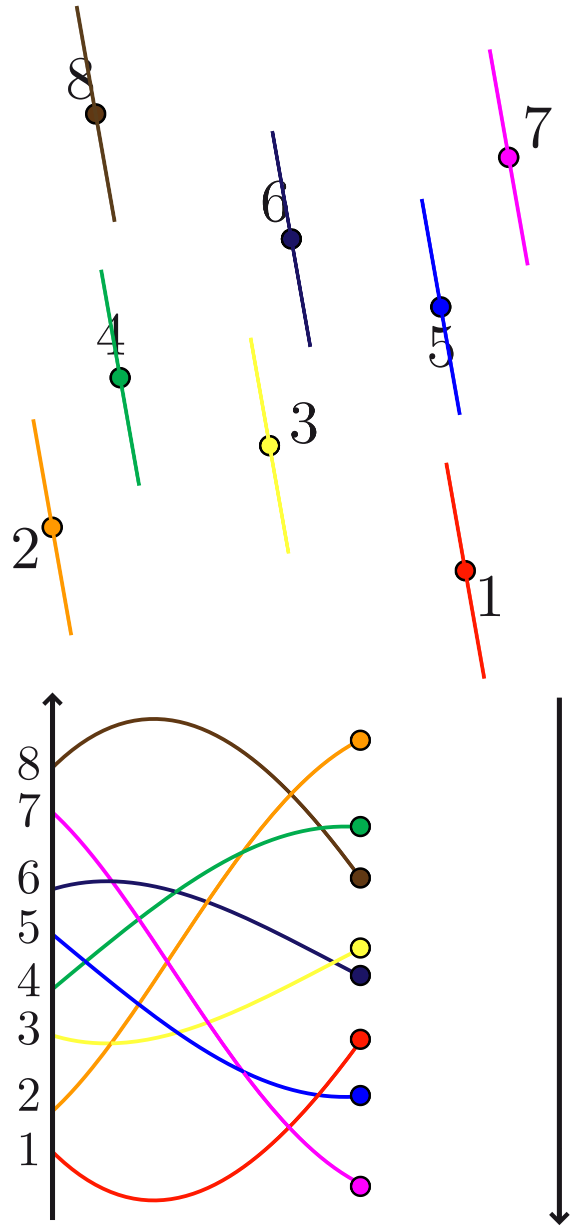
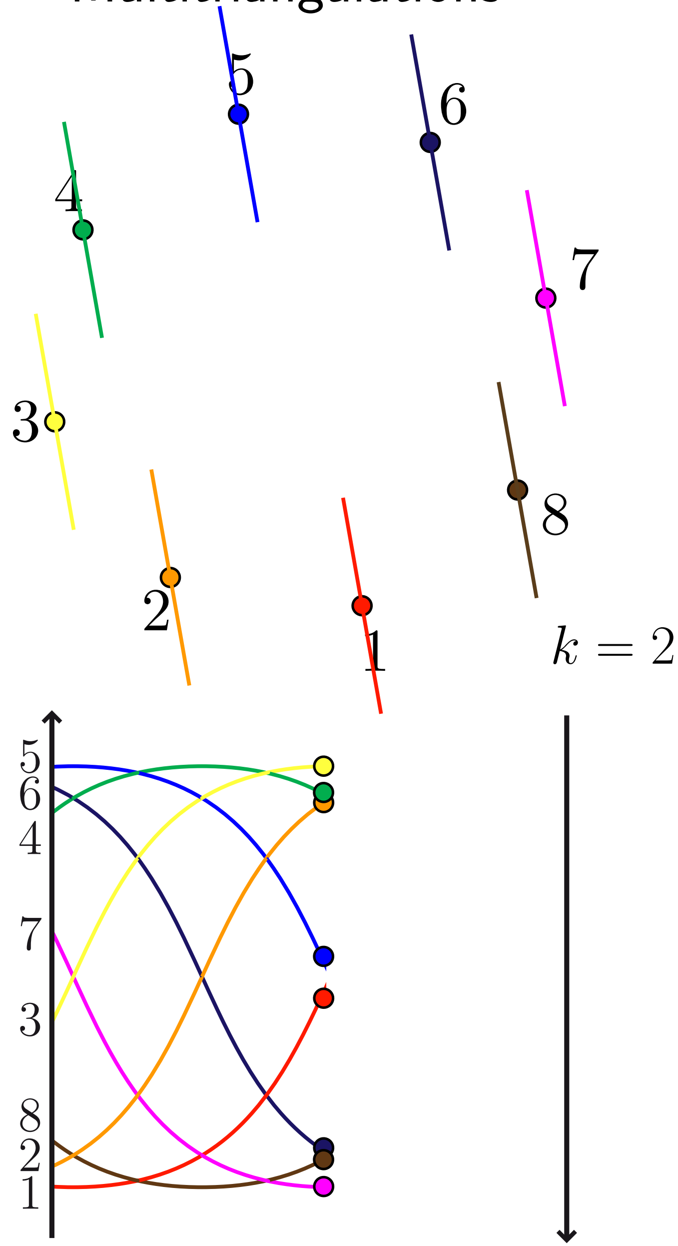


Multitriangulations



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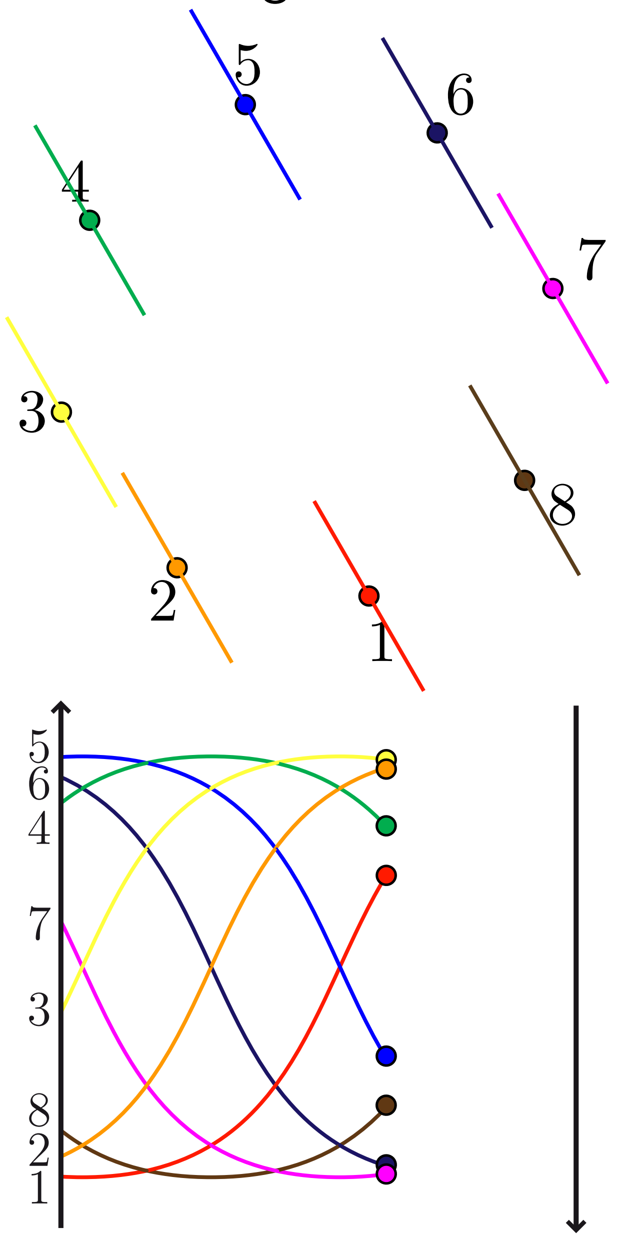

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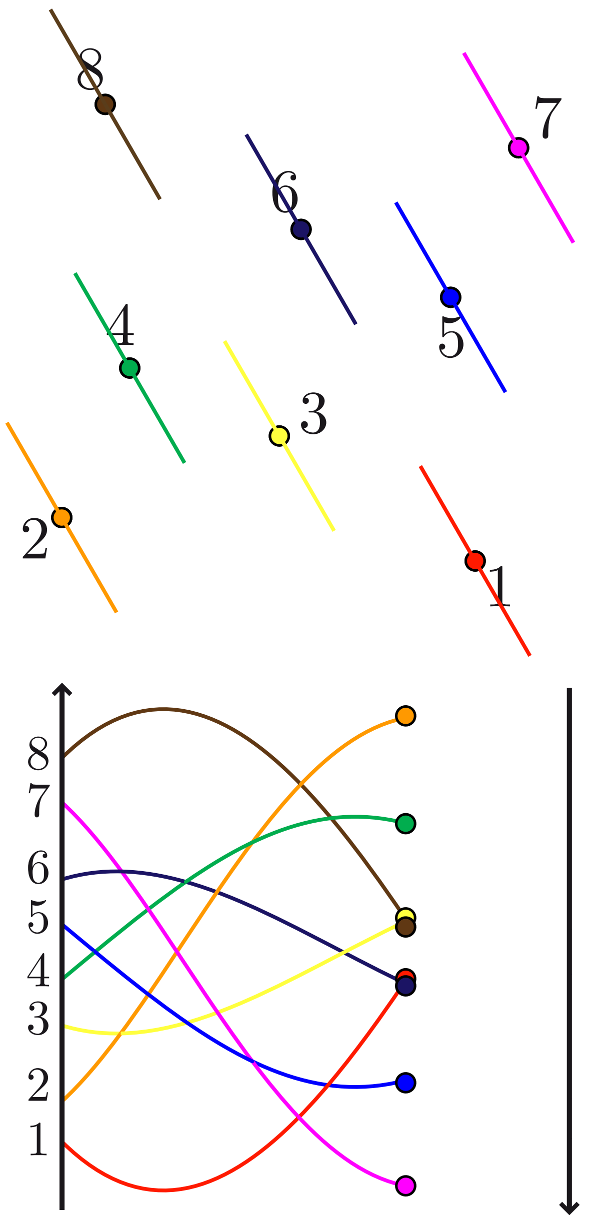

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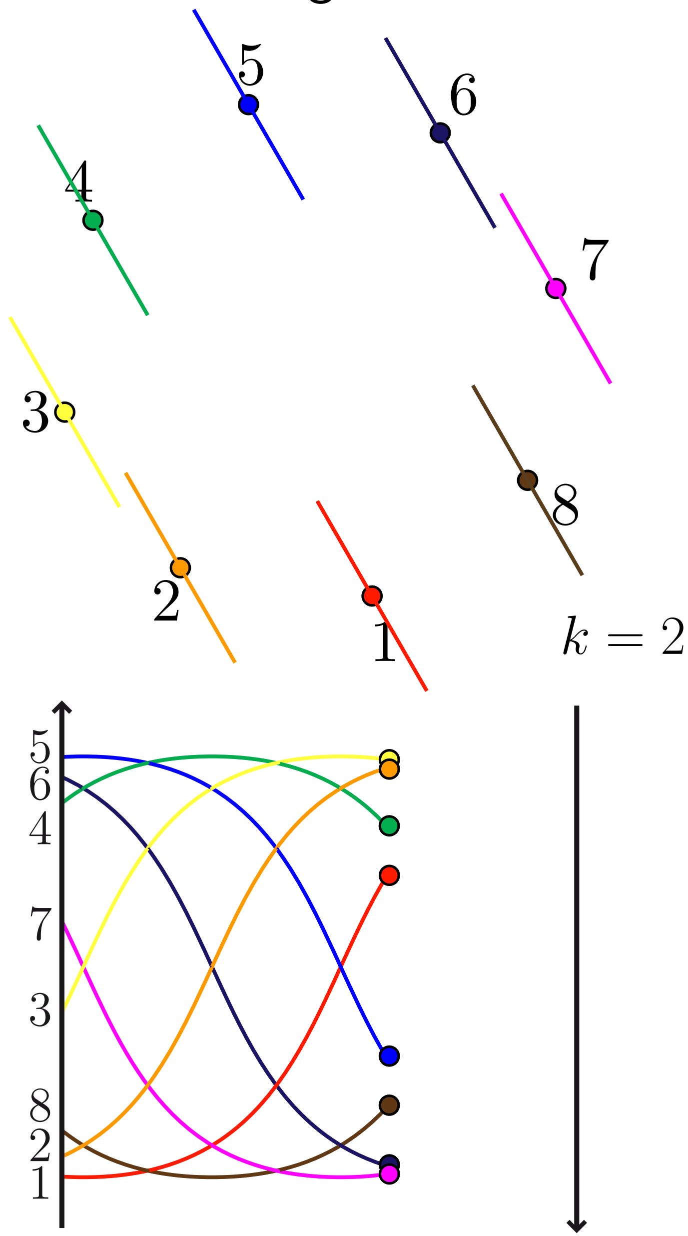

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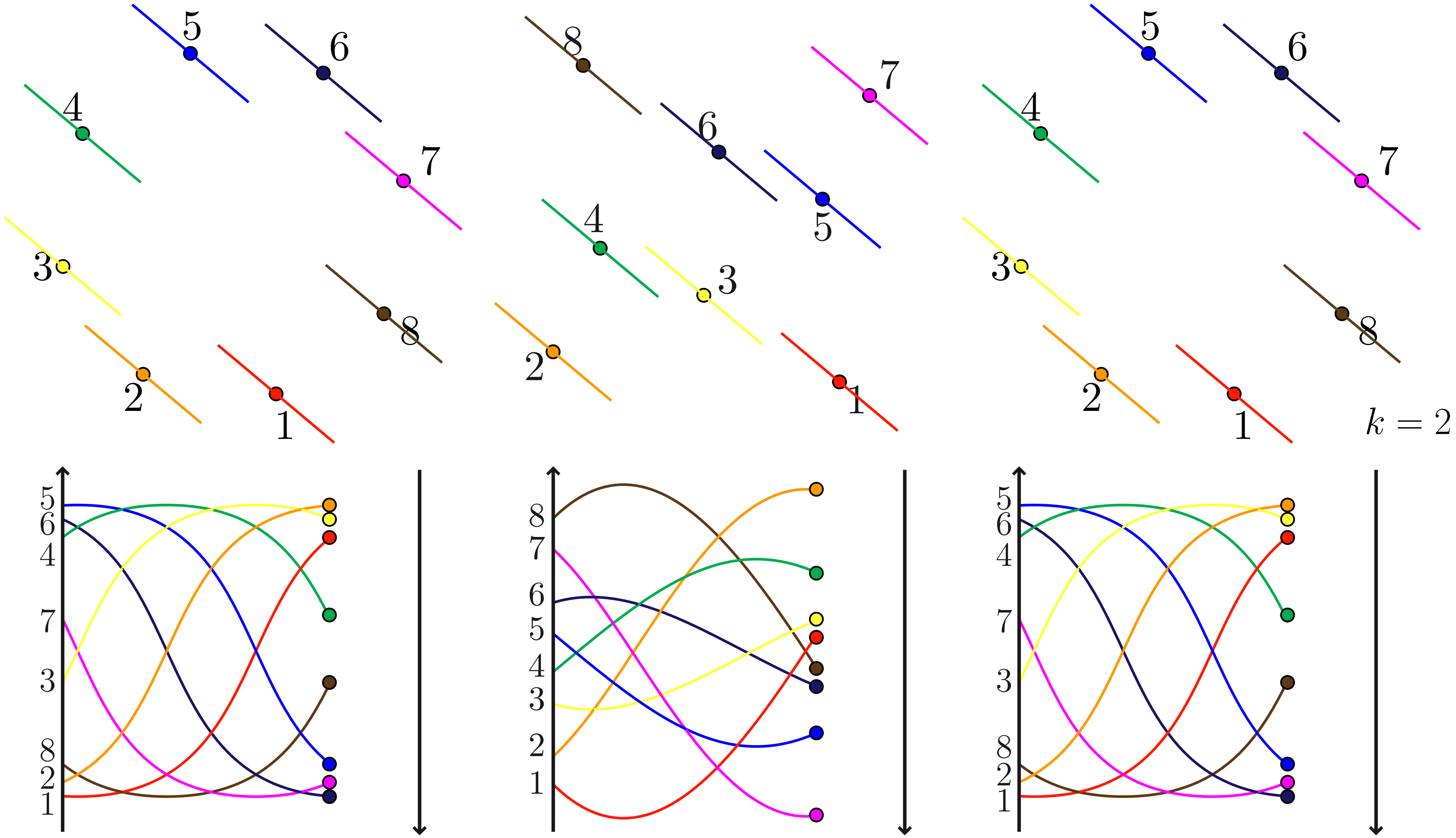

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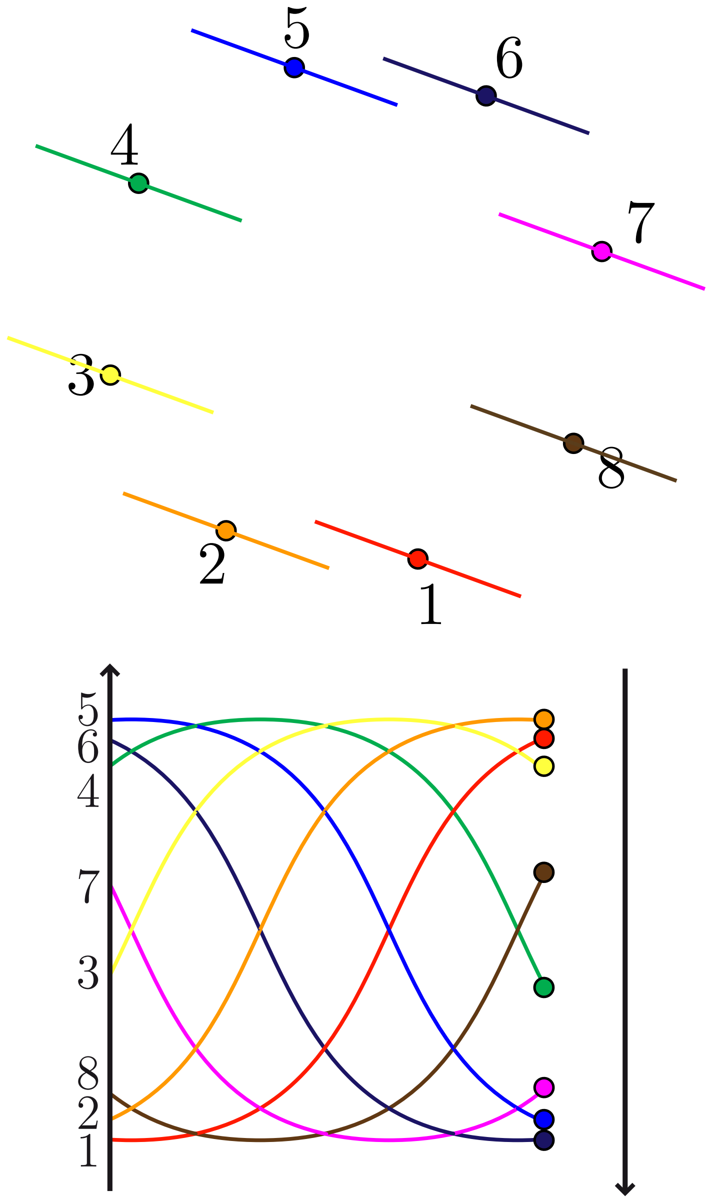

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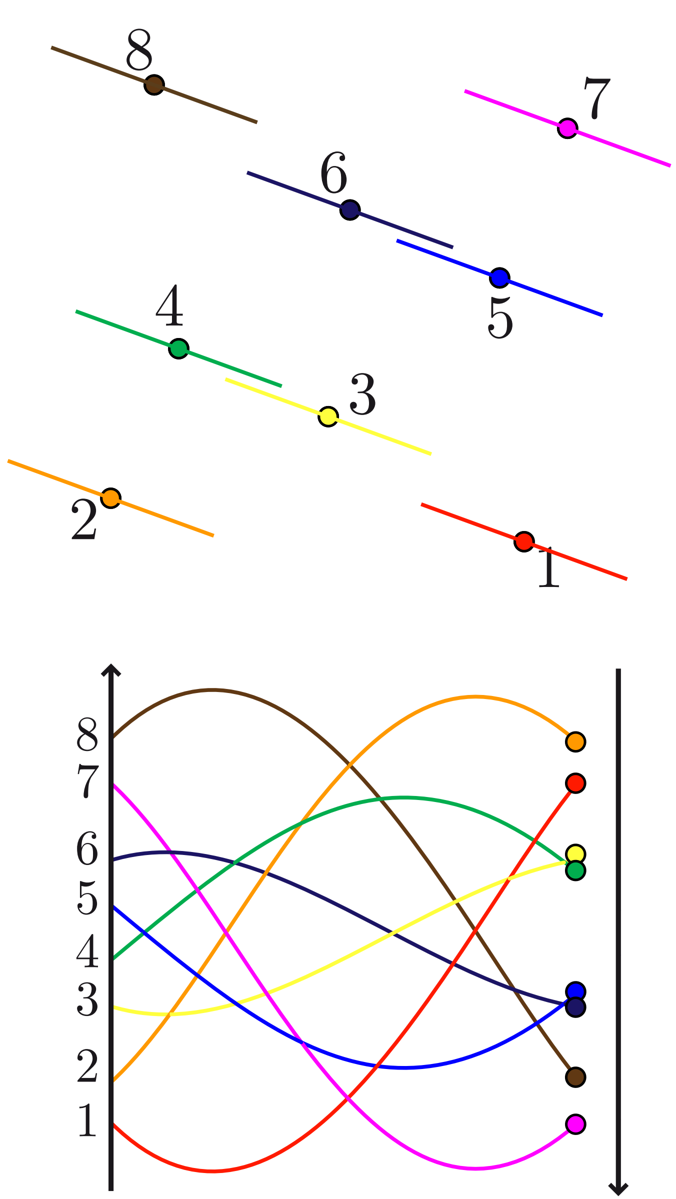

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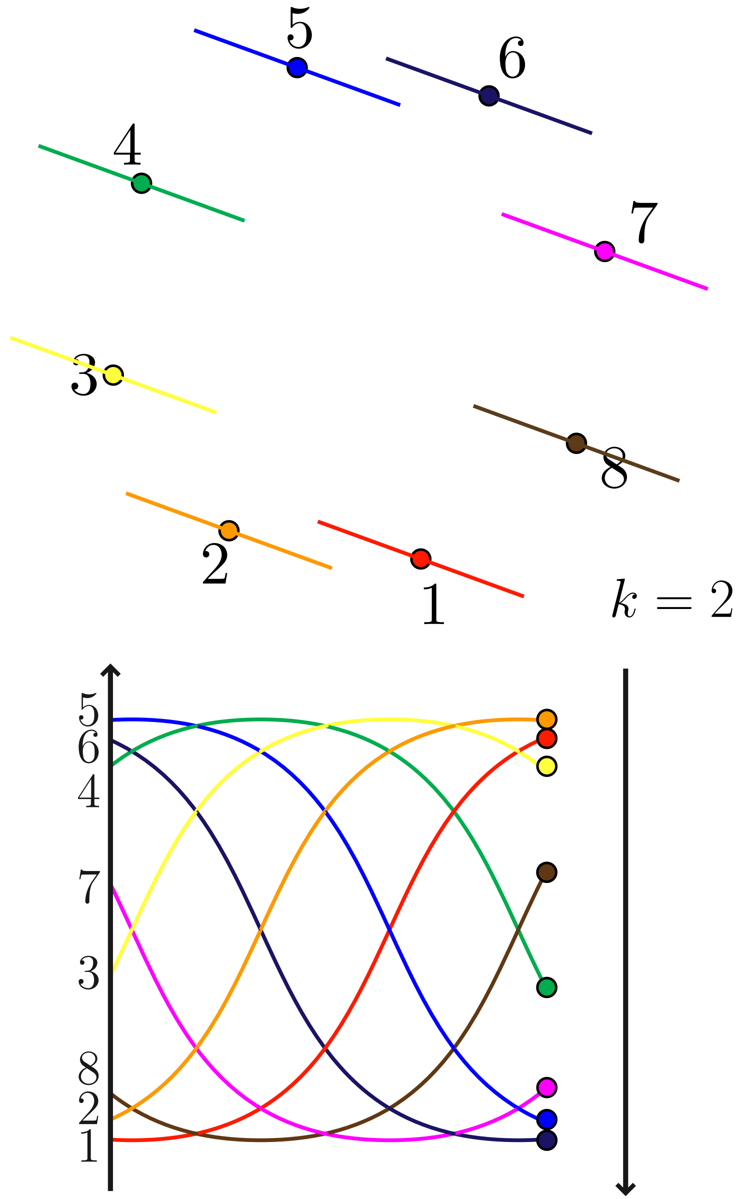

