Lower bounds for the measurable chromatic number of Euclidean space

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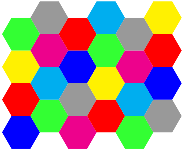
Outline

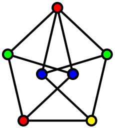
- The chromatic number of Euclidean space
- Frankl and Wilson intersection theorems (1981)
- A theta like bound (F. M. de Oliveira Filho, F. Vallentin 2010)
- Combining the approaches: numerical results in small dimensions (joined work (in progress) with F. Oliveira, F. Vallentin).

$\chi(\mathbb{R}^n)$

- $\chi(\mathbb{R}^n)$ is the smallest number of colors needed to color every point of $\mathbb{R}^n$, such that two points at distance 1 receive different colors. (E. Nelson, 1950, introduced $\chi(\mathbb{R}^2)$)
- Easy: $\chi(\mathbb{R}) = 2$. No other value is known!

- For $n = 2$: $4 \leq \chi(\mathbb{R}^2) \leq 7$ (Nelson, Isbell, 1950).





$\chi(\mathbb{R}^n)$

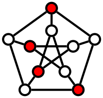
- $\chi(\mathbb{R}^n) \geq \chi(G)$ for all finite graph $G = (V, E)$ embedded in $\mathbb{R}^n$ ($G \hookrightarrow \mathbb{R}^n$) i.e. such that $V \subset \mathbb{R}^n$ and the edges have length 1.
- De Bruijn and Erdös (1951):

$$\chi(\mathbb{R}^n) = \max_{\substack{G \text{ finite} \\ G \hookrightarrow \mathbb{R}^n}} \chi(G)$$

- Good sequences of graphs: Raiski (1970), Larman and Rogers (1972), Frankl and Wilson (1981), Székely and Wormald (1989).
- Erdös conjectured an exponential growth for $\chi(\mathbb{R}^n)$. Frankl and Wilson show exponential growth (1981). Raigorodskii (2000) slight improvement.

How to estimate $\chi(G)$, G finite

- The independence number $\alpha(G)$ of $G = (V, E)$ is the maximum number of pairwise unconnected vertices.



- Every color class of an admissible coloring is an independent set so

$$\chi(G) \geq \frac{|V|}{\alpha(G)}.$$

Example: $G =$  the Moser's spindle, $\alpha = 2$ so $\chi(G) \geq 7/2$.

- How to upper bound $\alpha(G)$: two general methods: Lovász theta number $\vartheta(G)$ and the linear algebra method.

The unit distance graph

- The unit distance graph

$$V = \mathbb{R}^n \qquad E = \{(x,y) : \|x-y\| = 1\}.$$

- Its independent sets are the 1-avoiding sets, i.e. the sets S not containing pairs of elements at distance 1.
- Problem: to have a substitute for $|S|$ and for $\alpha(G)$. Requires S is measurable.
- Next problem: to upper bound it.

$\chi_m(\mathbb{R}^n)$, $m_1(\mathbb{R}^n)$

- The measurable chromatic number $\chi_m(\mathbb{R}^n)$: the color classes are required to be measurable.
- $\chi_m(\mathbb{R}^n) \geq \chi(\mathbb{R}^n)$. Falconer (1981): $\chi_m(\mathbb{R}^n) \geq n+3$. In particular $\chi_m(\mathbb{R}^2) \geq 5$!
- $m_1(\mathbb{R}^n)$ is the supremum of the density of a measurable subset of $\mathbb{R}^n$ containing no pair of points at distance 1:

$$m_1(\mathbb{R}^n) = \sup \left\{ \delta(S) \ : \ S \subset \mathbb{R}^n, \ S \text{ avoids } 1 \right\}$$

where $\delta(S) = \limsup_{r \to +\infty} \operatorname{vol}(S \cap B_n(r)) / \operatorname{vol}(B_n(r))$.

- Larman and Rogers (1972):

$$\chi_m(\mathbb{R}^n) \geq \frac{1}{m_1(\mathbb{R}^n)} \qquad m_1(\mathbb{R}^n) \leq \frac{\alpha(G)}{|V|} \quad \text{for all } G \hookrightarrow \mathbb{R}^n.$$

Frankl and Wilson Intersection Theorem

Theorem (FW, 1981)

$n \geq 1$, $w \leq n/2$, q a prime power such that $w/2 < q < w$.
$S \subset \{0,1\}^{n,w}$ such that for all $(u,v) \in S^2$, $|u \cap v| \neq w - q$. Then

$$|S| \leq \binom{n}{q-1}.$$

- S is an independent set of the graph $G = (V, E)$,

  $$V = \{0,1\}^{n,w}, \qquad E = \{(u,v) : |u \cap v| = w - q\}.$$

  The theorem gives an upper bound for $\alpha(G)$.
- Proof uses the linear algebra method.
- Better than the upper bound given by Lovász ϑ.

Frankl and Wilson Intersection Theorem

Sketch of proof:

- To every $u \in S$ is associated $f_u \in L(\{0,1\}^{n,w})$.

$$f_u(v) = \frac{1}{(q-1)!} \prod_{\ell=1}^{q-1} (|u \cap v| - w + \ell).$$

- These f_u fall into a subspace of dimension $\binom{n}{q-1}$, the subspace of functions of "degree" at most (q-1).
- Moreover they are linearly independent. Indeed, they take integral values, and, if $q = p^k$, p prime,

$$\Big(f_u(v)\Big)_{u,v \in S} = \mathrm{Id} \mod p.$$

Applications to chromatic numbers

- Using $\{0,1\}^{n,w} \subset \mathbb{R}^n$, $\|u-v\|^2 = 2w - 2|u \cap v|$. So avoiding one intersection value means avoiding one distance. So

$$\chi(\mathbb{R}^n) \geq \frac{|V|}{\alpha(G)} = \frac{\binom{n}{w}}{\binom{n}{q-1}}.$$

With $w = \min(2q-1, n/2)$ and $q \approx \alpha n$, optimizing α, leads to (Frankl and Wilson 1981):

$$\chi(\mathbb{R}^n) \gtrapprox (1.207)^n \qquad m_1(\mathbb{R}^n) \lessapprox (0.829)^n$$

- Raigorodskii: using $\{0,1,-1\}^{n,w_1,w_2}$ improves to $(1.239)^n$.

Lovász theta number

- The theta number $\vartheta(G)$ (L. Lovász, 1979) satisfies the Sandwich Theorem:
$$\alpha(G) \le \vartheta(G) \le \chi(\overline{G})$$
- It is the optimal value of a semidefinite program:

$$\vartheta(G) = \max \Big\{ \sum_