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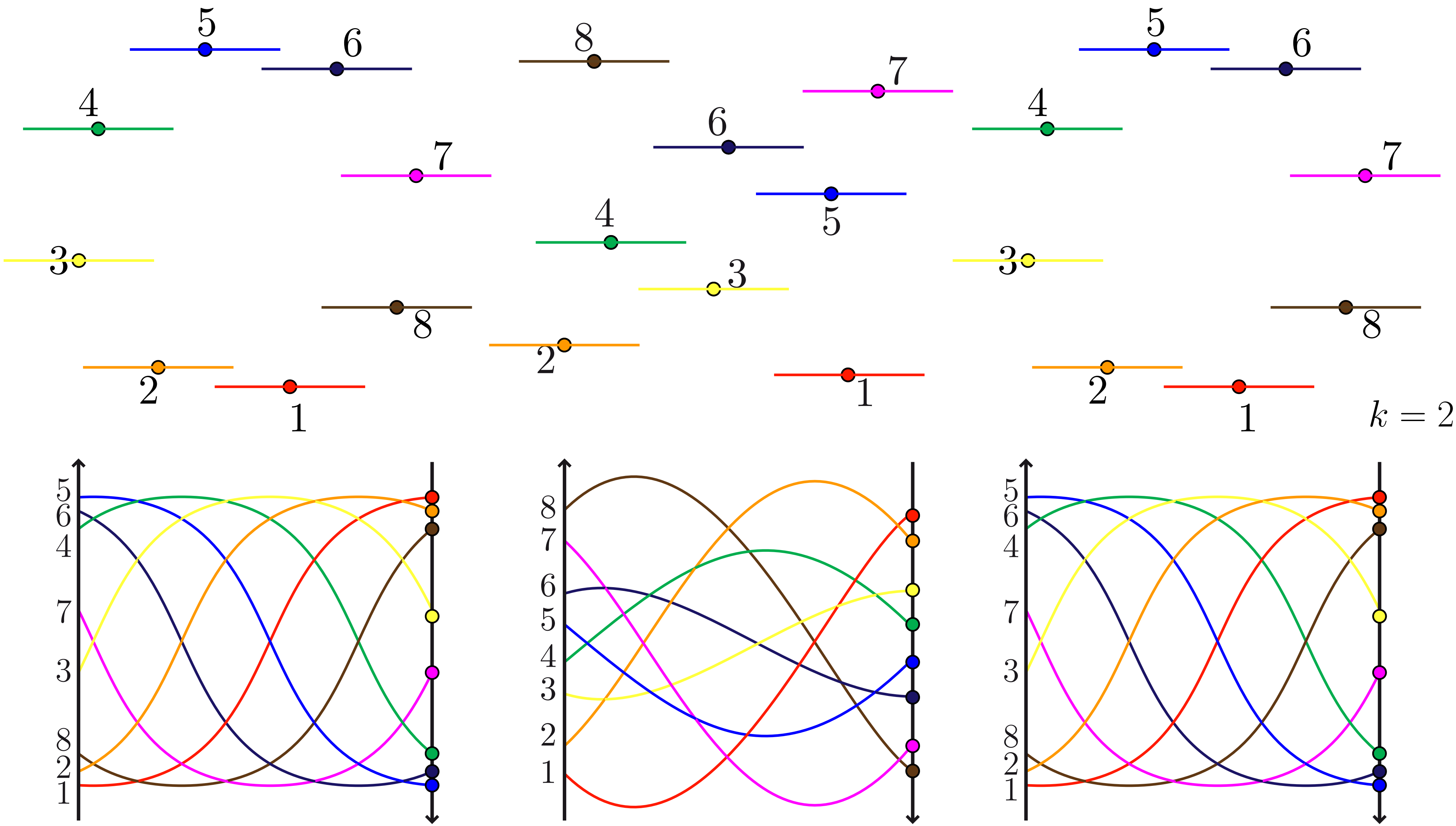
Triangulations | Pseudotriangulations | Multitriangulations



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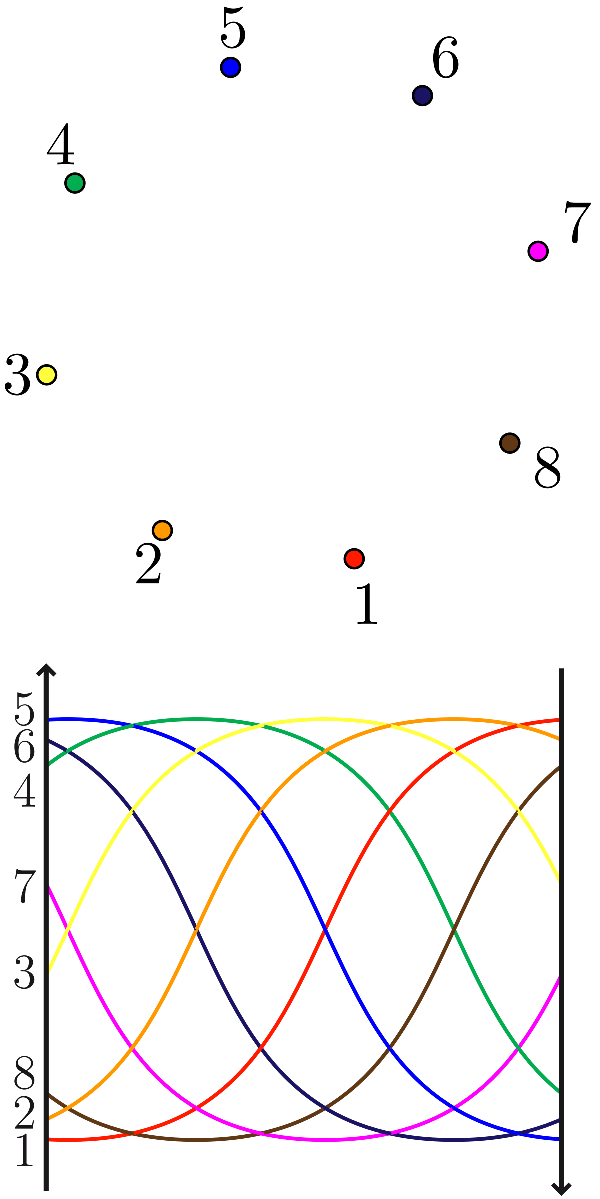
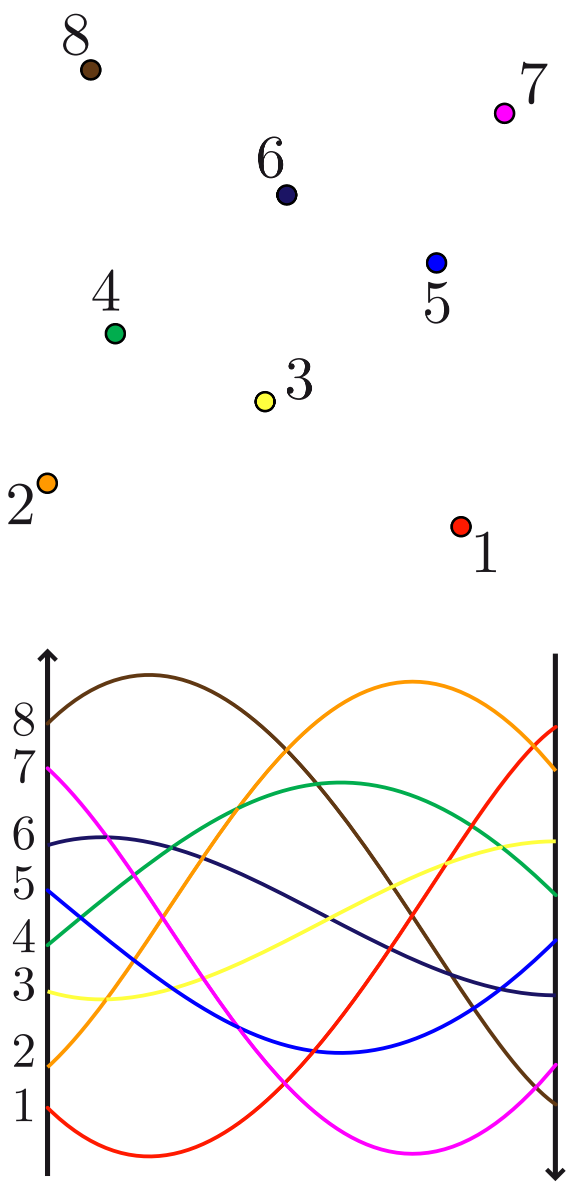
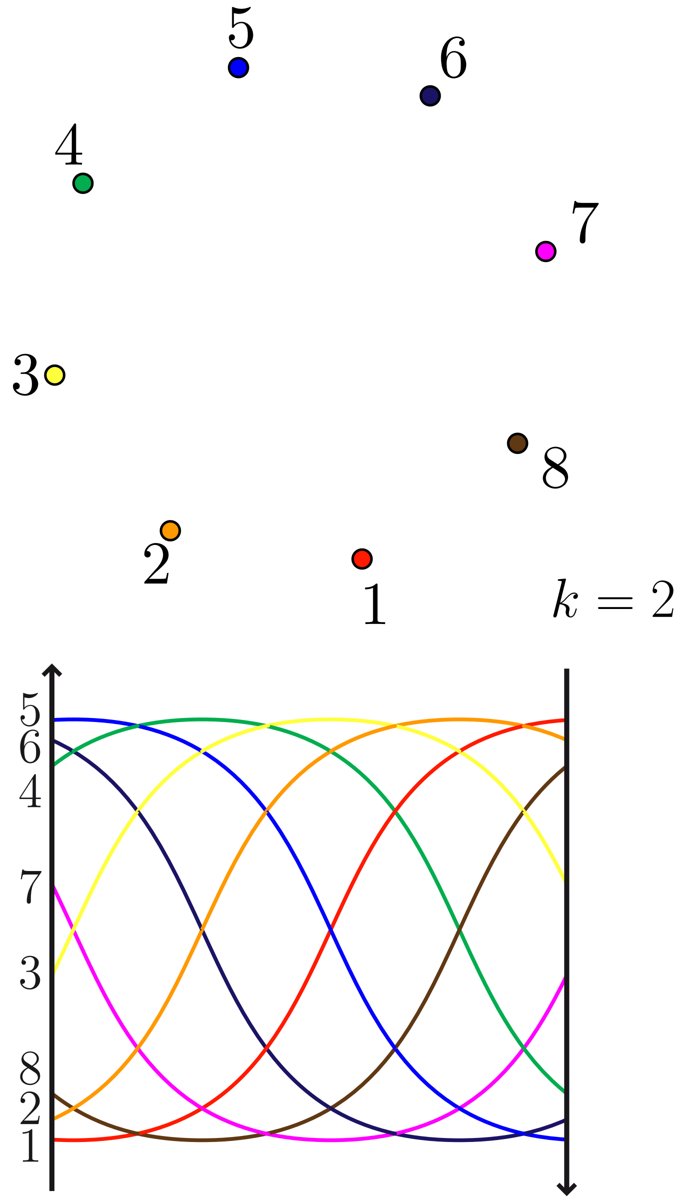


Pseudotriangulations



Multitriangulations



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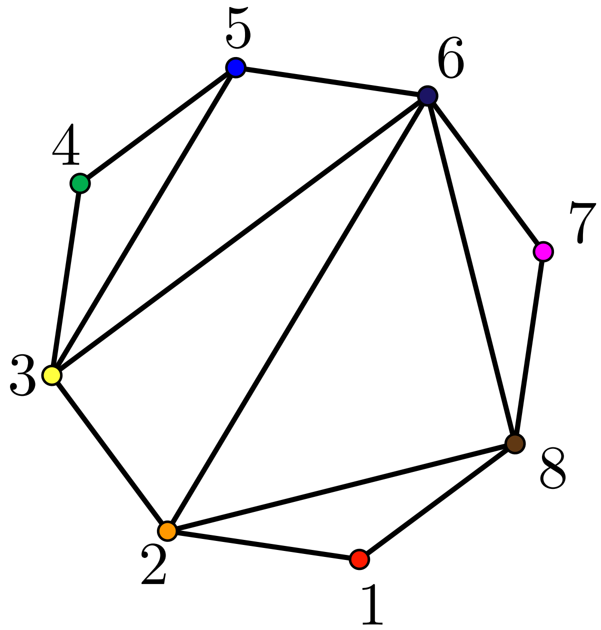
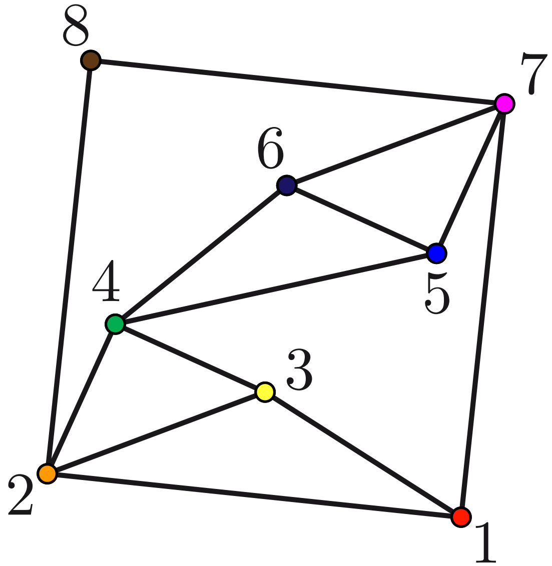
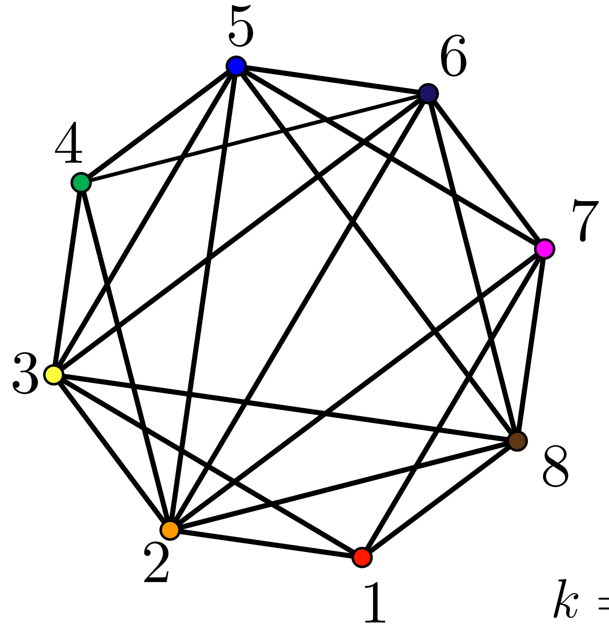
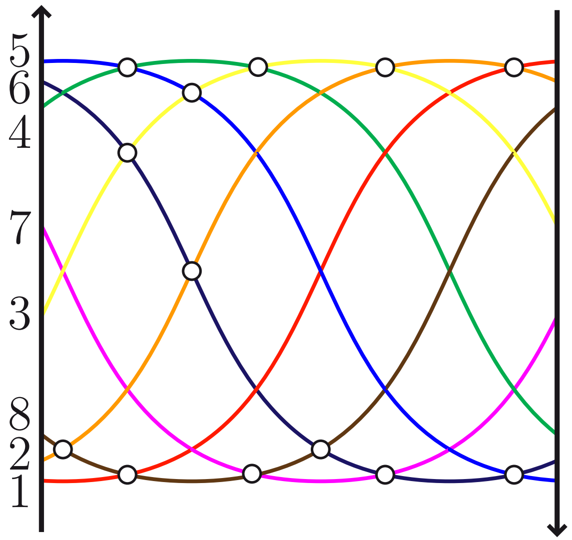
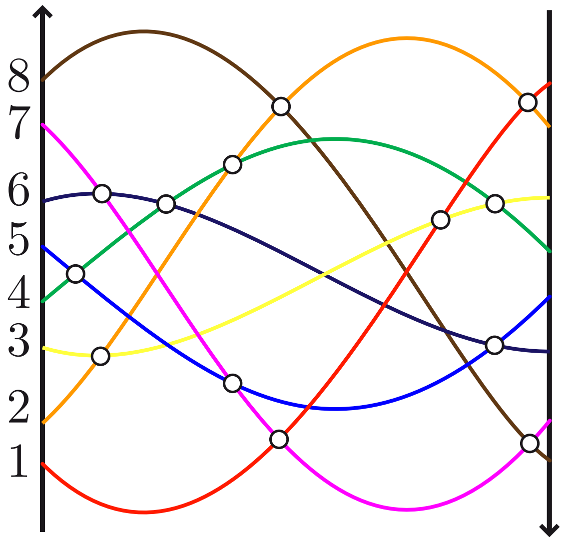
Triangulations	Pseudotriangulations	Multitriangulations

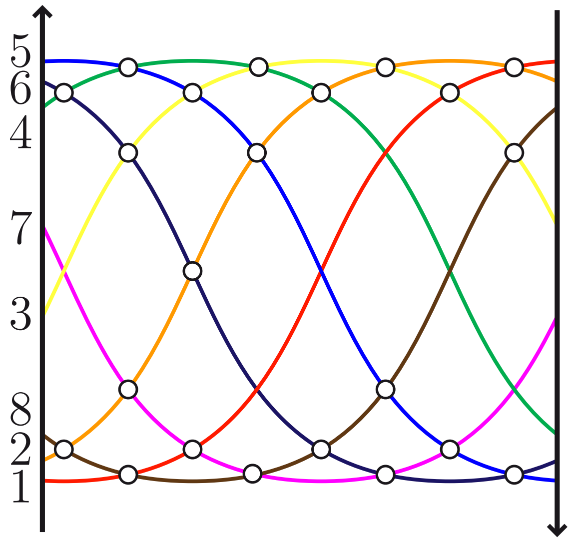












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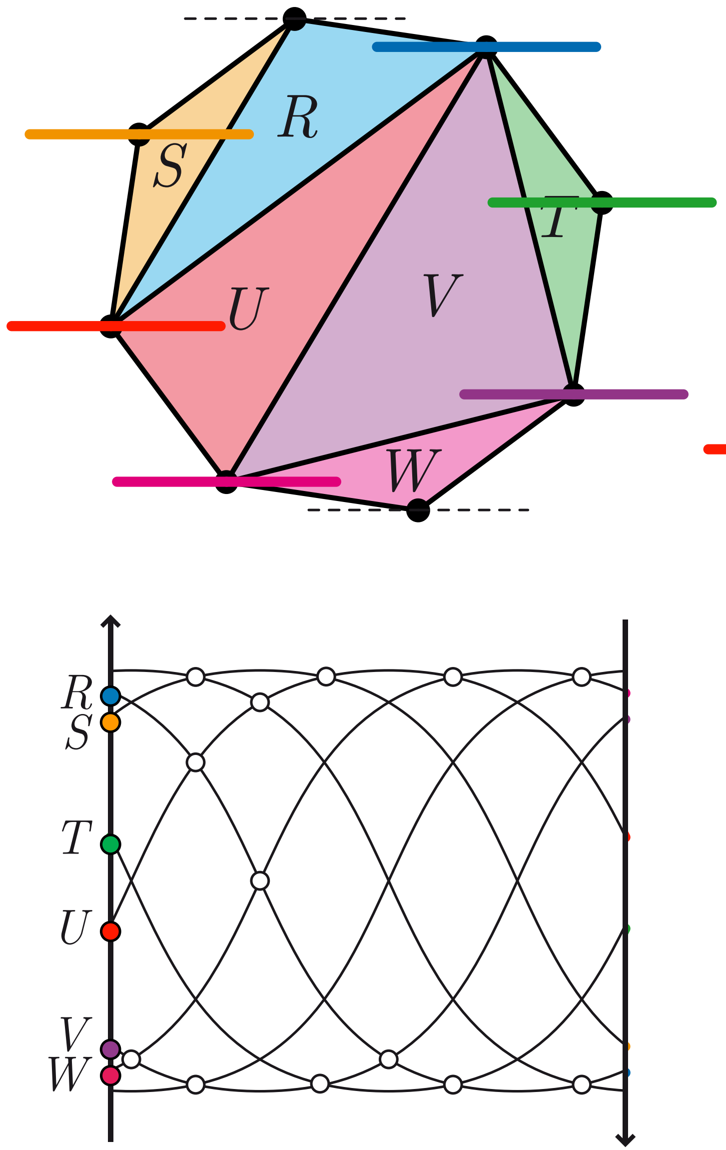
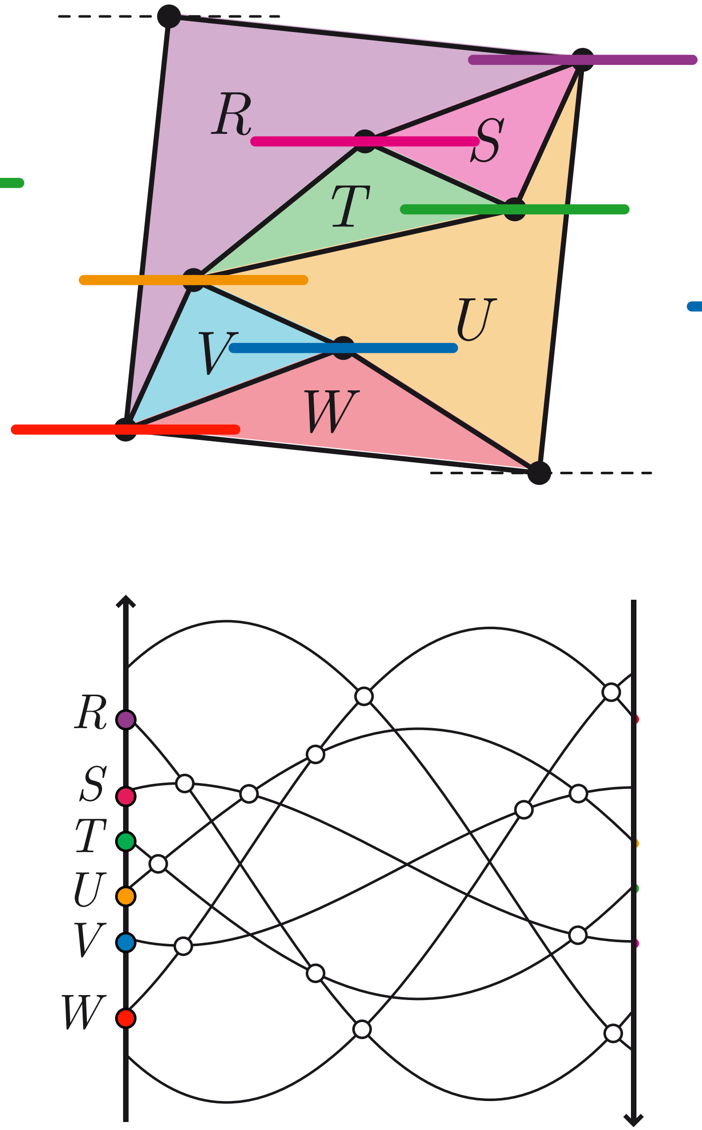
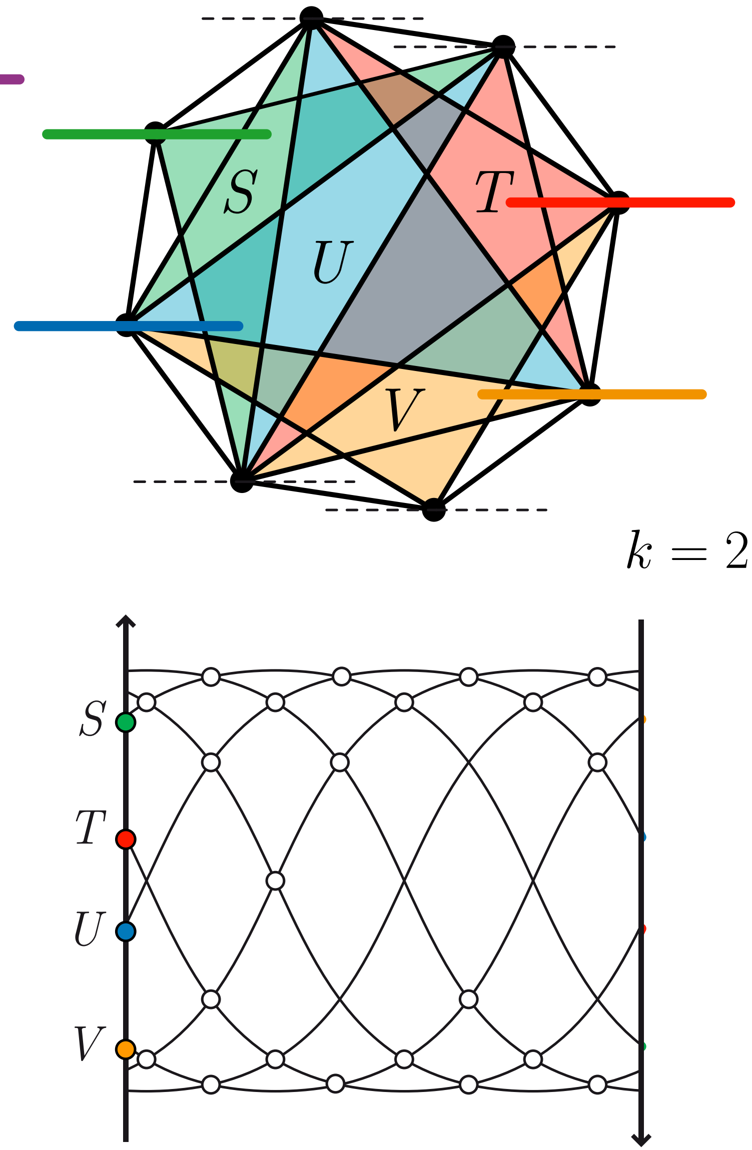


Pseudotriangulations

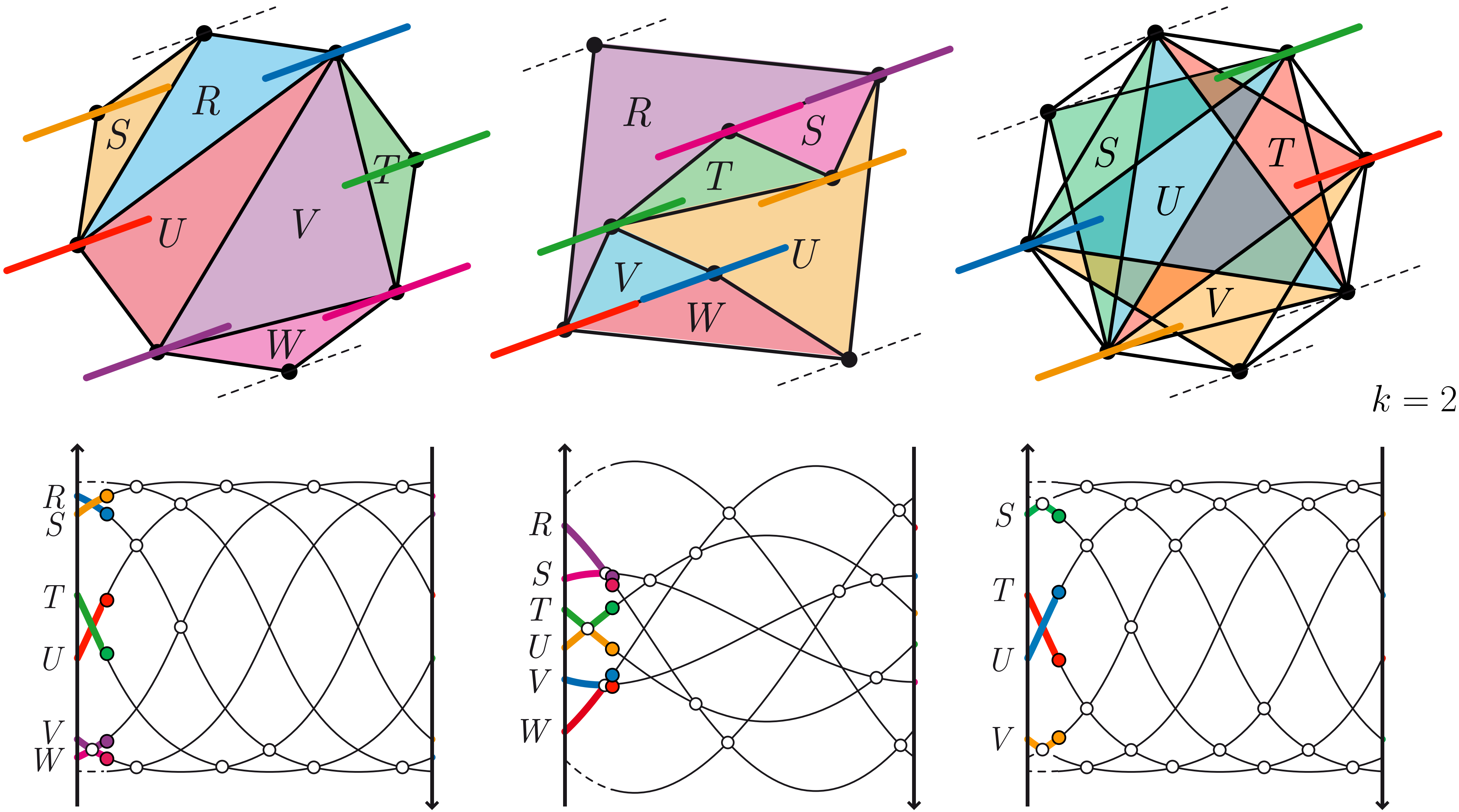


Multitriangulations



DUALITY

Triangulations | Pseudotriangulations | Multitriangulations



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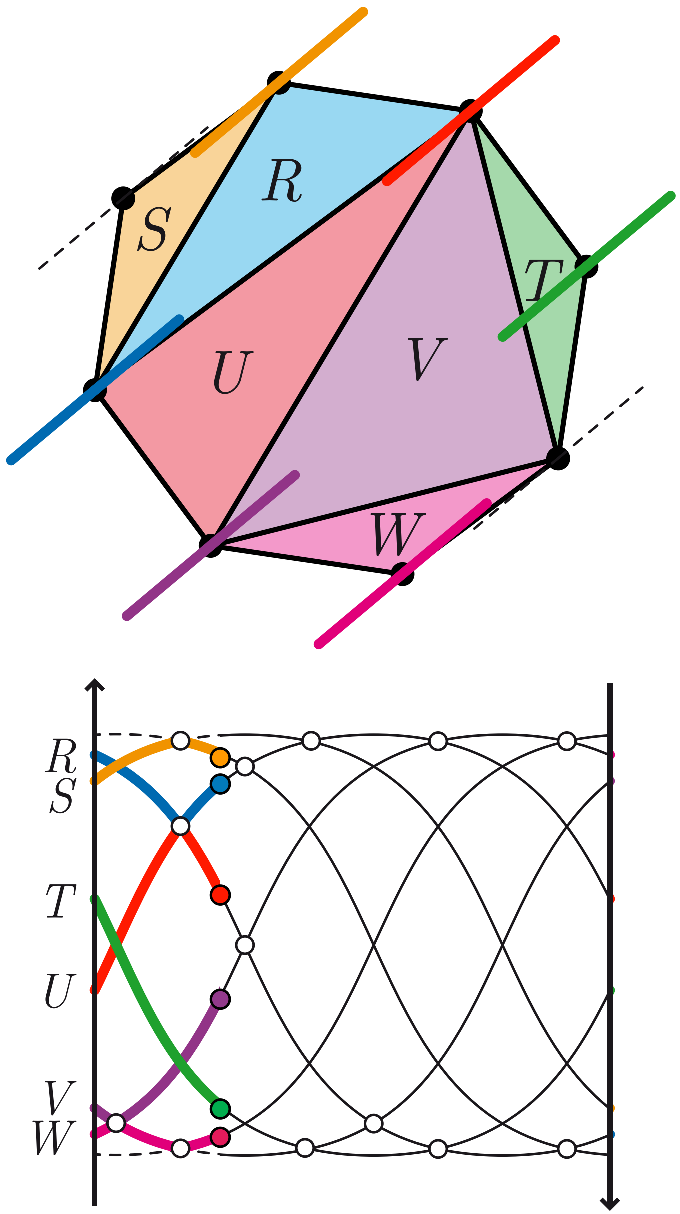

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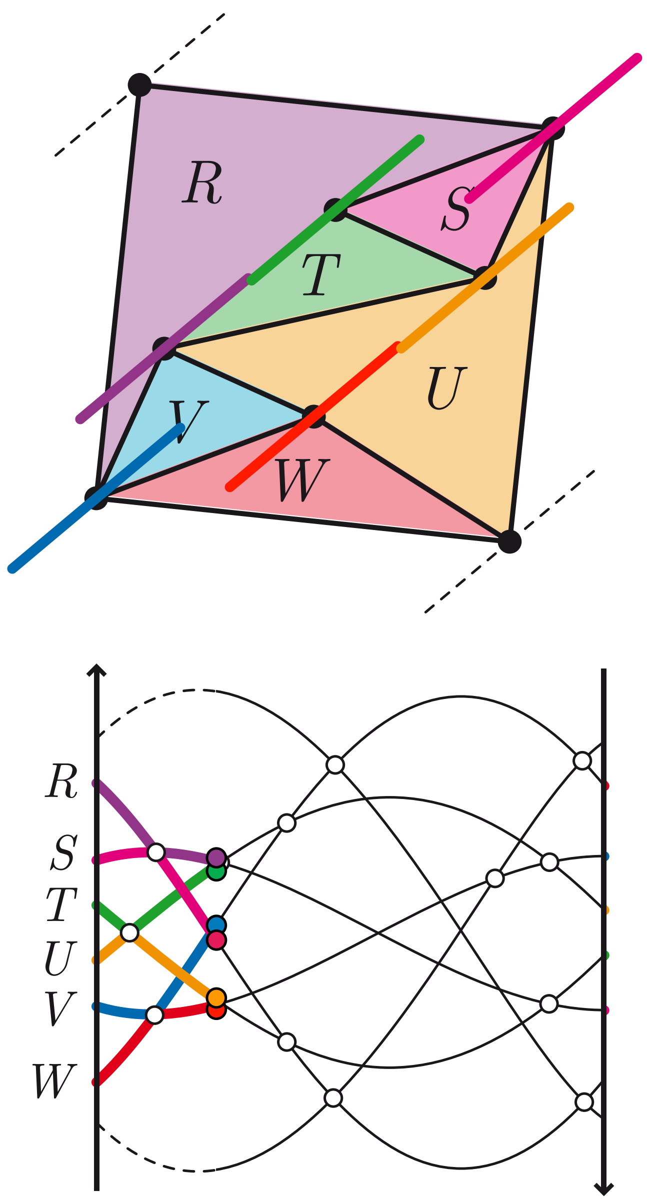
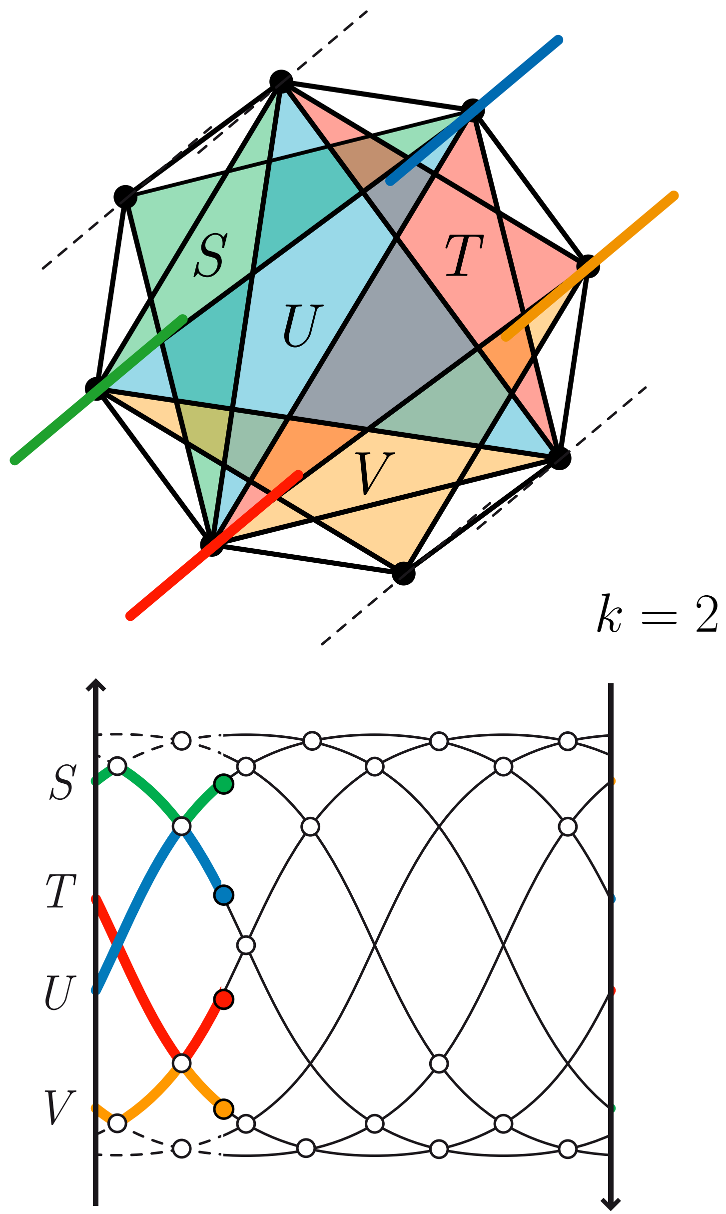


Multitriangulations



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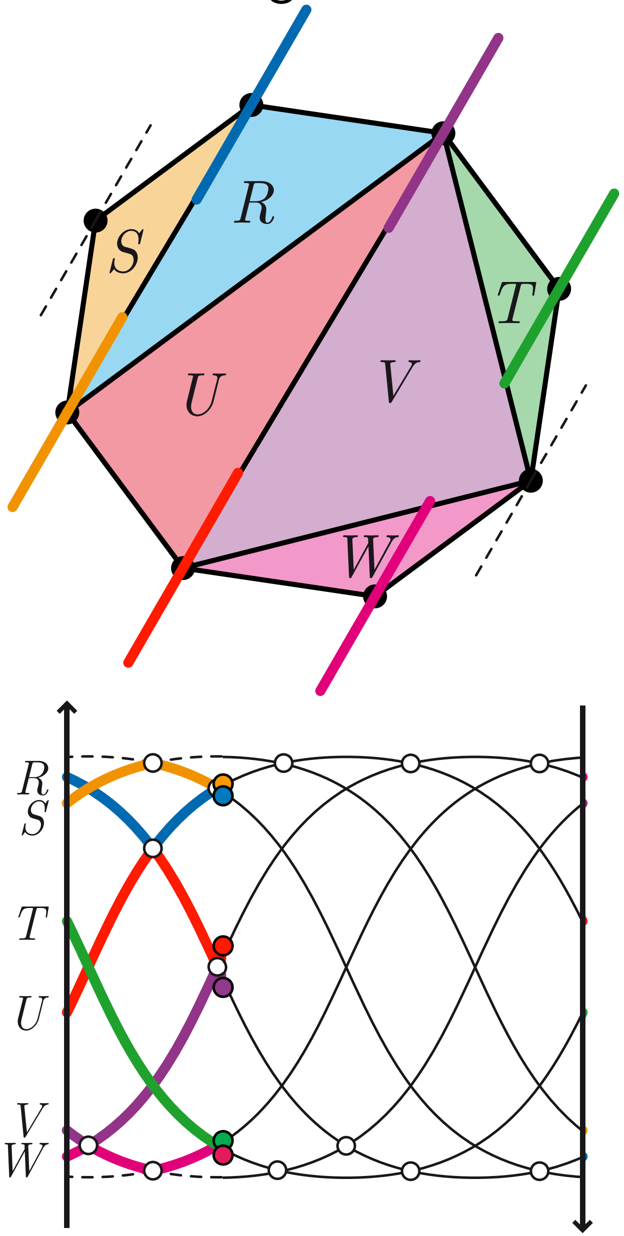

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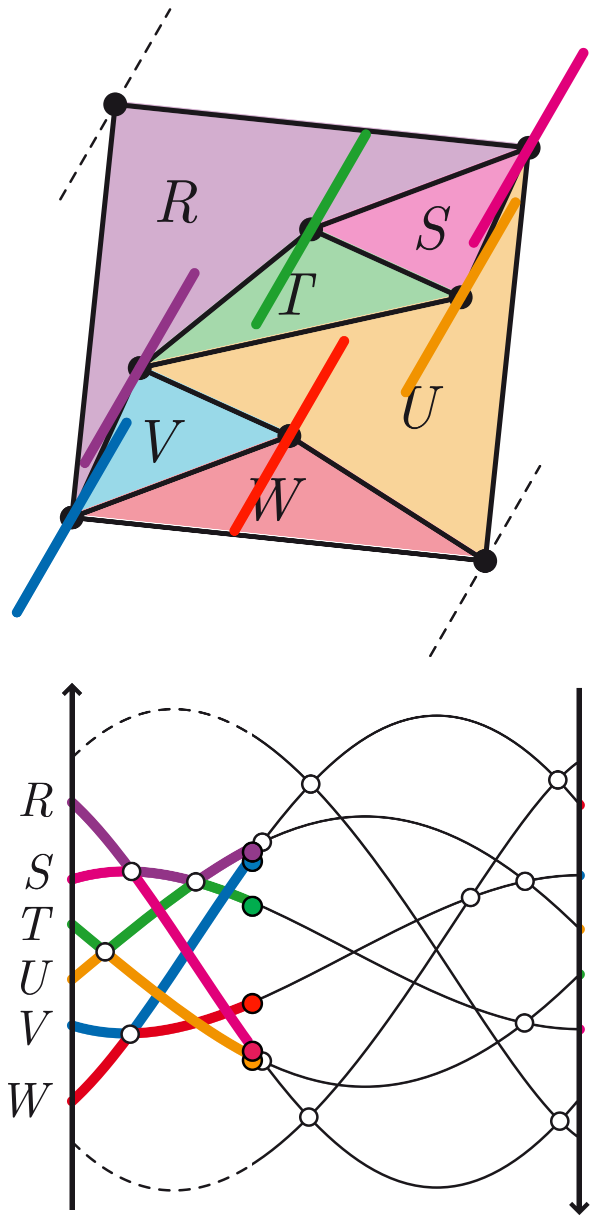
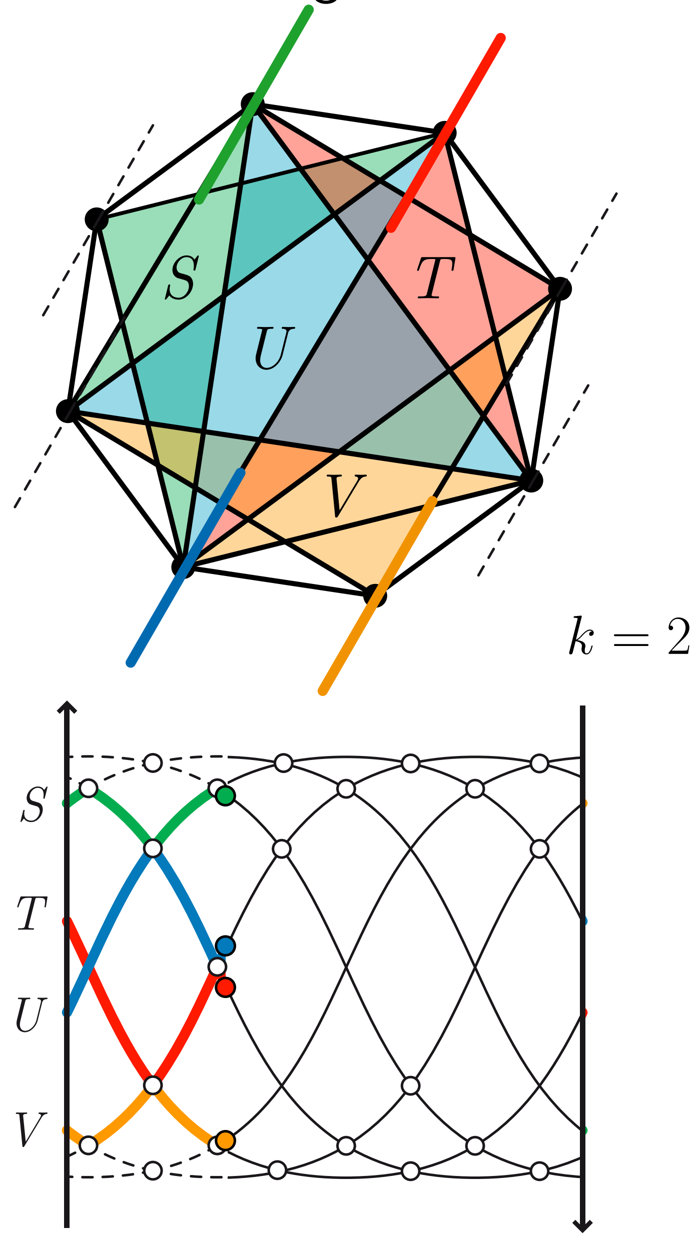


Multitriangulations



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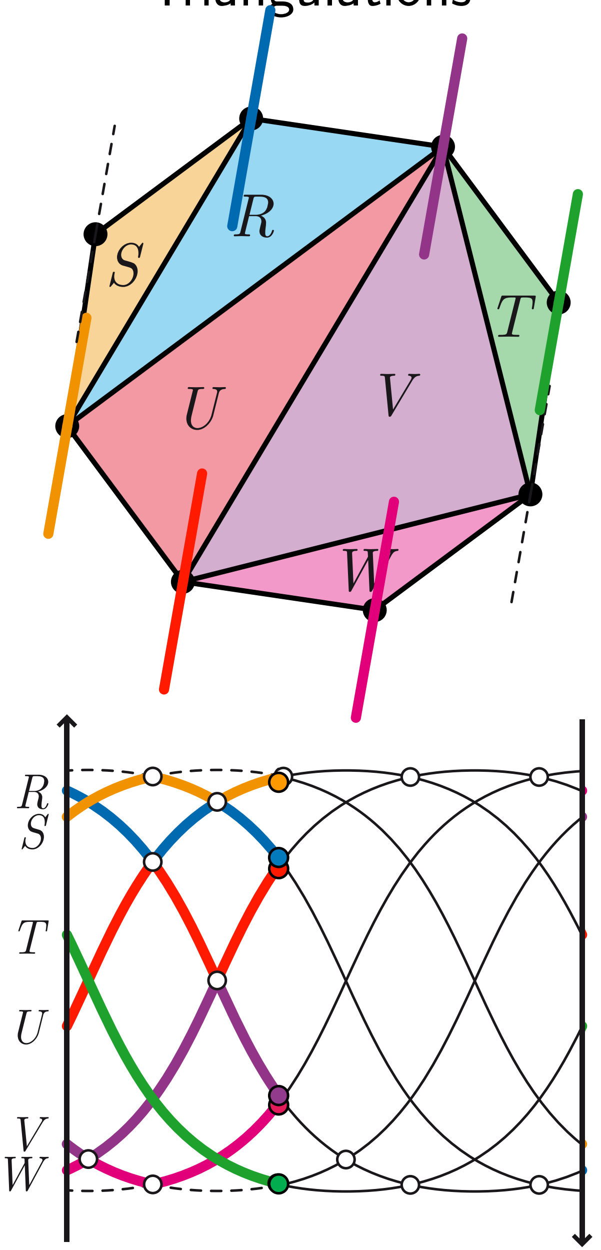
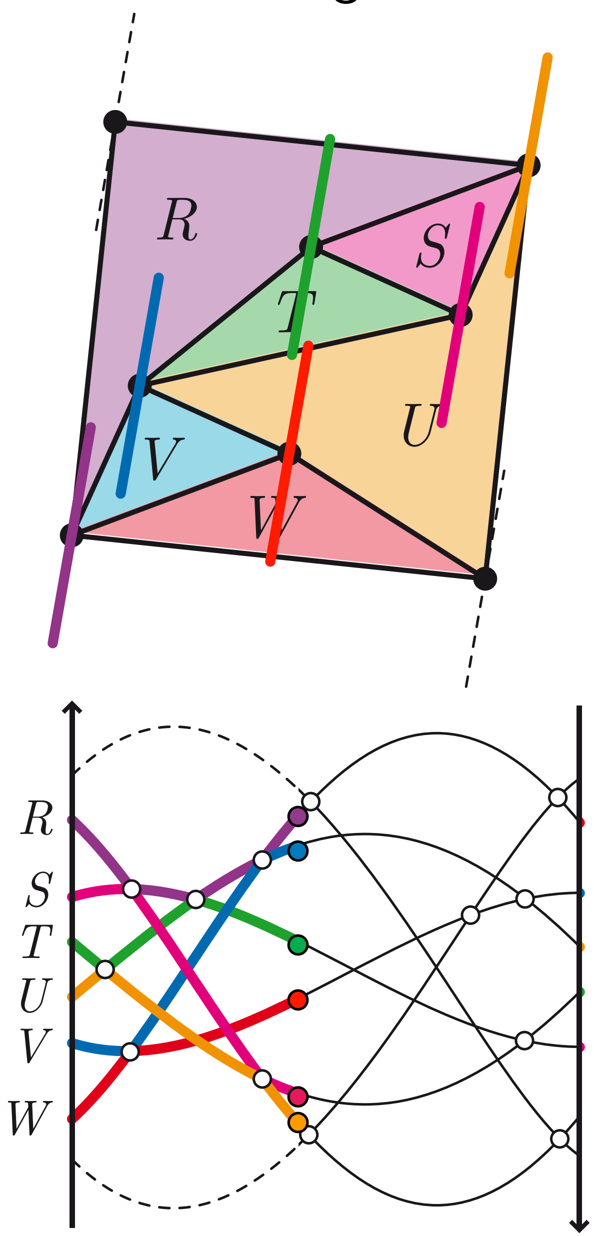
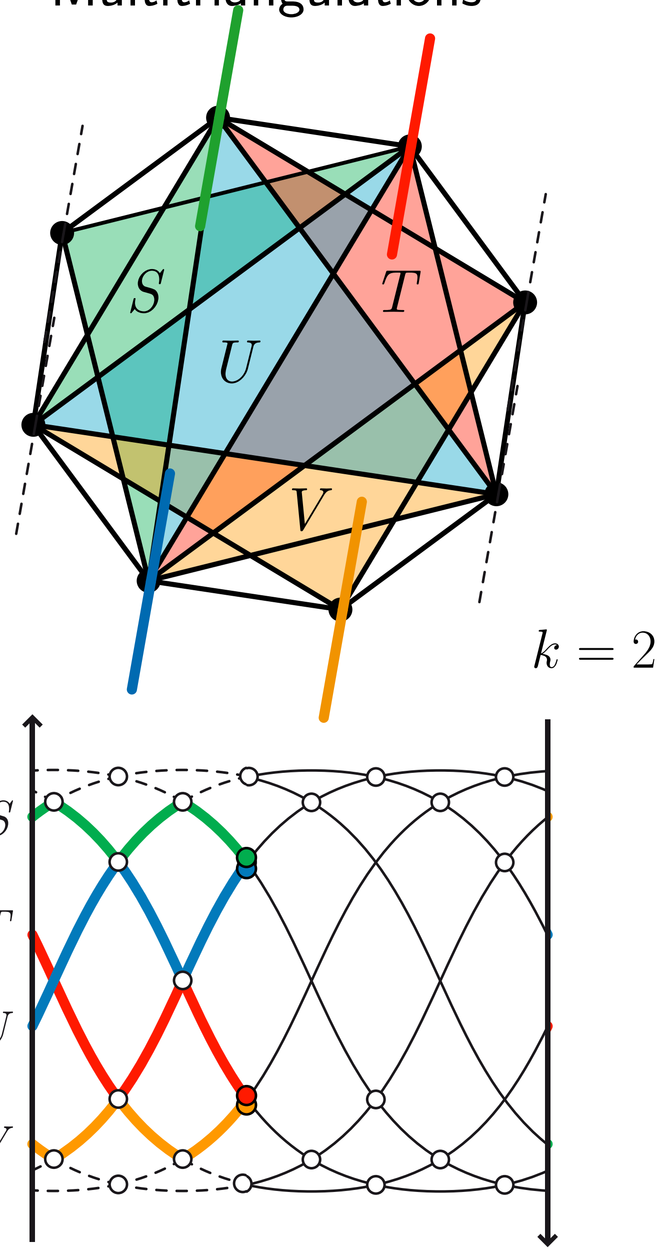


Pseudotriangulations



Multitriangulations



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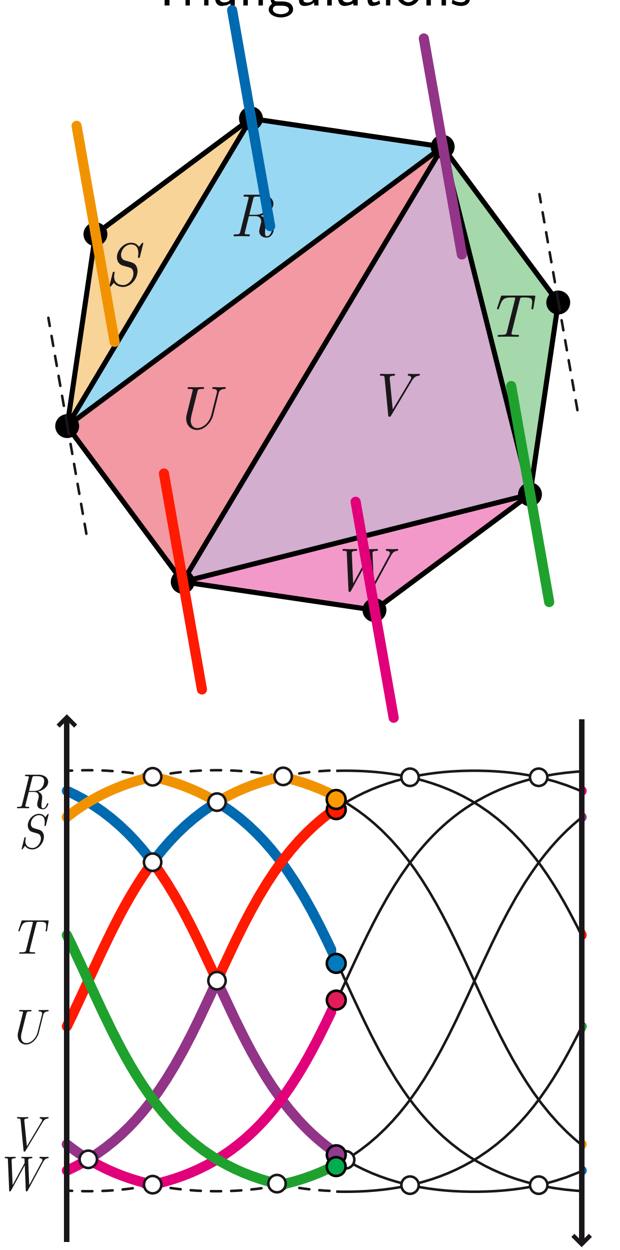

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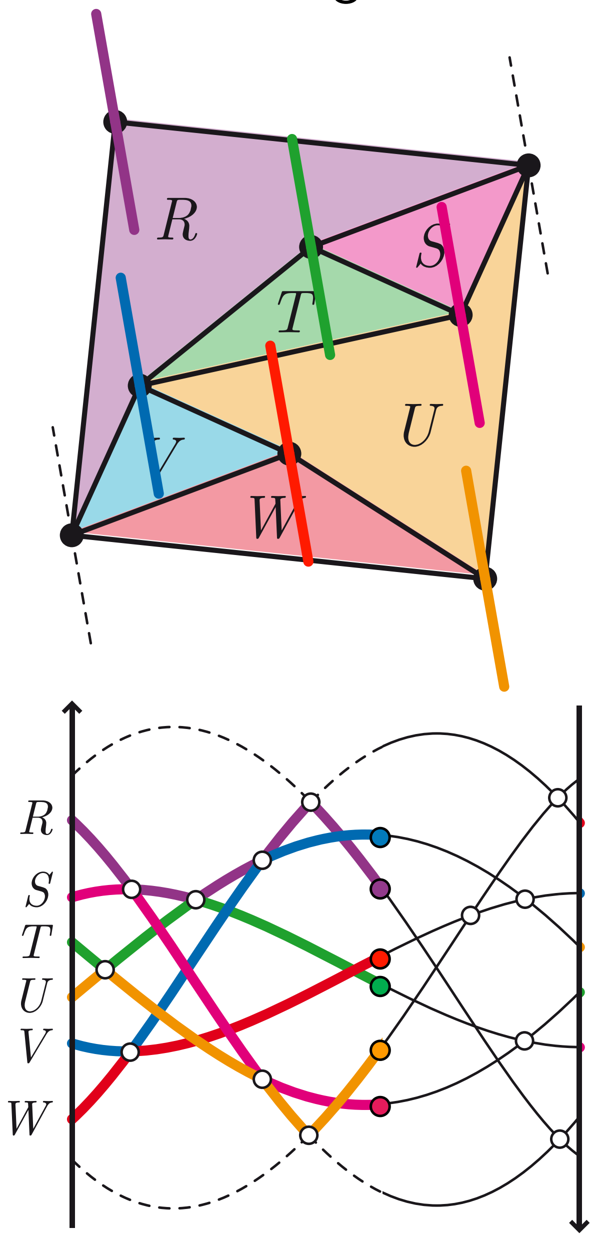
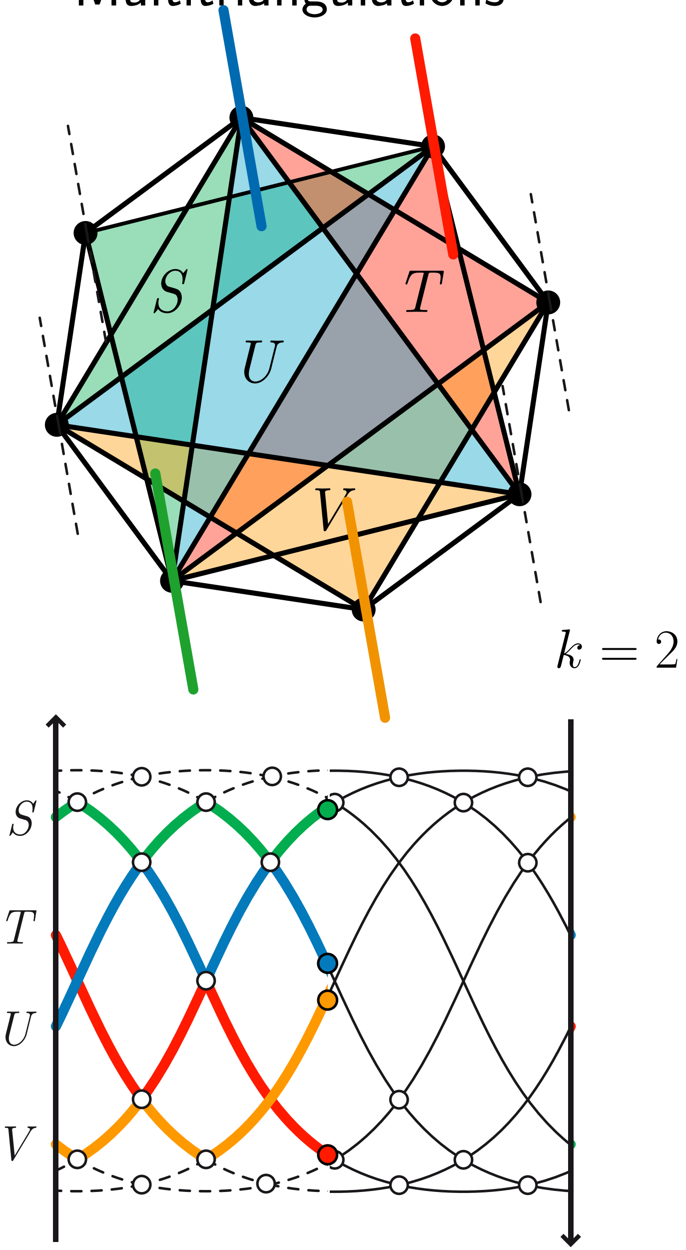


Multitriangulations



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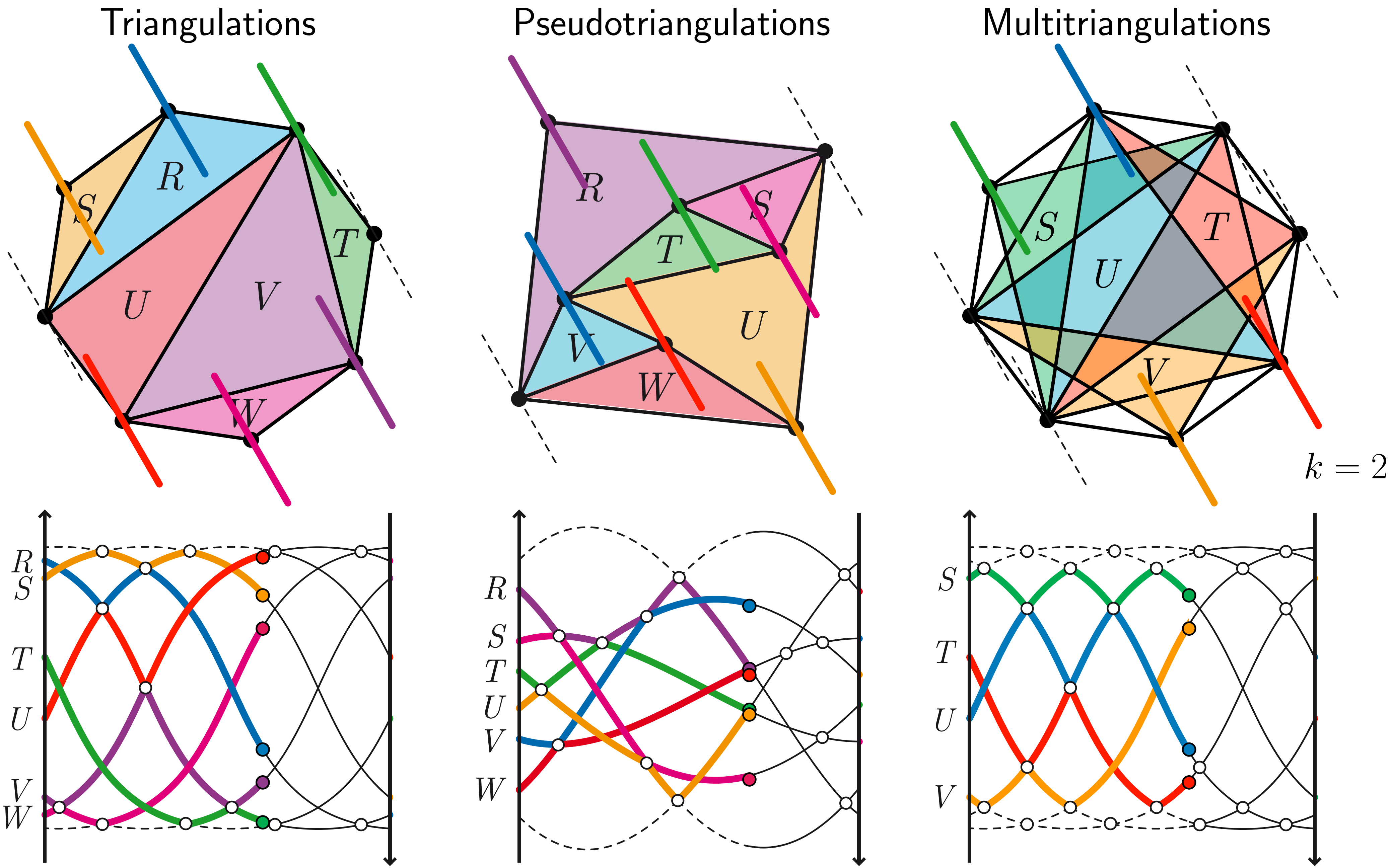
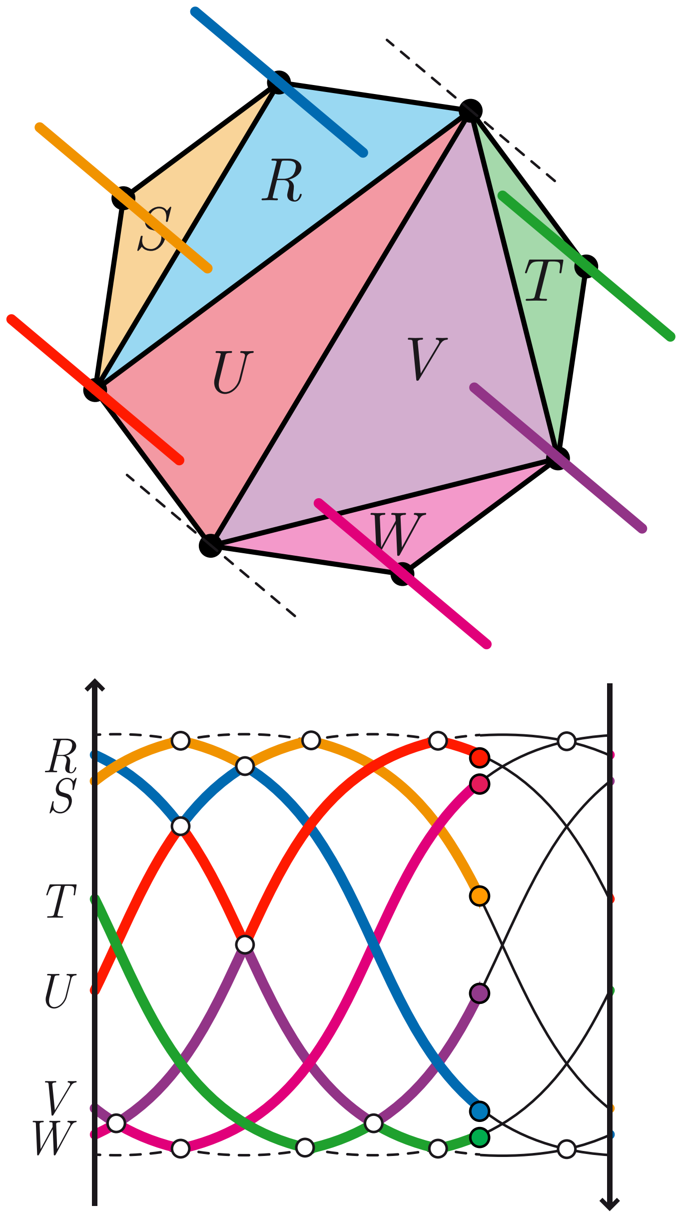

Triangulations
Pseudotriangulations
Multitriangulations
R
S
T
U
V
W
k = 2

DUALITY

Triangulations



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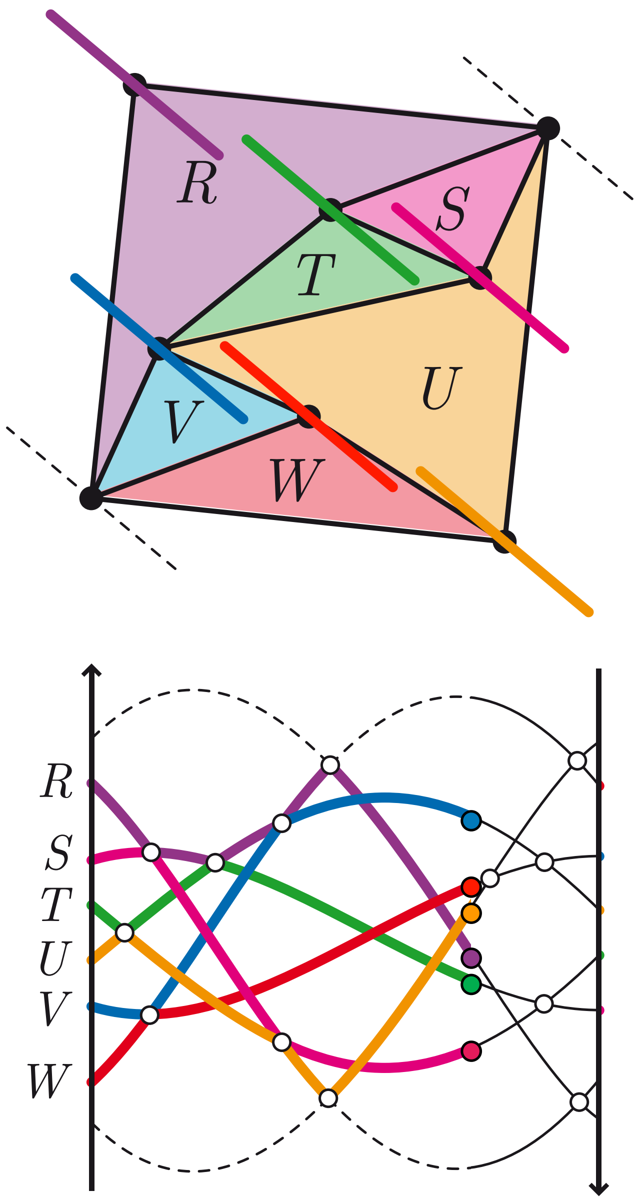
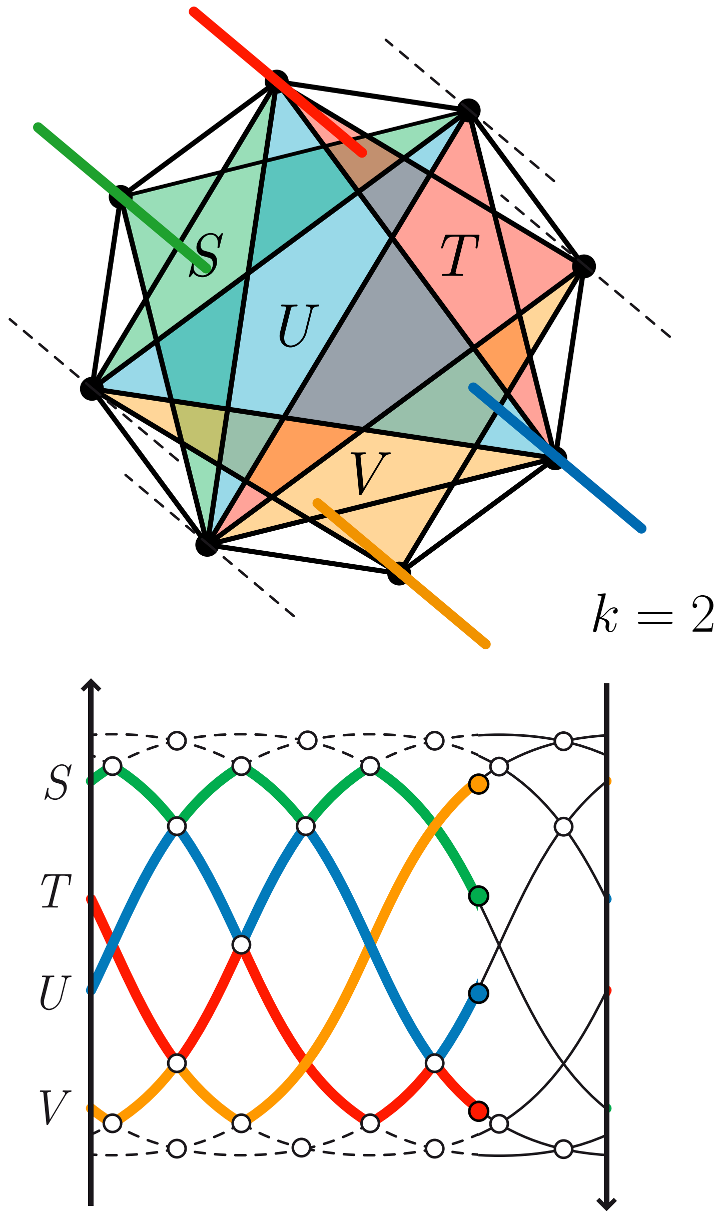
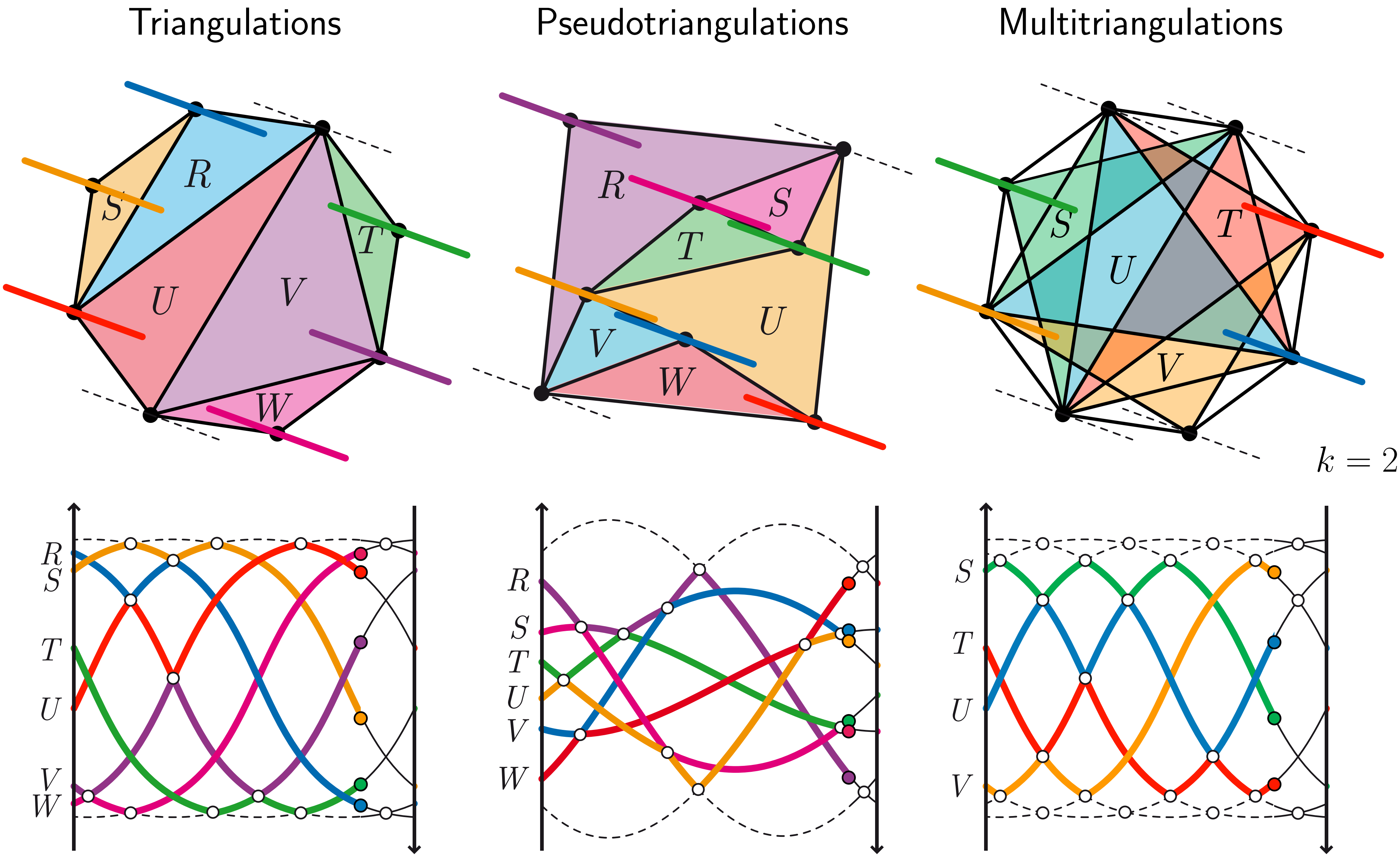


Multitriangulations



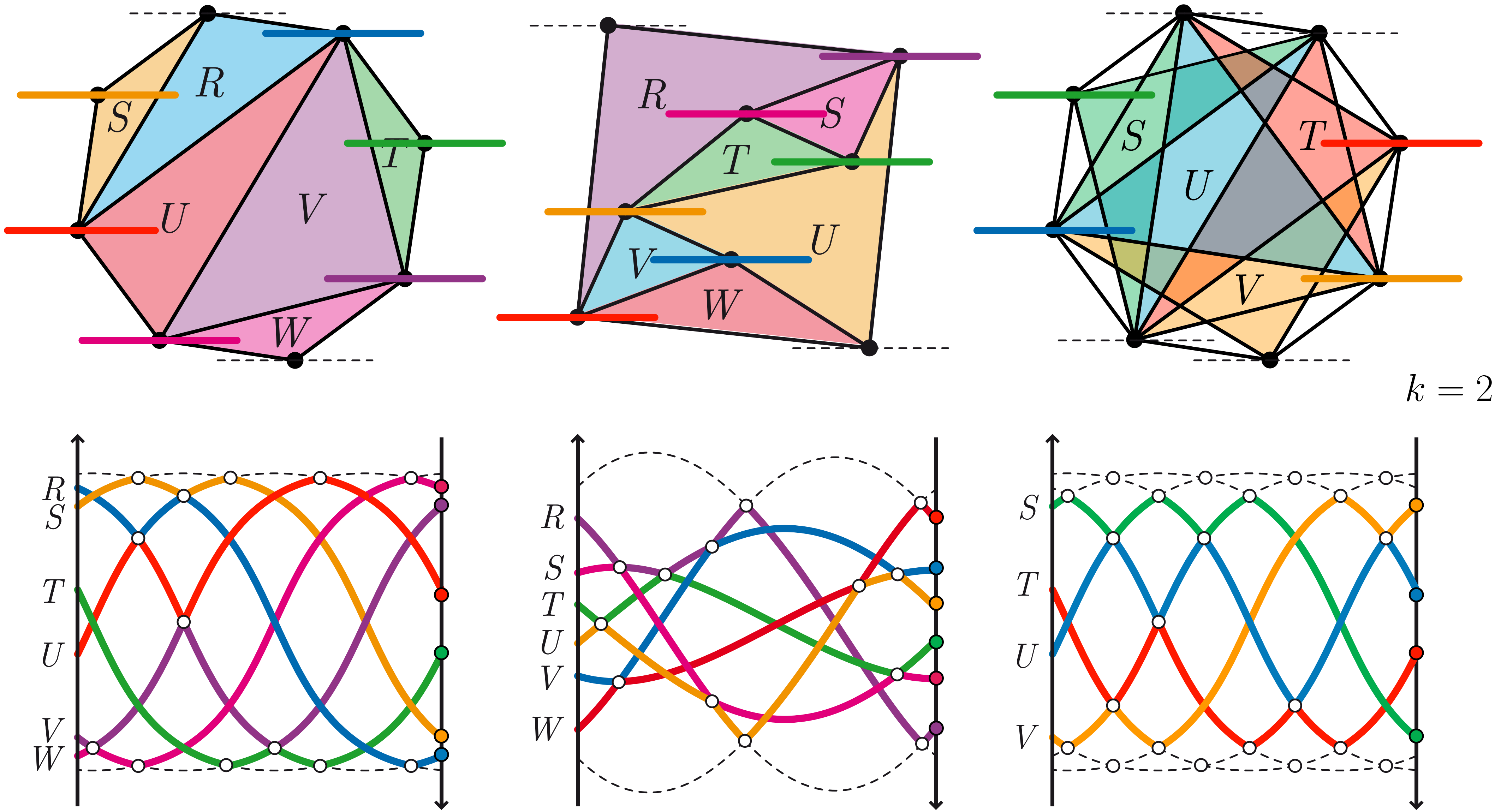
DUALITY



Triangulations
Pseudotriangulations
Multitriangulations
R
S
T
U
V
W
k = 2

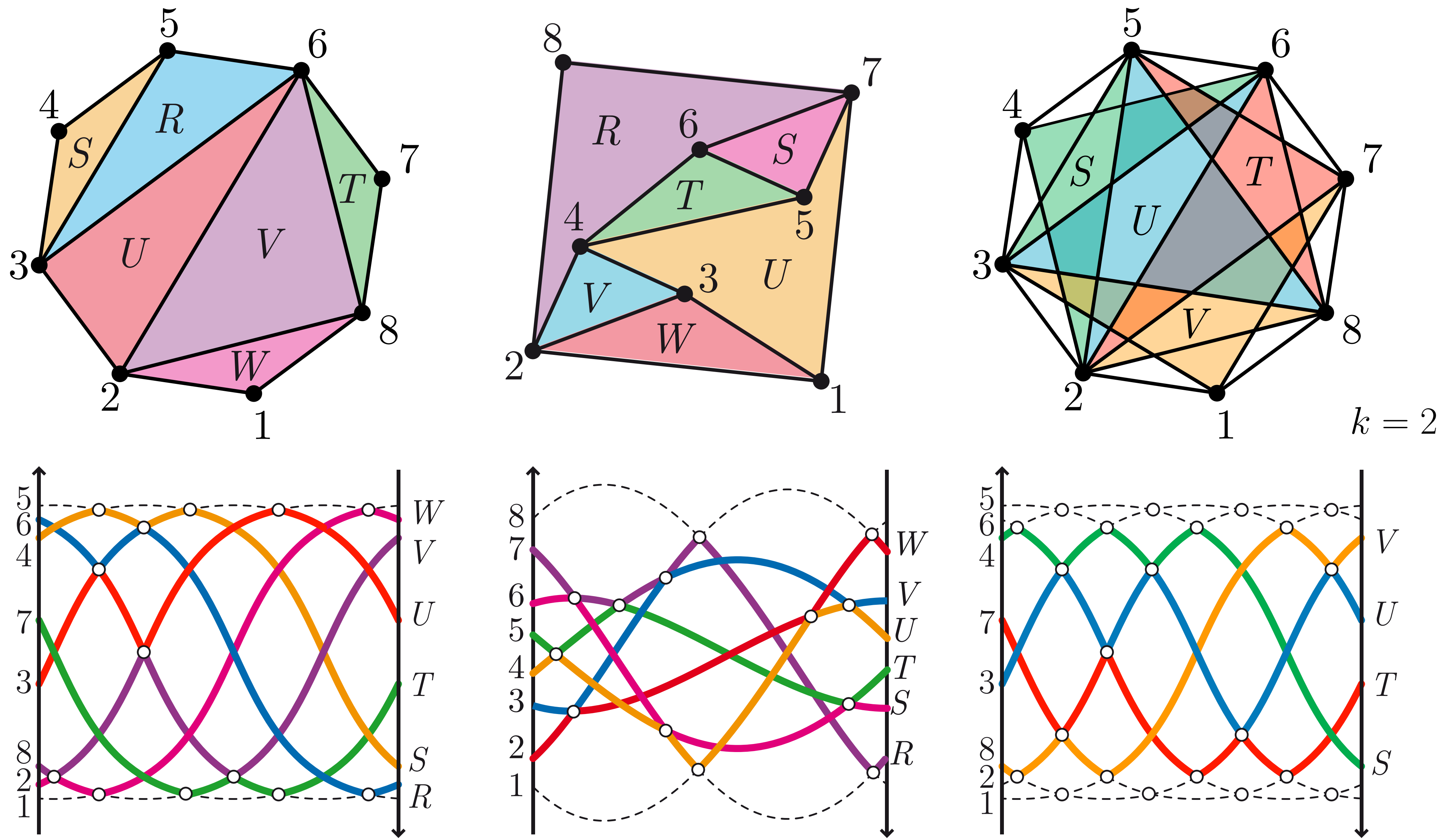
DUALITY

Triangulations | Pseudotriangulations | Multitriangulations



DUALITY

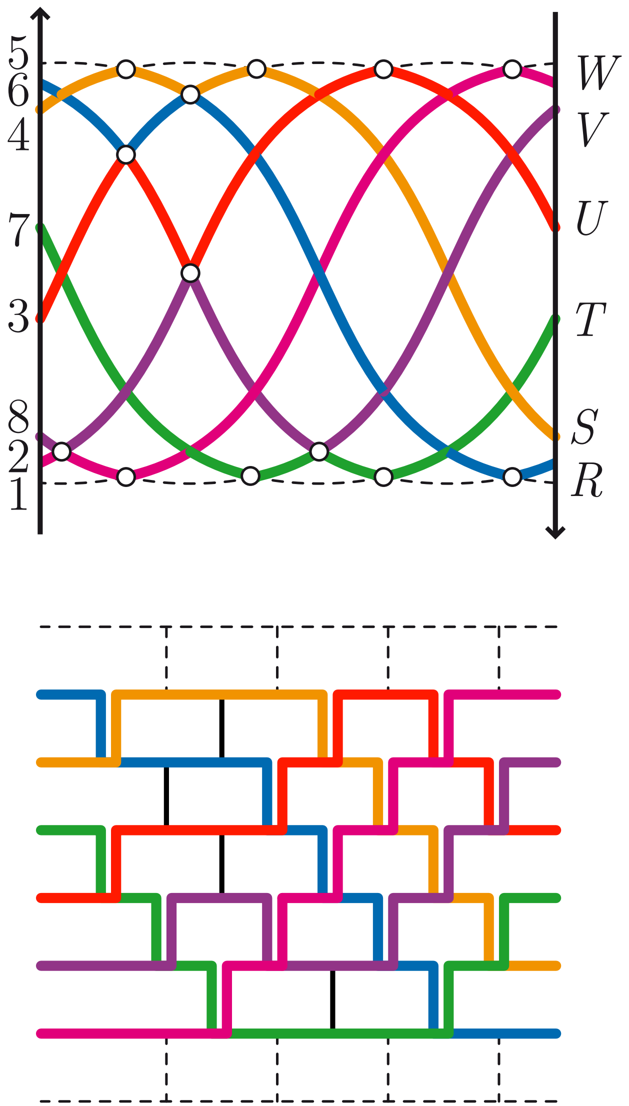
Triangulations | Pseudotriangulations | Multitriangulations

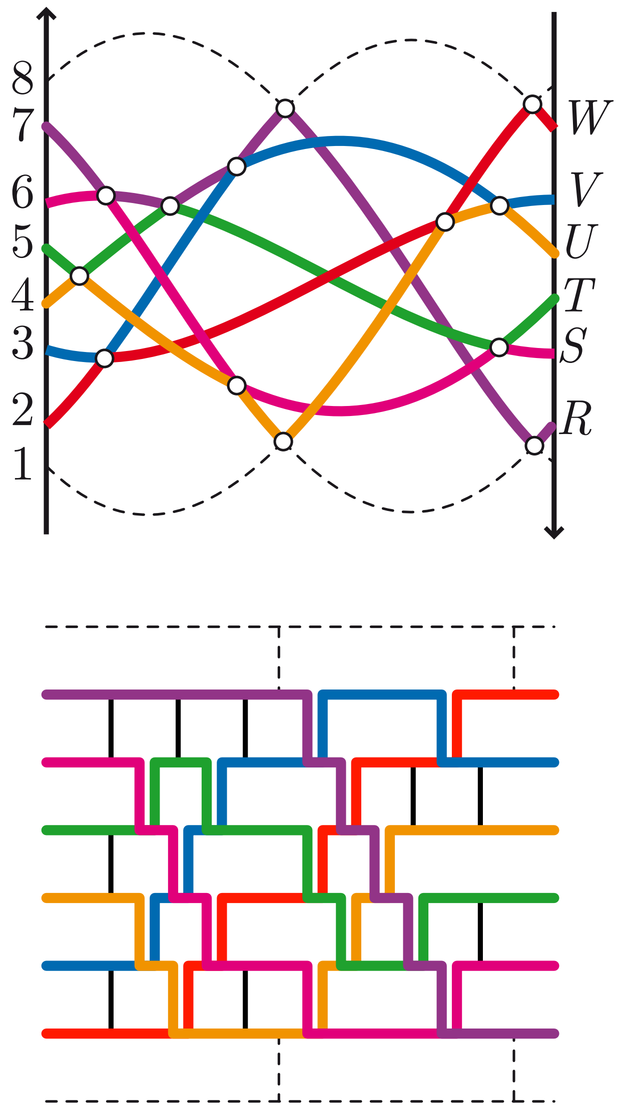


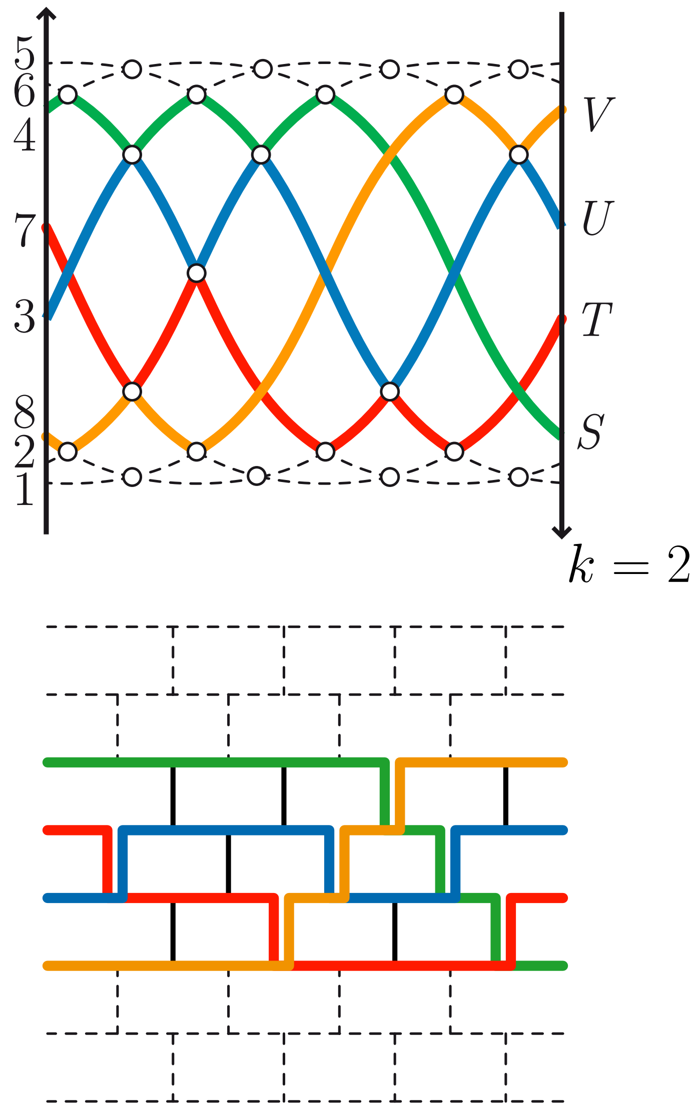
VP & M. Pocchiola, Multitriangulations, pseudotriangulations and primitive sorting networks, 2010^+.

DUALITY

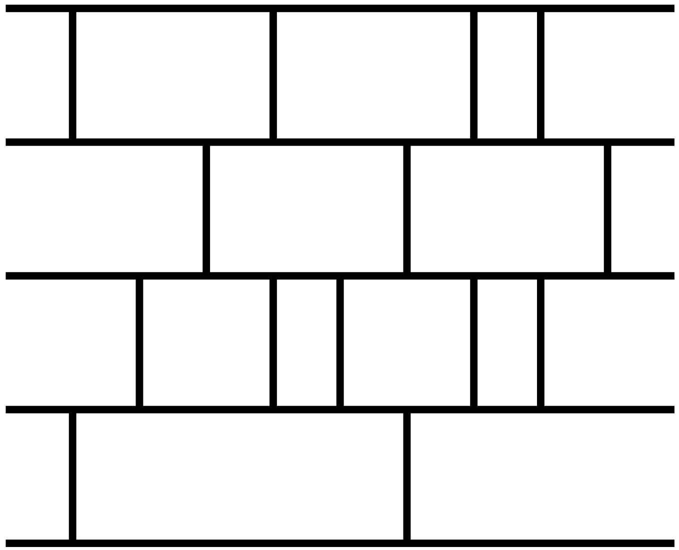
Triangulations	Pseudotriangulations	Multitriangulations





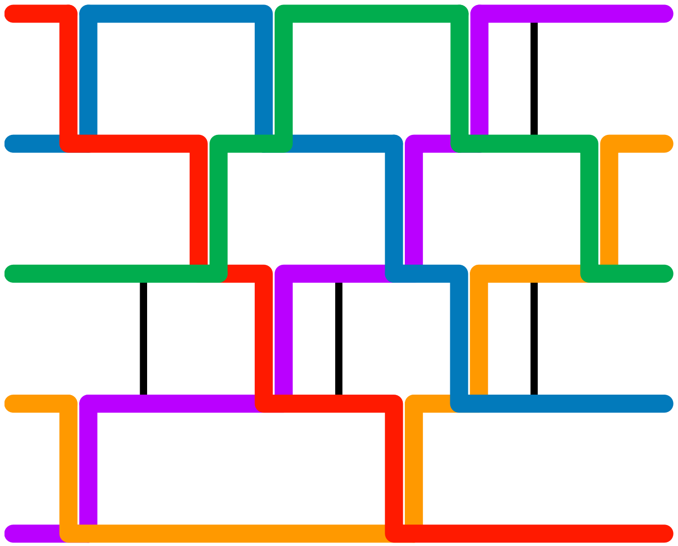


NETWORKS & PSEUDOLINE ARRANGEMENTS



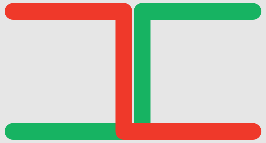
network $\mathcal{N}$ = n horizontal levels and m vertical commutators.
bricks of $\mathcal{N}$ = bounded cells.

NETWORKS & PSEUDOLINE ARRANGEMENTS



network $\mathcal{N} = n$ horizontal levels and m vertical commutators.
bricks of $\mathcal{N}$ = bounded cells.

pseudoline = x-monotone path which starts at a level l and ends at the level $n+1-l$.

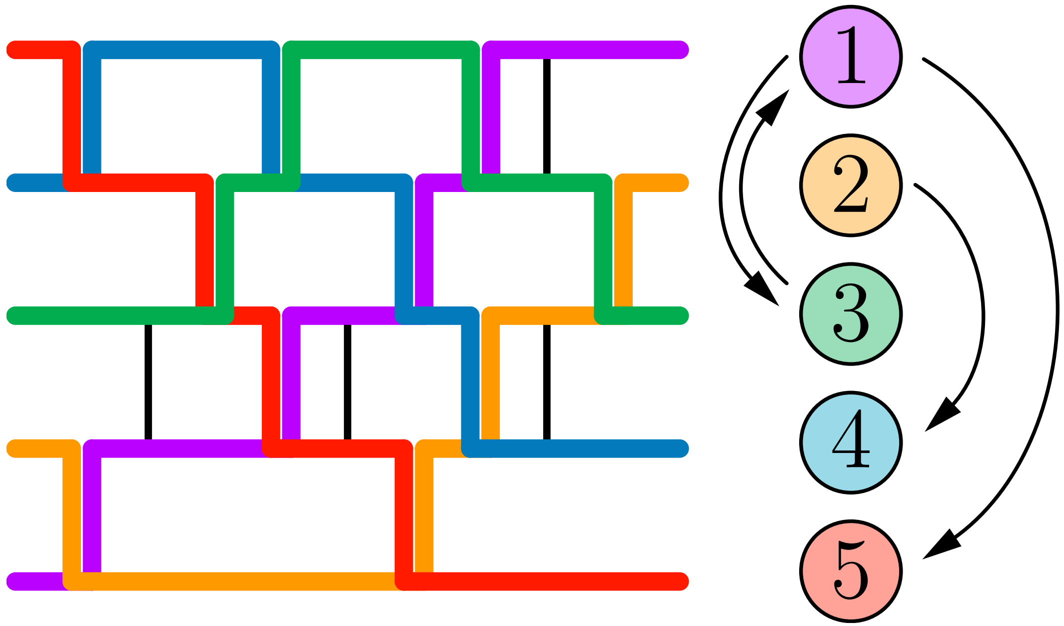
crossing =  contact = 

pseudoline arrangement (with contacts) = n pseudolines supported by $\mathcal{N}$ which have pairwise exactly one crossing, eventually some contacts, and no other intersection.

CONTACT GRAPH OF A PSEUDOLINE ARRANGEMENT

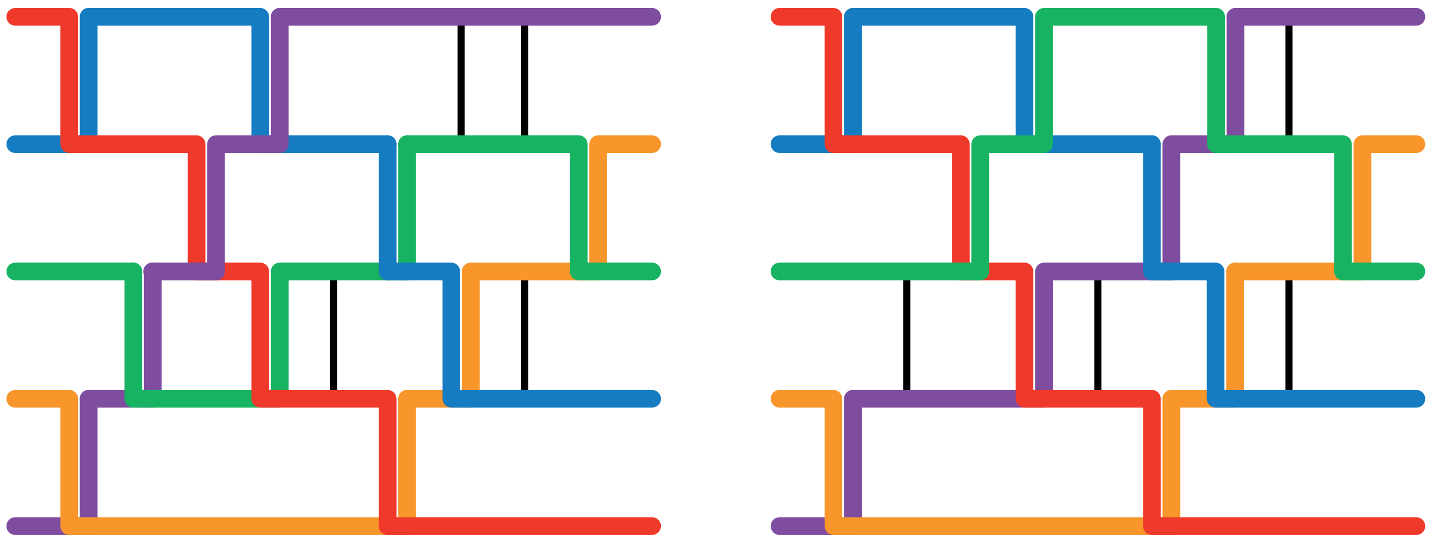
Contact graph $\Lambda^{\#}$ of a pseudoline arrangement Λ =

- a node for each pseudoline of Λ, and
- an arc for each contact point of Λ oriented from top to bottom.



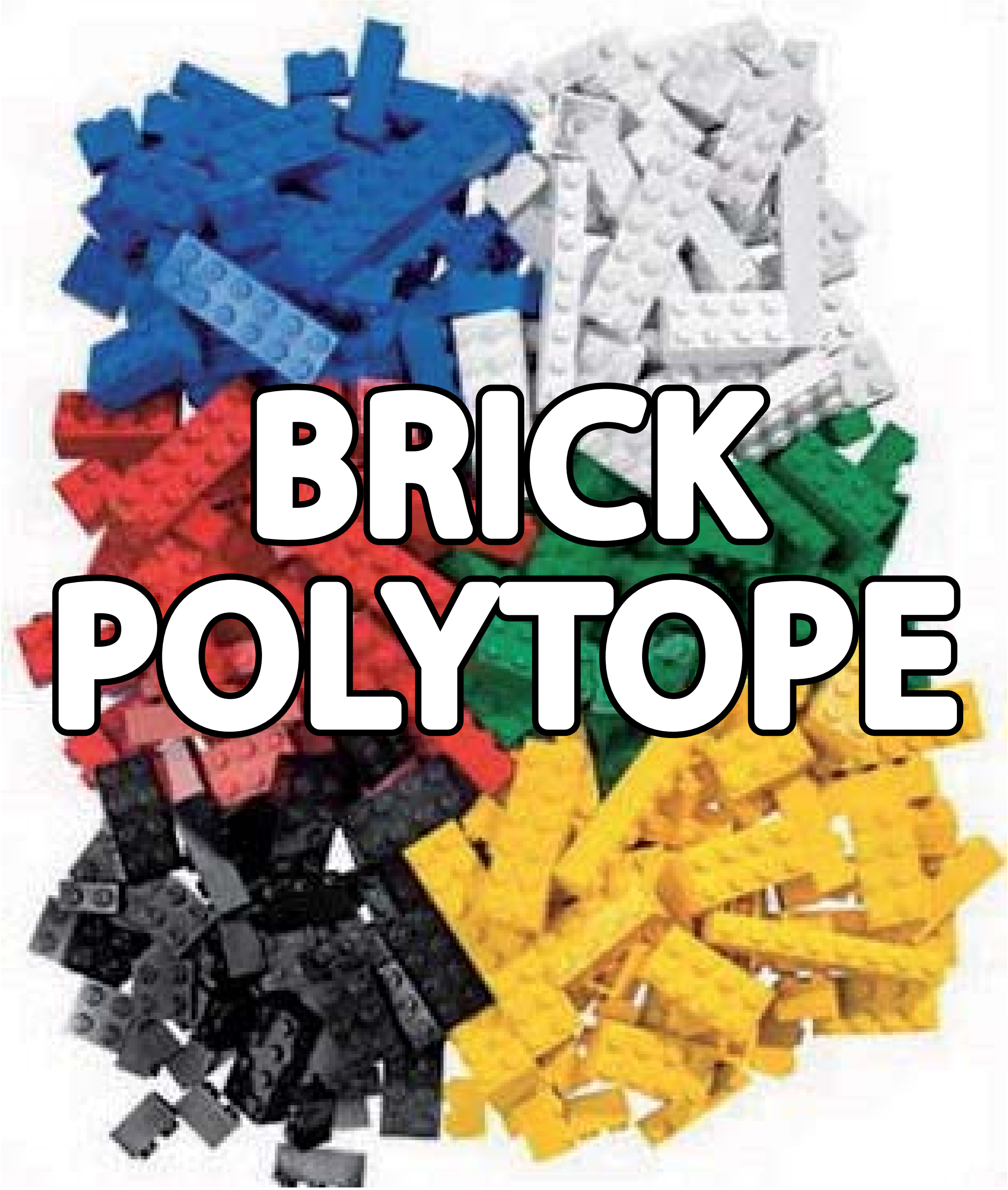
FLIPS

flip $=$ exchange a contact with the corresponding crossing.



THEOREM. Let $\mathcal{N}$ be a sorting network with n levels and m commutators. The graph of flips $G(\mathcal{N})$ is $\left(m - \binom{n}{2}\right)$-regular and connected.

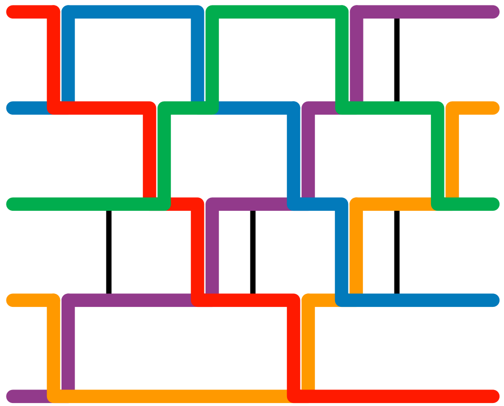
QUESTION. Is $G(\mathcal{N})$ the graph of a simple $\left(m - \binom{n}{2}\right)$-dimensional polytope?


BRICK
POLYTOPE

BRICK POLYTOPE

Λ pseudoline arrangement supported by $\mathcal{N}$ $\longmapsto$ brick vector $\omega(\Lambda) \in \mathbb{R}^n$.

$\omega(\Lambda)_j$ = number of bricks of $\mathcal{N}$ below the jth pseudoline of Λ.



Brick polytope $\Omega(\mathcal{N}) = \text{conv}\{\omega(\Lambda) \mid \Lambda \text{ pseudoline arrangement supported by } \mathcal{N}\}$.

BRICK POLYTOPE

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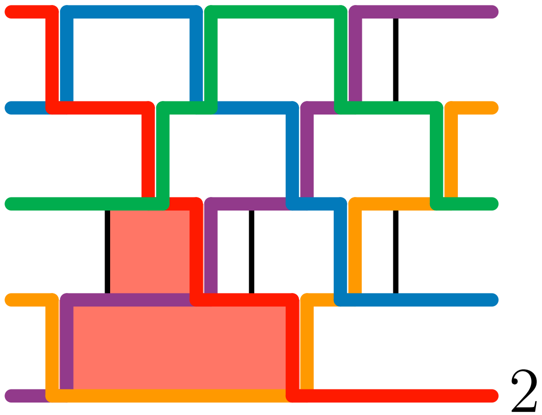

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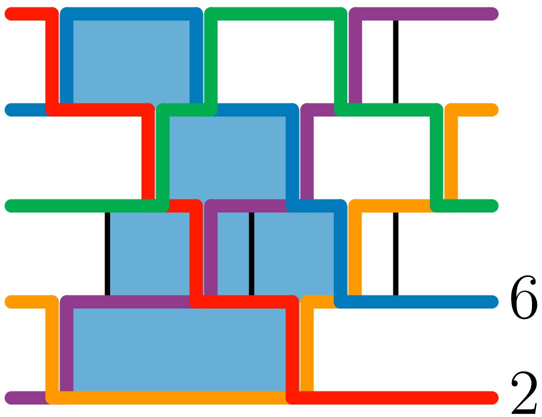

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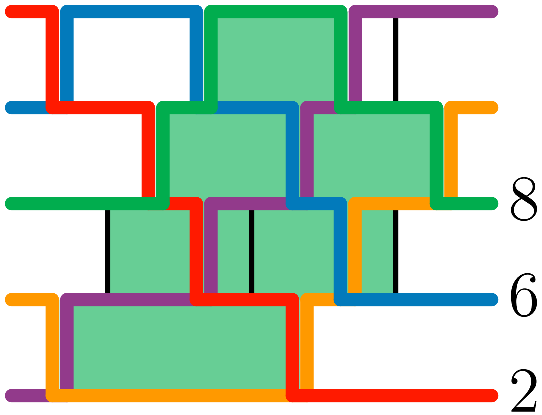

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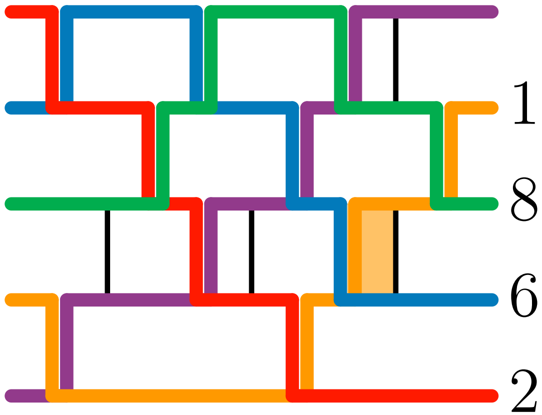

Brick polytope $\Omega(\mathcal{N}) = \text{conv}\,\{\omega(\Lambda) \mid \Lambda \text{ pseudoline arrangement supported by } \mathcal{N}\}$.

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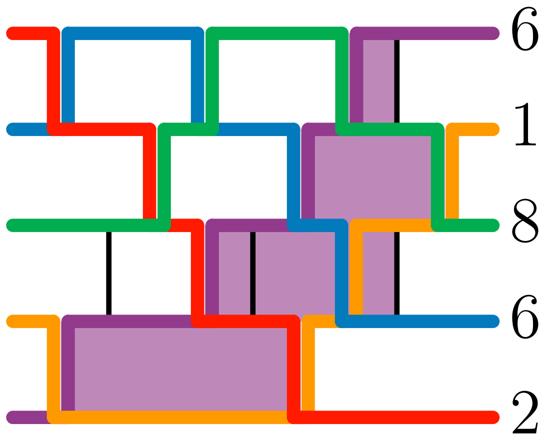

Brick polytope $\Omega(\mathcal{N}) = \text{conv}\,\{\omega(\Lambda) \mid \Lambda$ pseudoline arrangement supported by $\mathcal{N}\}$.

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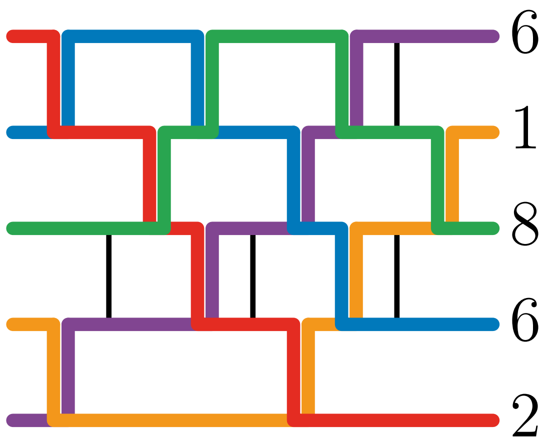


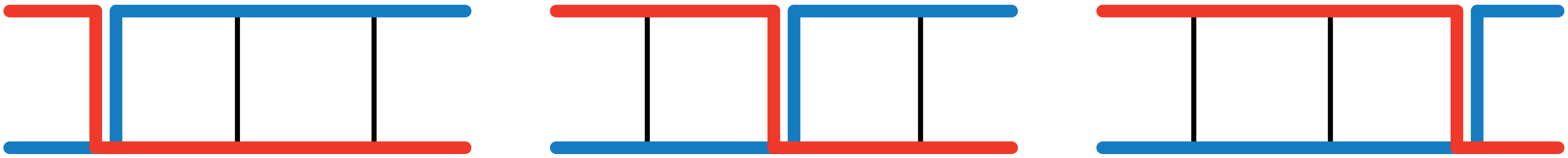
Brick polytope $\Omega(\mathcal{N}) = \text{conv}\{\omega(\Lambda) \mid \Lambda \text{ pseudoline arrangement supported by } \mathcal{N}\}$.

REMARK. The brick polytope is not full-dimensional:

$$\Omega(\mathcal{N}) \subset \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \;\middle|\; \sum_{i=1}^{n} x_i = \sum_{b \text{ brick of } \mathcal{N}} \text{depth}(b) \right\}.$$

EXAMPLE: 2-LEVELS NETWORKS

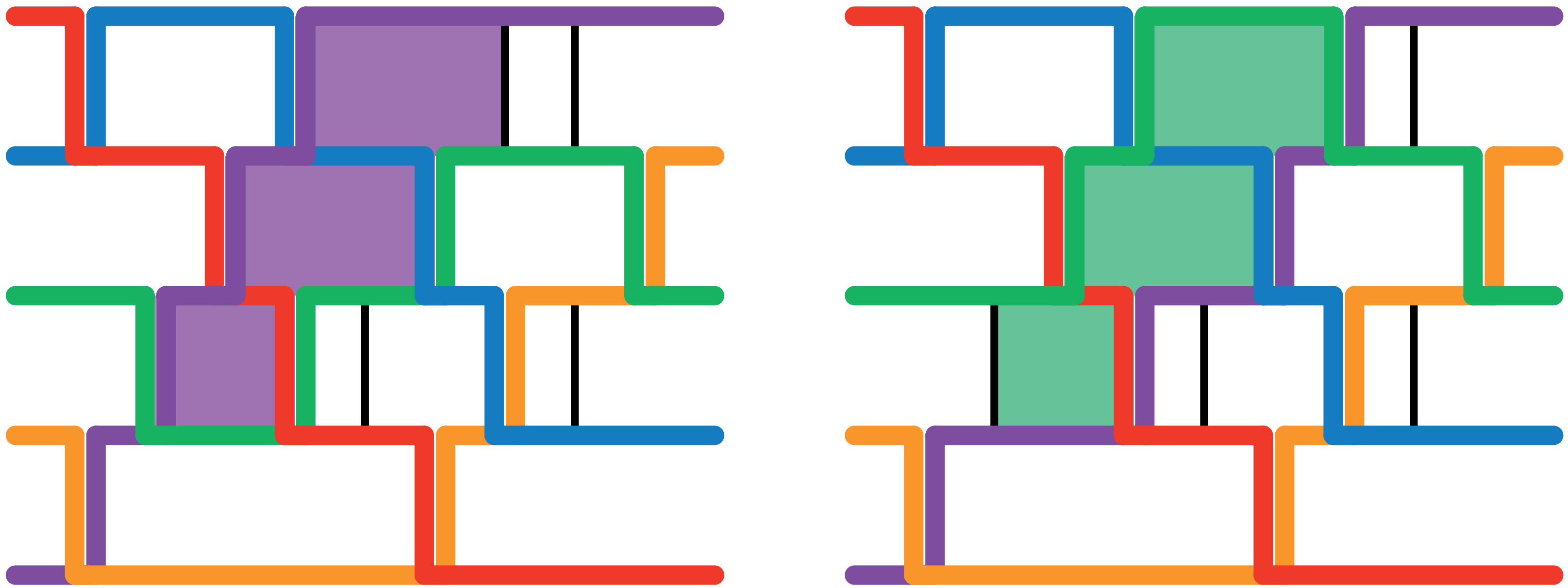
$\mathcal{X}_m$ = network with two levels and m commutators.



Graph of flips $G(\mathcal{X}_m)$ = complete graph K_m.

Brick polytope $\Omega(\mathcal{X}_m) = \text{conv}\left\{ \begin{pmatrix} m-i \\ i-1 \end{pmatrix} \;\middle|\; i \in [m] \right\} = \left[\begin{pmatrix} m-1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ m-1 \end{pmatrix} \right]$.

BRICK VECTORS AND FLIPS



REMARK. If Λ and Λ' are two pseudoline arrangements supported by $\mathcal{N}$ and related by a flip between their ith and jth pseudolines, then $\omega(\Lambda) - \omega(\Lambda') \in \mathbb{N}_{>0}\,(e_j - e_i)$.

INCIDENCE CONE OF A DIRECTED MULTIGRAPH

G directed (multi)graph $\longmapsto$ Incidence configuration $I(G) = \{e_j - e_i \mid (i,j) \in G\}$,
$\longmapsto$ Incidence cone $C(G)$ = cone generated by $I(G)$.

REMARK.
independant sets in $I(G)$ $\longleftrightarrow$ forests in G,
spanning sets of $\langle \mathbb{1} \mid x \rangle = 0$ $\longleftrightarrow$ connected spanning subgraphs of G,
basis of $\langle \mathbb{1} \mid x \rangle = 0$ $\longleftrightarrow$ spanning trees of G,
circuits in $I(G)$ $\longleftrightarrow$ simple cycles in G,
cocircuits in $I(G)$ $\longleftrightarrow$ minimal cuts in G,
and signs correspond to the orientations of the edges.

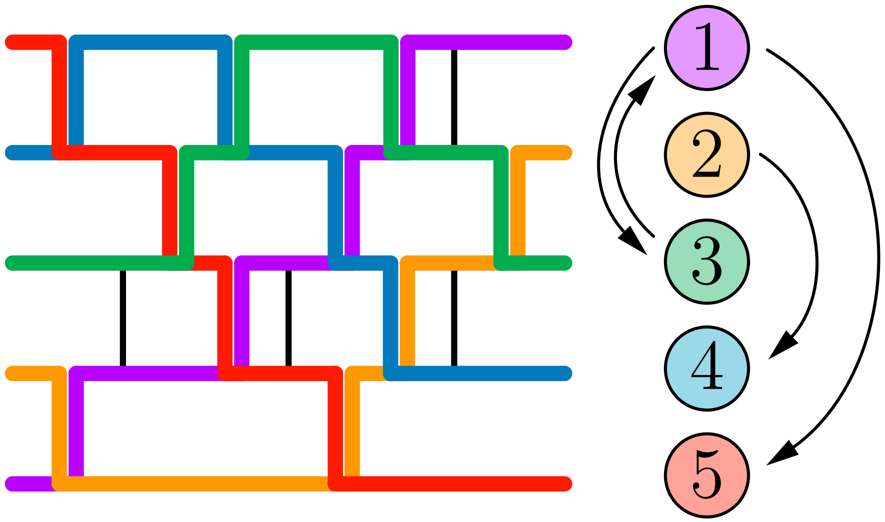
REMARK. H subgraph of G. Then $I(H)$ forms a k-face of $C(G)$ $\Longleftrightarrow$ H has $n-k$ connected components and G/H is acyclic. In particular:

$C(G)$ is pointed $\longleftrightarrow$ G is acyclic,
facets of $C(G)$ $\longleftrightarrow$ complements of the minimal directed cuts of G.

CONTACT GRAPH OF A PSEUDOLINE ARRANGEMENT

Contact graph $\Lambda^{\#}$ of a pseudoline arrangement Λ =

- a node for each pseudoline of Λ, and
- an arc for each contact point of Λ oriented from top to bottom.



THEOREM. The cone of the brick polytope $\Omega(\mathcal{N})$ at the brick vector $\omega(\Lambda)$ is the incidence cone $C(\Lambda^{\#}) = \text{cone}\left\{e_j - e_i \;\middle|\; (i,j) \in \Lambda^{\#}\right\}$ of the contact graph of Λ.

COMBINATORIAL DESCRIPTION

THEOREM. The cone of the brick polytope $\Omega(\mathcal{N})$ at the brick vector $\omega(\Lambda)$ is the incidence cone $C(\Lambda^{\#})$ of the contact graph of Λ:

$$\text{cone}\,\{\omega(\Lambda') - \omega(\Lambda) \mid \Lambda' \text{ supported by } \mathcal{N}\} = \text{cone}\left\{e_j - e_i \,\middle|\, (i,j) \in \Lambda^{\#}\right\}.$$

VERTICES OF $\Omega(\mathcal{N})$

The brick vector $\omega(\Lambda)$ is a vertex of $\Omega(\mathcal{N}) \iff$ the contact graph $\Lambda^{\#}$ is acyclic.

GRAPH OF $\Omega(\mathcal{N})$

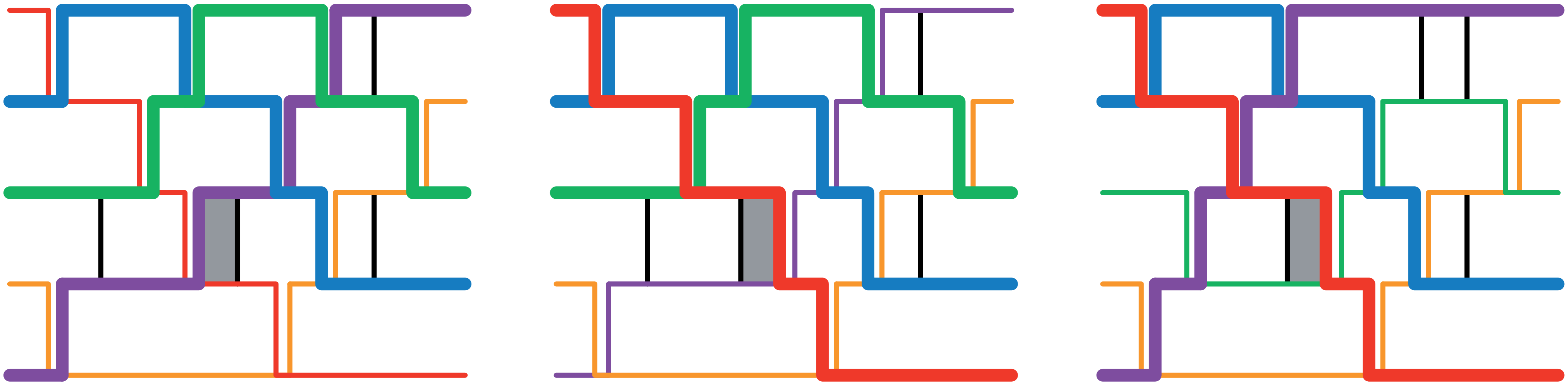
The graph of the brick polytope is a subgraph of $G(\mathcal{N})$ whose vertices are the pseudoline arrangements with acyclic contact graphs.

FACETS OF $\Omega(\mathcal{N})$

The facets of $\Omega(\mathcal{N})$ correspond to the minimal directed cuts of the contact graphs of the pseudoline arrangements supported by $\mathcal{N}$.

BRICK POLYTOPES AND MINKOWSKI SUMS

$\mathcal{N}$ network with n levels, b a brick of $\mathcal{N}$, Λ pseudoline arrangement supported by $\mathcal{N}$.
$\omega(\Lambda, b) \in \mathbb{R}^n$ characteristic vector of the pseudolines of Λ passing above b.
$\Omega(\mathcal{N}, b) = \text{conv}\,\{\omega(\Lambda, b) \mid \Lambda \text{ pseudoline arrangement supported by } \mathcal{N}\} \subset \mathbb{R}^n$.



THEOREM. $\Omega(\mathcal{N}) = \text{conv}_\Lambda \sum_b \omega(\Lambda, b) = \sum_b \text{conv}_\Lambda \omega(\Lambda, b) = \sum_b \Omega(\mathcal{N}, b)$.

BRICK POLYTOPES AND GENERALIZED PERMUTOHEDRA

Generalized permutohedra = polytope whose inequality description is of the form

$$Z\left(\{z_I\}_{I\in[n]}\right) = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} x_i = z_{[n]} \text{ and } \sum_{i\in I} x_i \ge z_I \text{ for } I \subset [n] \right\}$$

for some tight values $\{z_I\}_{I\subset[n]} \in \mathbb{R}^{2^{[n]}}$ satisfying $z_I + z_J \le z_{I\cup J} + z_{I\cap J}$.

A. Postnikov, Permutohedra, associahedra and beyond, 2009.

THEOREM. Any generalized permutahedron is a Minkowski sum of simplices:

$$Z\left(\{z_I\}_{I\in[n]}\right) = \sum_{I\subset[n]} y_I \Delta_I \quad \text{where} \quad y_I = \sum_{J\subset I} (-1)^{|I\smallsetminus J|} z_J \quad \left(\text{ie.} \quad z_I = \sum_{J\subset I} y_J\right).$$

F. Ardila, C. Benedetti & J. Doker, Matroid polytopes and their volumes, 2010.

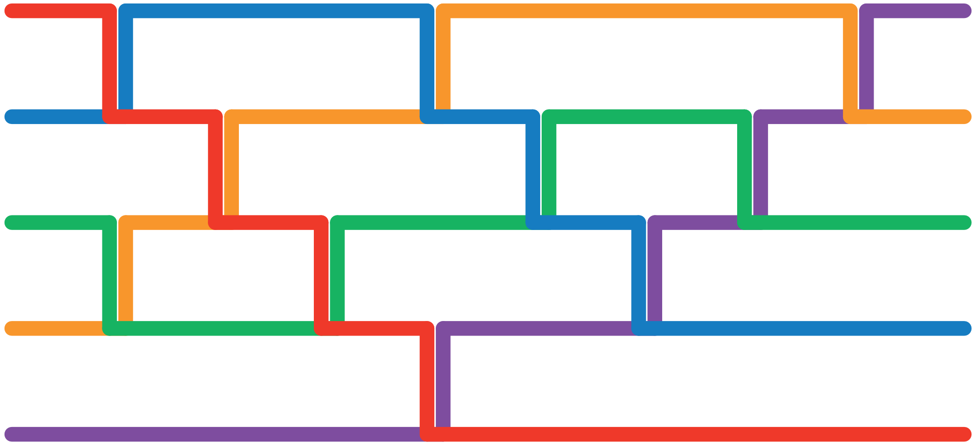
REMARK. All brick polytopes are generalized permutohedra. Compute $\{y_I\}_{I\subset[n]}$.
Which generalized permutohedra are brick polytopes?

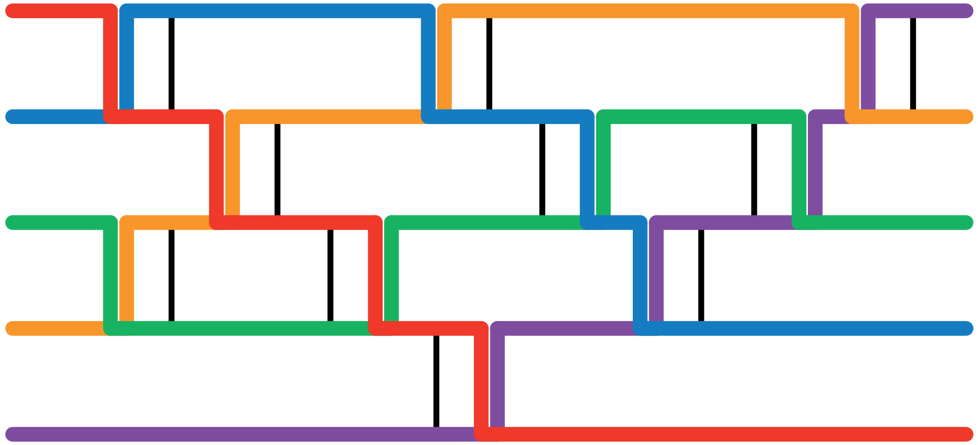
ASSOCIAHEDRA & PERMUTAHEDRA

DUPLICATED NETWORKS: PERMUTAHEDRA

Reduced network = network with n levels and $\binom{n}{2}$ commutators.
It supports only one pseudoline arrangement.

Duplicated network Π = network with n levels and $2\binom{n}{2}$ commutators obtained by duplicating each commutator of a reduced network.

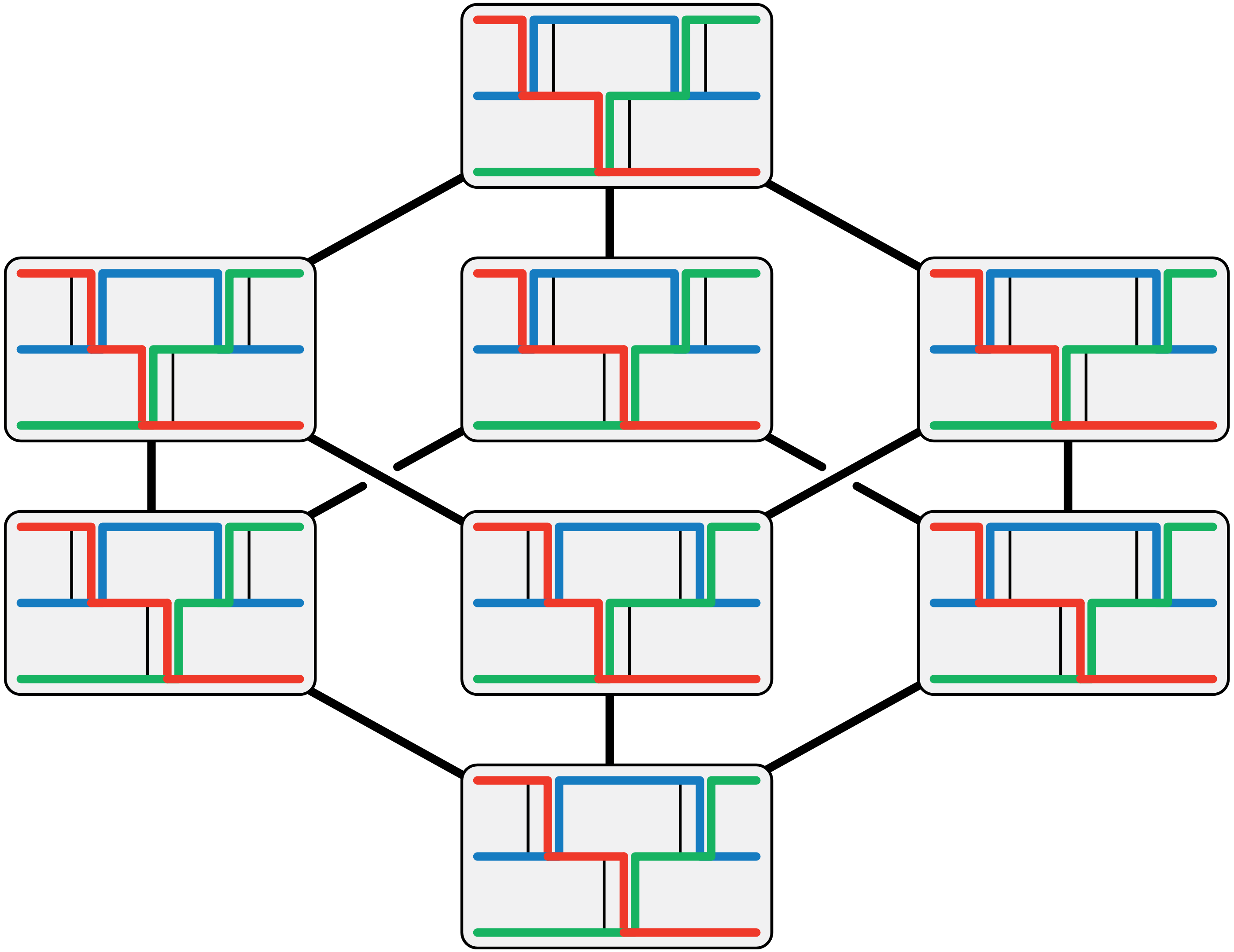




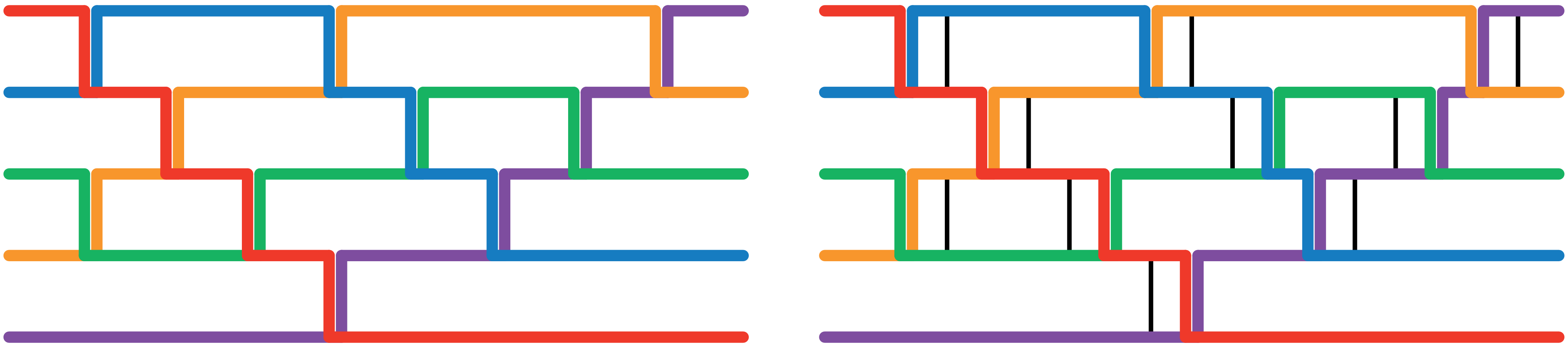
Any pseudoline arrangement supported by Π has one contact and one crossing among each pair of duplicated commutators.

DUPLICATED NETWORKS: PERMUTAHEDRA

Graph of flips $G(\Pi) = \binom{n}{2}$-dimensional cube.



DUPLICATED NETWORKS: PERMUTAHEDRA



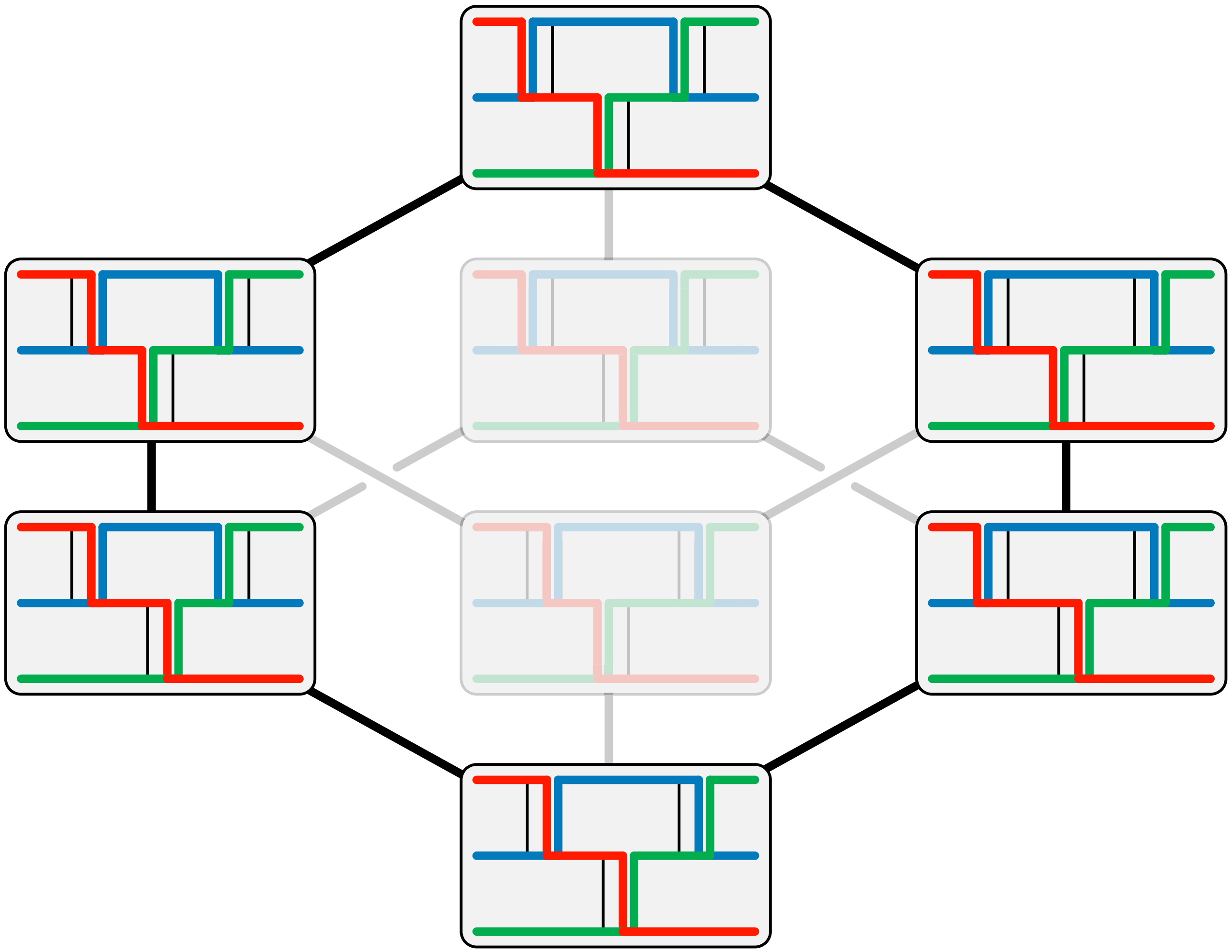
Any pseudoline arrangement supported by Π has one contact and one crossing among each pair of duplicated commutators. $\Longrightarrow$ The contact graph $\Lambda^{\#}$ is a tournament.

Vertices of $\Omega(\Pi)$	$\Longleftrightarrow$	acyclic tournaments	$\Longleftrightarrow$	permutations of $[n]$,
Facets of $\Omega(\Pi)$	$\Longleftrightarrow$	cuts in a tournament	$\Longleftrightarrow$	ordered bipartitions of $[n]$.

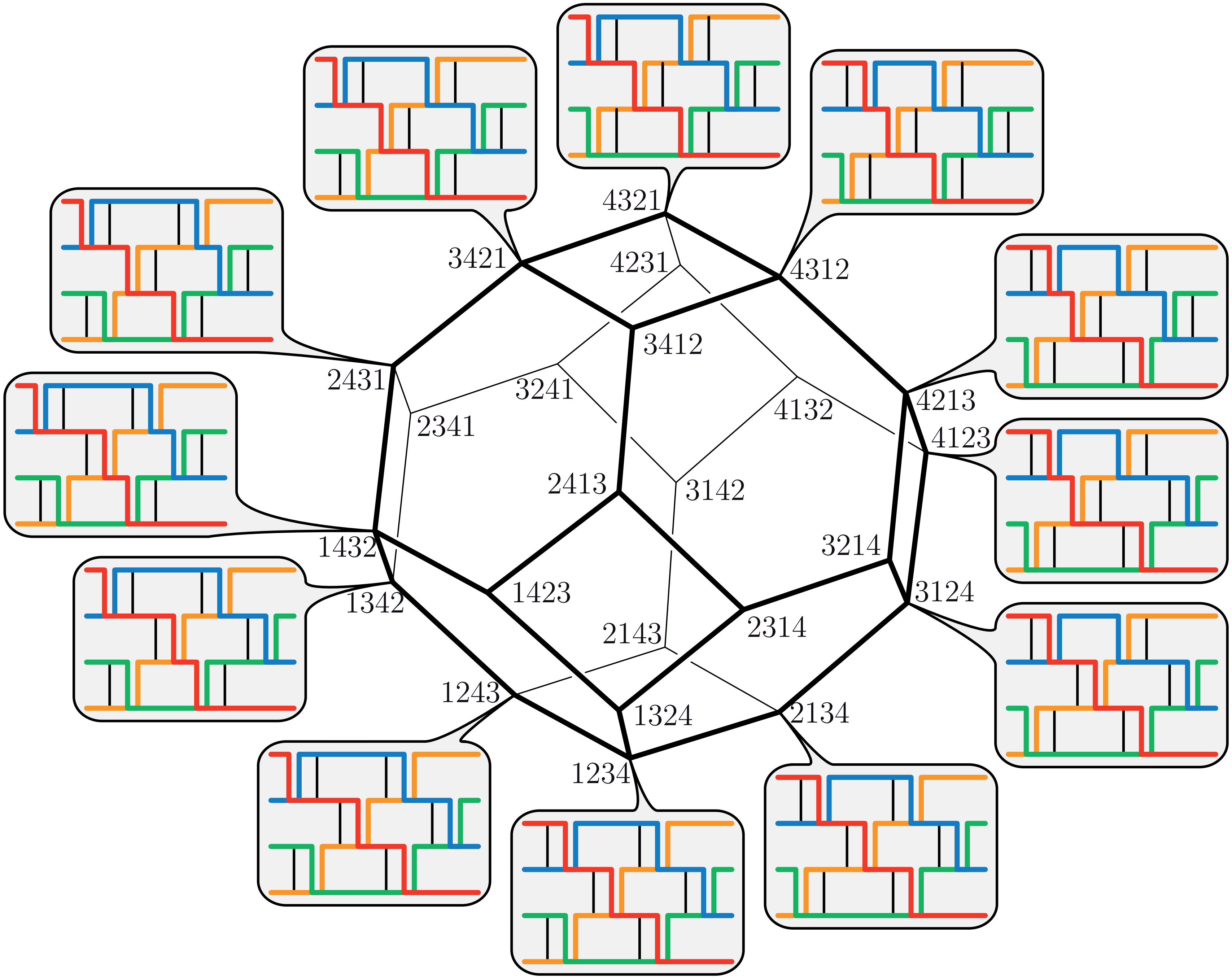
Brick polytope $\Omega(\Pi) =$ permutahedron.

DUPLICATED NETWORKS: PERMUTAHEDRA

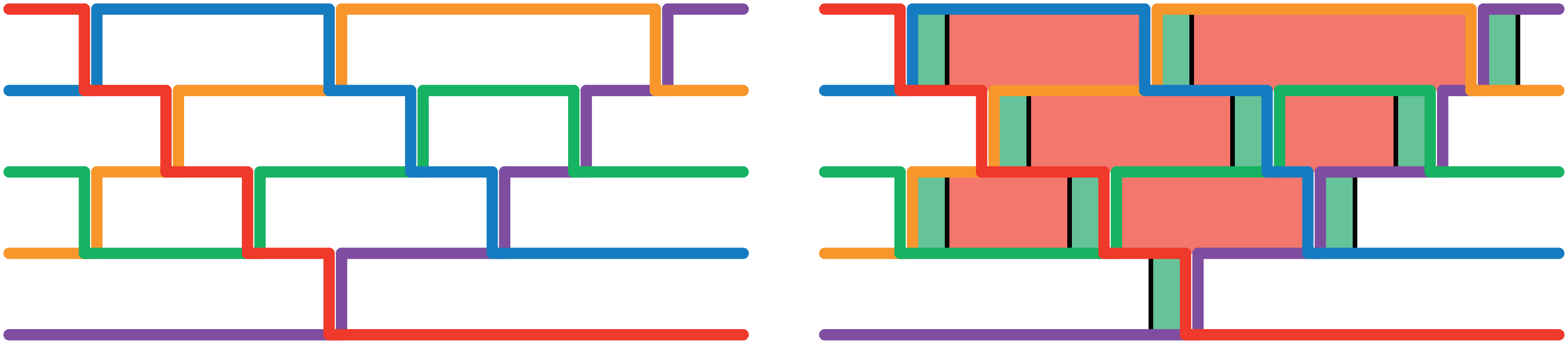
Brick polytope $\Omega(\Pi) =$ permutahedron.



DUPLICATED NETWORKS: PERMUTAHEDRA



DUPLICATED NETWORKS: PERMUTAHEDRA



Minkowski sum decomposition

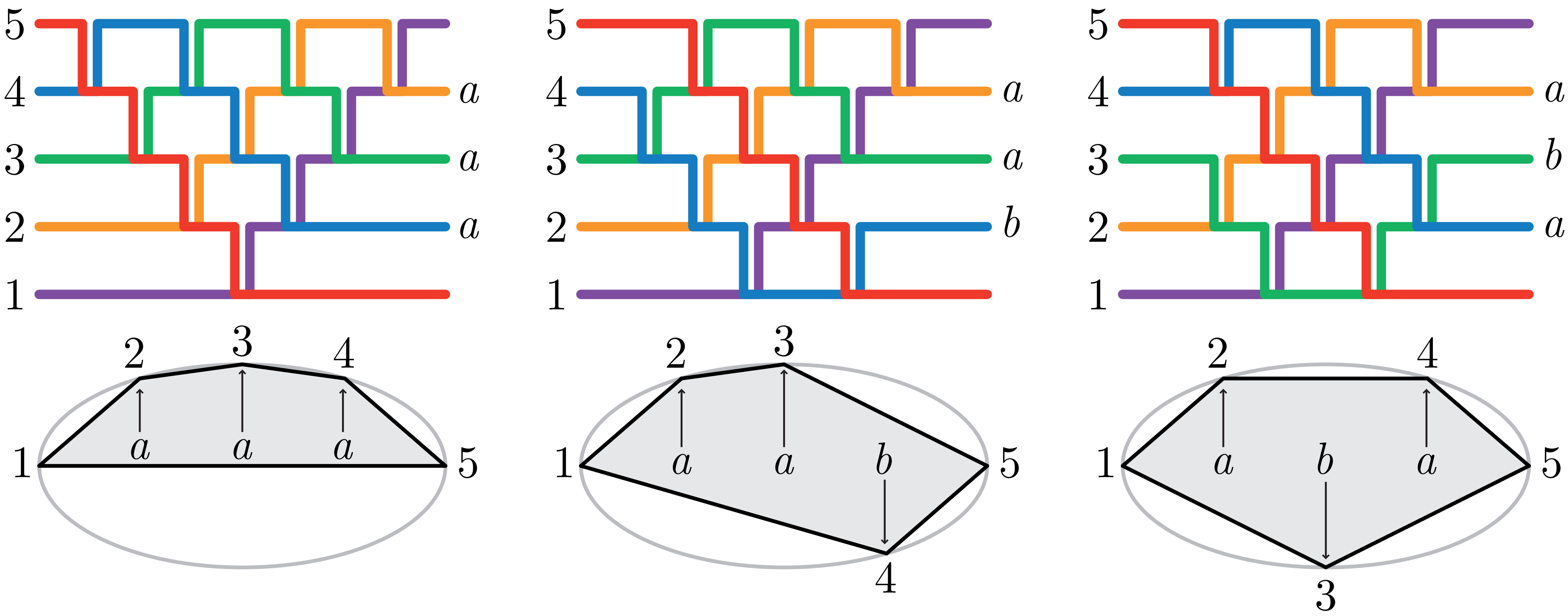
$$\Omega(\Pi) = \sum_{b \text{ brick of } \Pi} \Omega(\Pi, b) = \sum_{i<j} \text{segment } [e_i - e_j] + \sum \text{vertices} = \text{permutahedron}$$

$$P(0, 1, \dots, n-1) = \text{Newton}\left(\det \left[t_i^{j-1}\right]_{i,j \in [n]}\right) = \text{Newton}\left(\prod_{1 \le i < j \le n} (t_j - t_i)\right)$$

$$= \sum_{1 \le i < j \le n} \text{Newton}\,(t_j - t_i) = \sum_{1 \le i < j \le n} [e_j - e_i]$$

ALTERNATING NETWORKS: ASSOCIAHEDRA

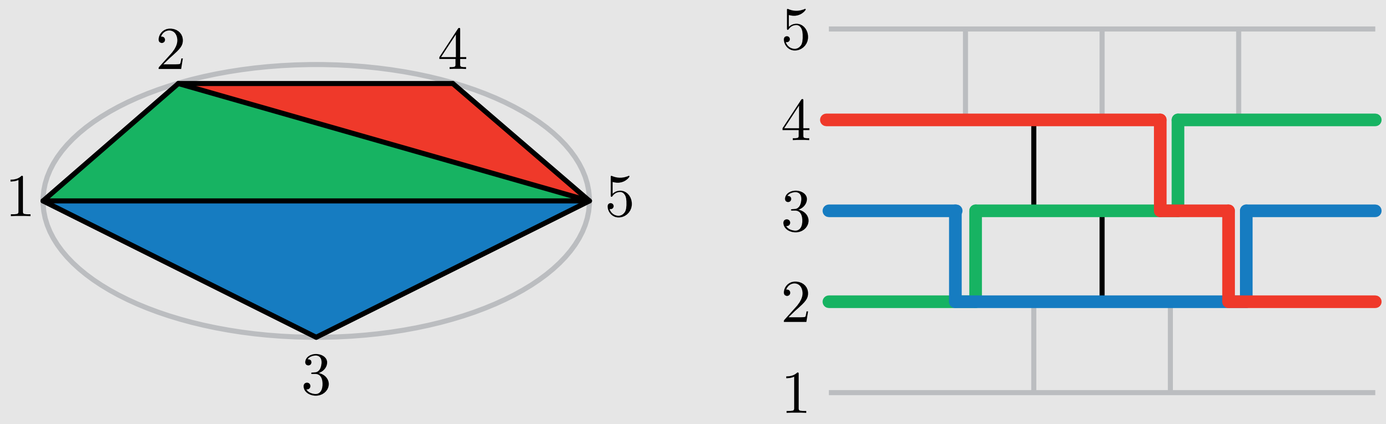
For $x \in \{a, b\}^{n-2}$, we define a reduced alternating network $\mathcal{N}_x$ and a polygon $\mathcal{P}_x$.



$\mathcal{N}_x$ is the dual pseudoline arrangement of the polygon $\mathcal{P}_x$.

ALTERNATING NETWORKS: ASSOCIAHEDRA

THEOREM. There is a duality between the pseudoline arrangements supported by $\mathcal{N}_x^1$ and the triangulations of the polygon $\mathcal{P}_x$.

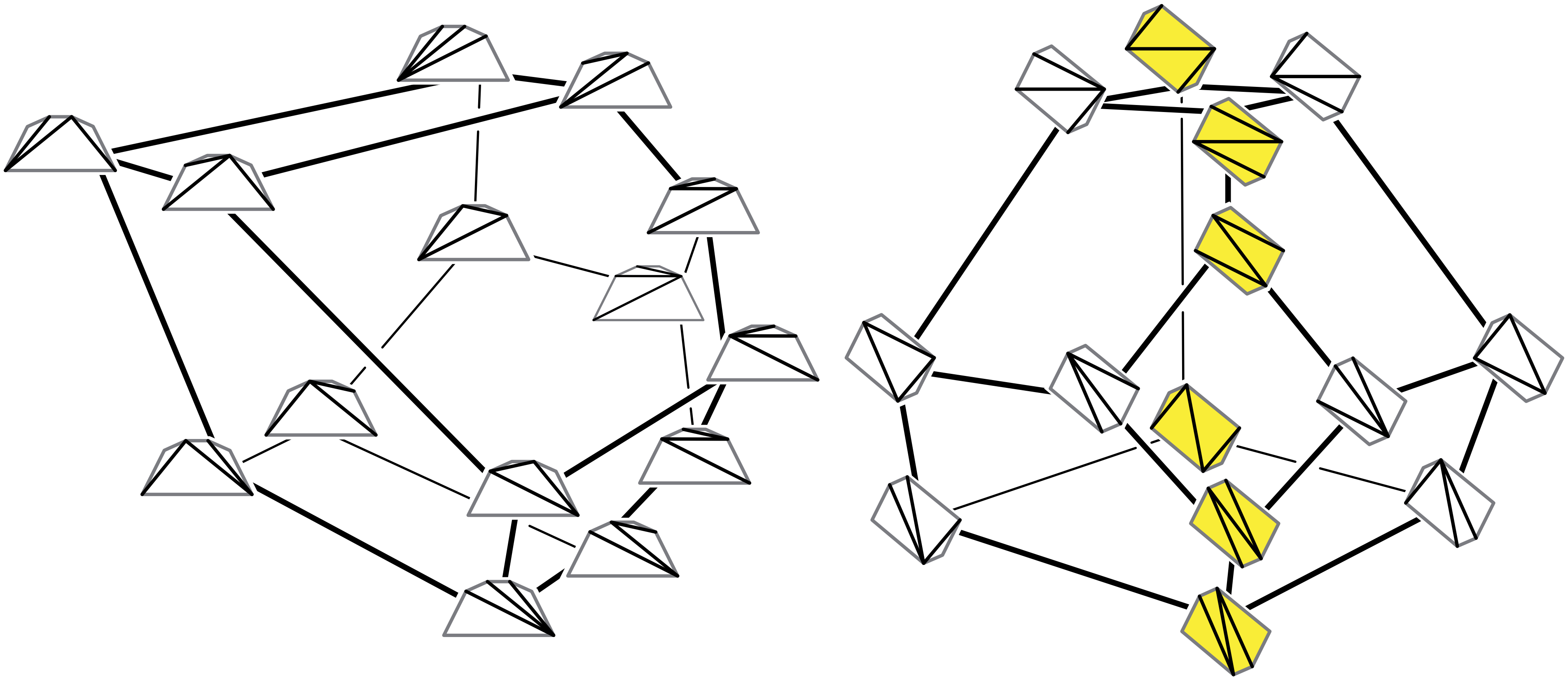


T triangulation of $\mathcal{P}_x$	$\longleftrightarrow$	T^* pseudoline arrangement supported by $\mathcal{N}_x^1$
Δ triangle of T	$\longleftrightarrow$	Δ^* pseudoline of T^*
e common edge of Δ and Δ'	$\longleftrightarrow$	e^* contact between Δ^* and Δ'^*
f common bissector of Δ and Δ'	$\longleftrightarrow$	f^* crossing between Δ^* and Δ'^*

COROLLARY. (i) The graph of flips $G(\mathcal{N}_x^1)$ is (isomorphic to) the graph of flips $G(\mathcal{P}_x)$.
(ii) The contact graph $(T^*)^\#$ is (isomorphic to) the dual binary tree of T.

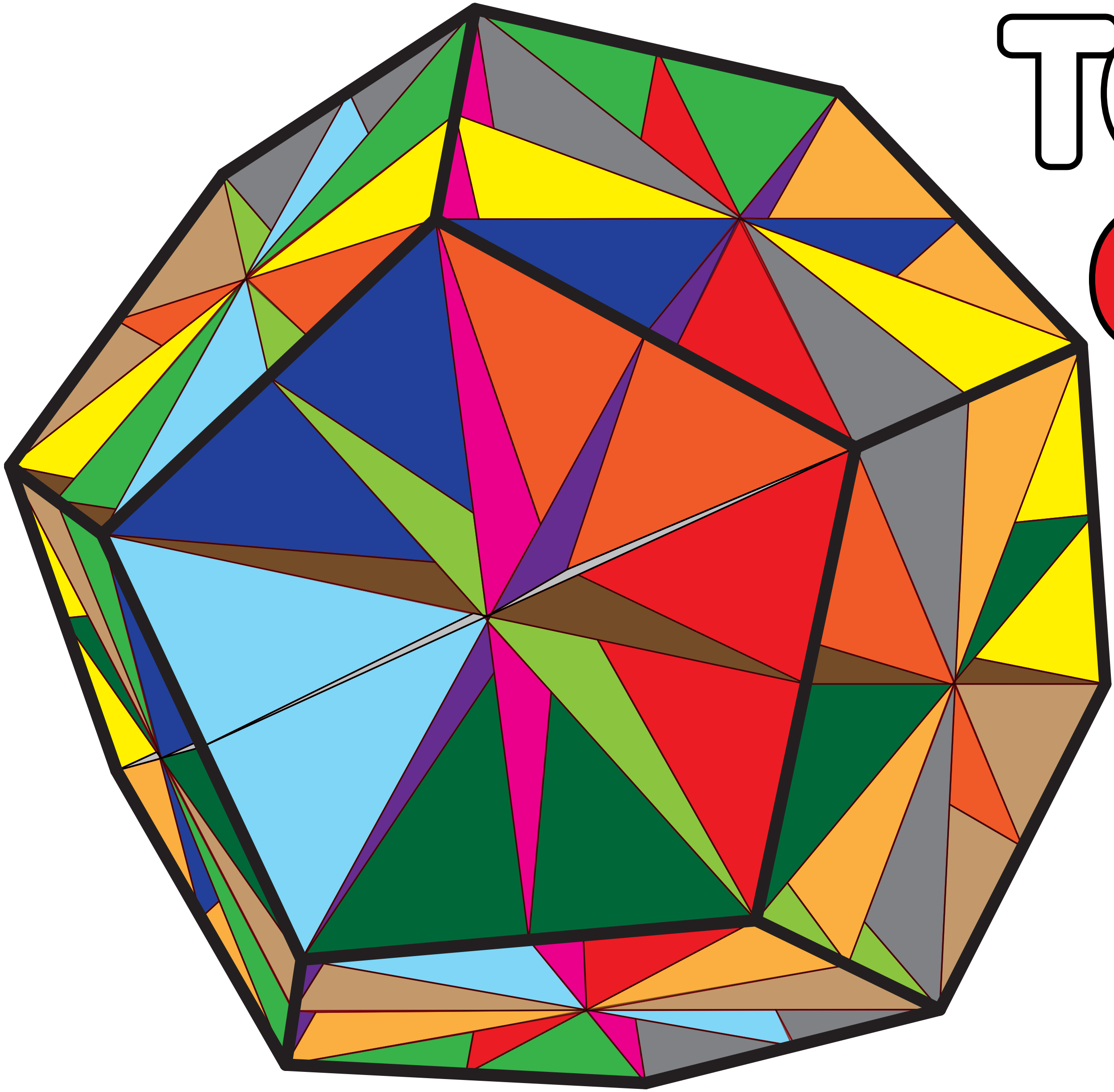
HOHLWEG & LANGE'S ASSOCIAHEDRA

THEOREM. For any word $x \in \{a, b\}^{n-2}$, the simplicial complex of crossing-free sets of internal diagonals of the convex n-gon $\mathcal{P}_x$ is (isomorphic to) the boundary complex of the polar of the brick polytope $\Omega(\mathcal{N}_x^1)$.



REMARK. Up to translation, we obtain Hohlweg & Lange's associahedra.

C. Hohlweg & C. Lange, Realizations of the associahedron and cyclohedron, 2007.



GENERALIZATION TO OTHER COXETER GROUPS

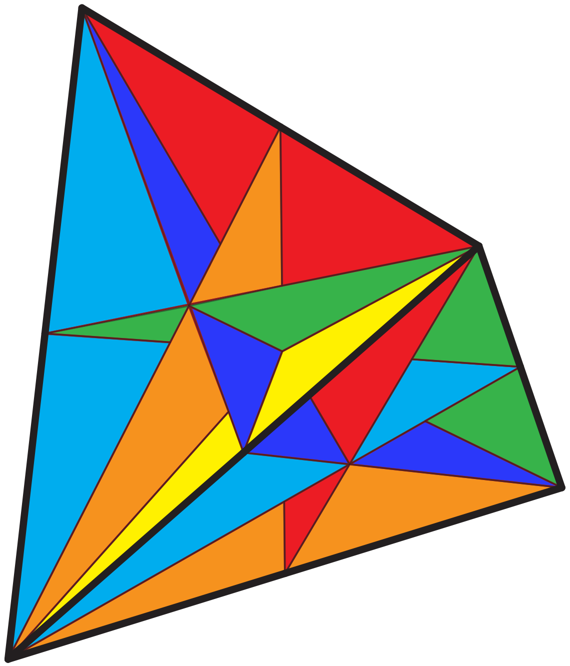
Christian STUMP
Université du Québec à Montréal

GEOMETRY OF COXETER GROUPS

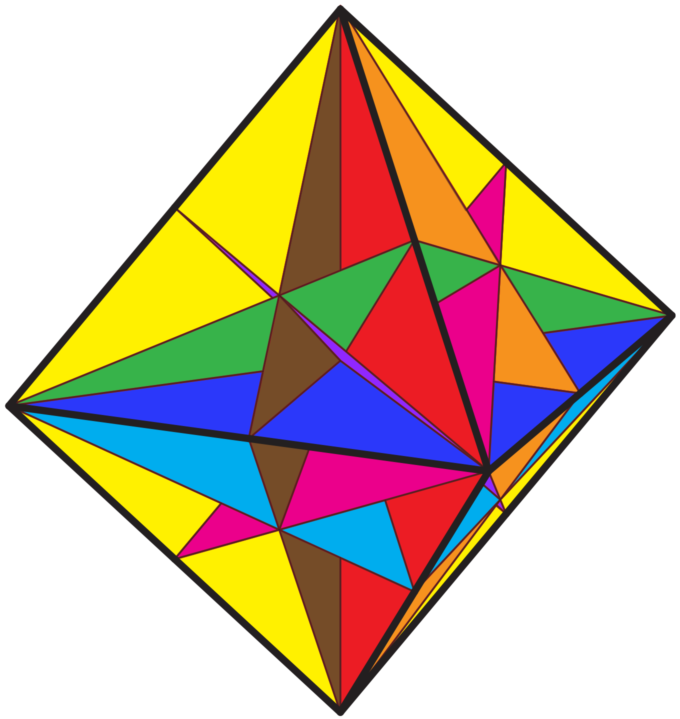
W = finite Coxeter group = finite group generated by orthogonal reflections.

S = simple system of generators = internal normal vectors of the fundamental chamber.

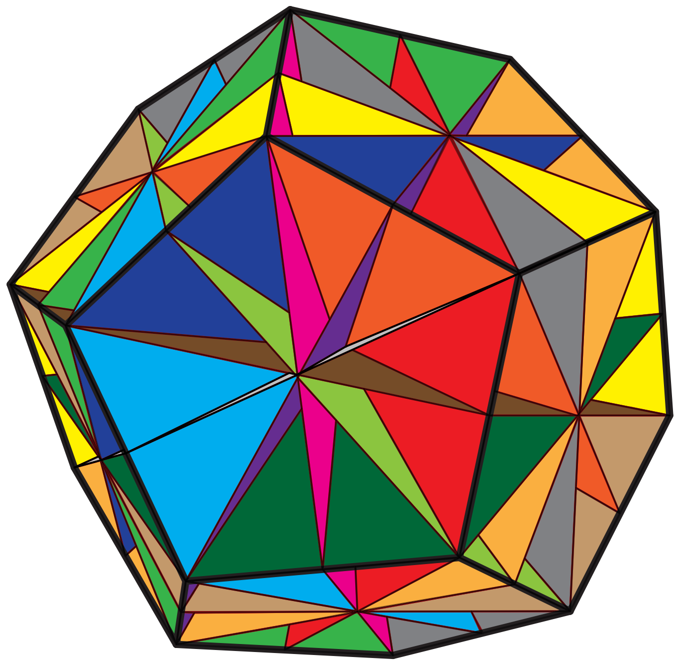
W-Permutahedron = convex hull of the W-orbit of a generic point.



Type $A_3 = \mathfrak{S}_4$



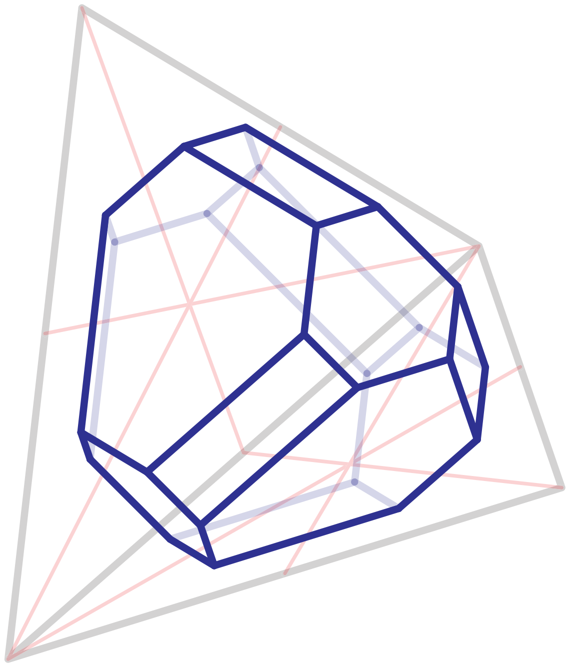
Type $B_3 = \mathfrak{S}_3 \rtimes (\mathbb{Z}_2)^3$



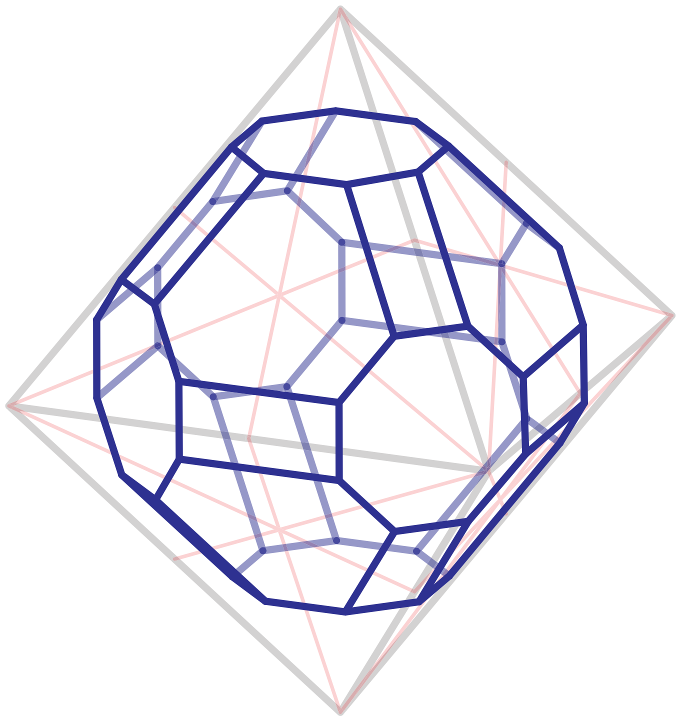
Type $H_3 = \mathfrak{A}_5$

GEOMETRY OF COXETER GROUPS

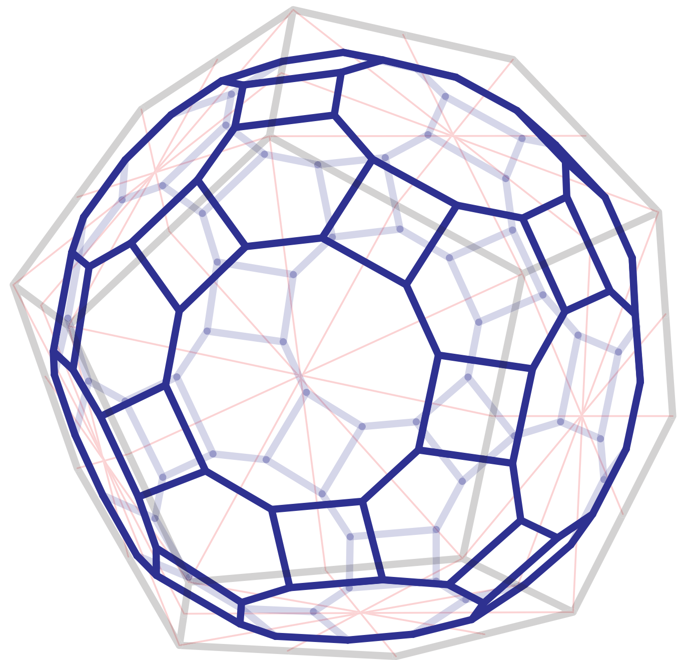
W = finite Coxeter group = finite group generated by orthogonal reflections.
S = simple system of generators = internal normal vectors of the fundamental chamber.
W-Permutahedron = convex hull of the W-orbit of a generic point.



Type $A_3 = \mathfrak{S}_4$



Type $B_3 = \mathfrak{S}_3 \rtimes (\mathbb{Z}_2)^3$



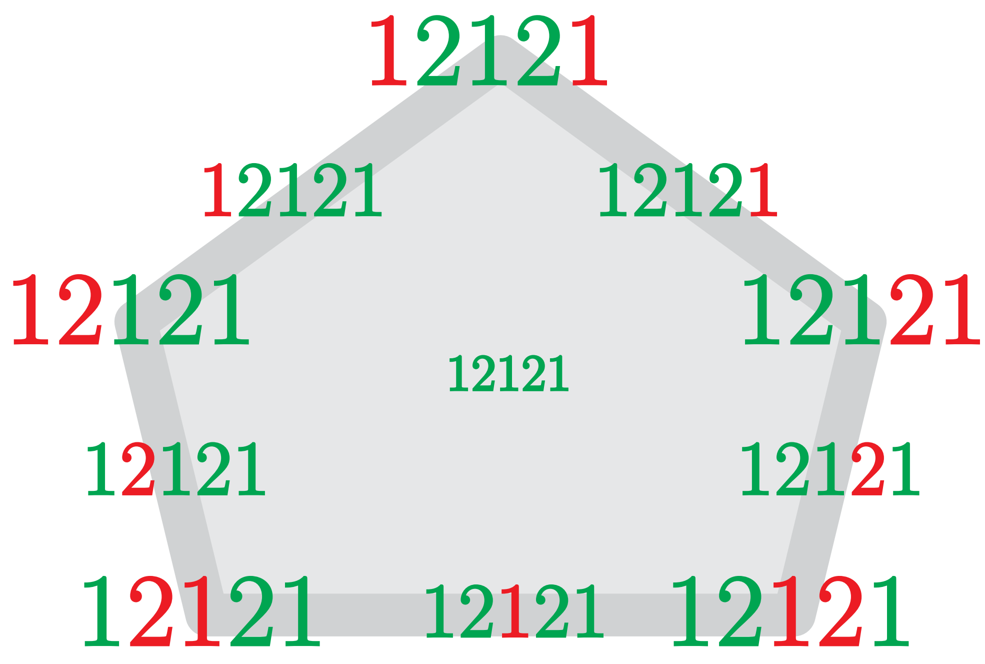
Type $H_3 = \mathfrak{A}_5$

SUBWORD COMPLEX

(W, S) a finite Coxeter system, Q a word on S and $\pi \in W$.
Subword complex $\Delta(Q, \pi)$ = simplicial complex of subsets of positions of Q whose complement contains a reduced expression of π.

A. Knutson & E. Miller, Subword complexes in Coxeter groups, 2004.

$W = A_2$
$Q = 12121$
$\pi = 121 = 212$



$\Delta(Q, \pi)$ spherical if and only if π = Kronecker product $\delta(Q)$. We assume $\pi = \delta(Q) = w_\circ$.

SUBWORD COMPLEX

(W, S) a finite Coxeter system, Q a word on S and $\pi \in W$.
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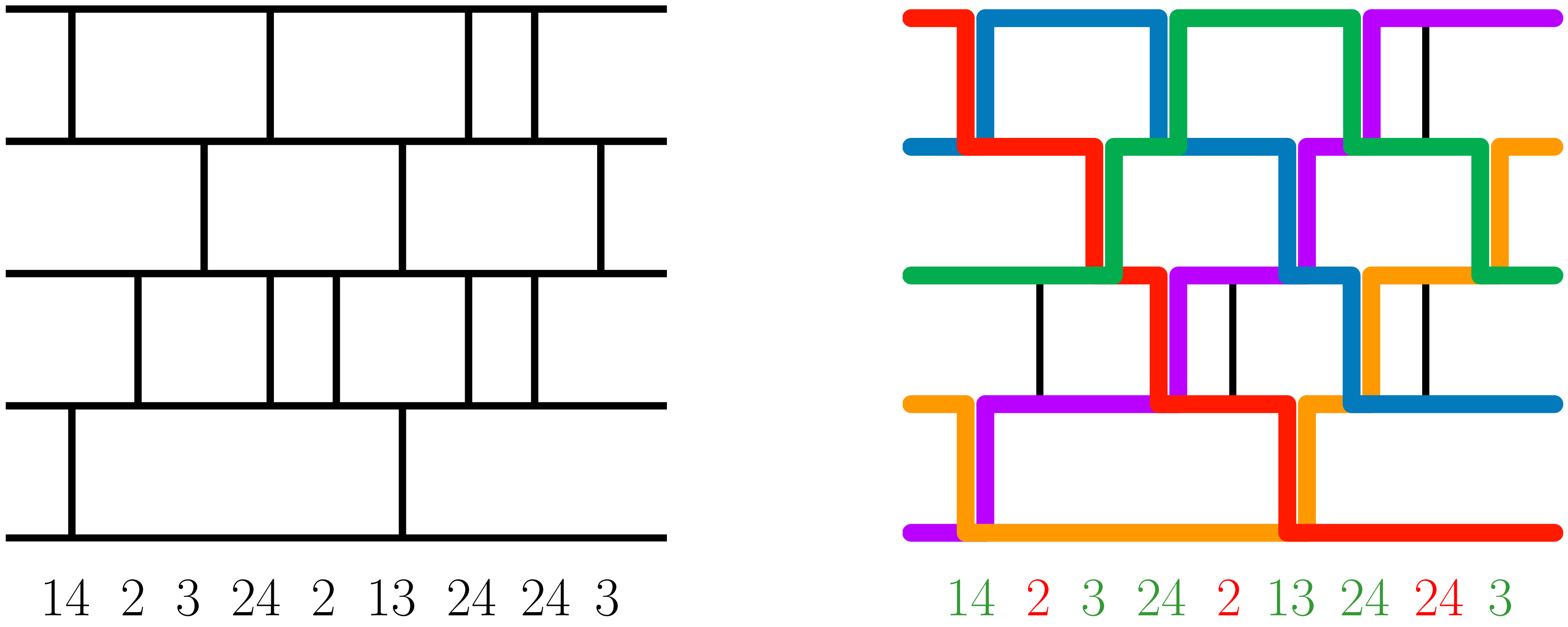

BRICK POLYTOPES OF SUBWORD COMPLEXES

(W, S) a finite Coxeter system, Q a word on S.
Brick polytope $\Omega(Q)$ = convex hull of the brick vectors of all facets of $\Delta(Q, w_\circ)$.

SIMILAR COMBINATORIAL PROPERTIES

GENERALIZATION OF THE CAMBRIAN LATTICES — N. Reading, Cambrian Lattices, 2006.

SIMILAR EXAMPLES:

- W-permutahedron for a duplicated word
- W-associahedra (vertex description, Minkowski decomposition, ...)

F. Chapoton, S. Fomin & A. Zelevinsky, Polytopal realizations of generalized associahedra, 2002.
C. Hohlweg, C. Lange & H. Thomas, Permutahedra and generalized associahedra, 2011.

THANK YOU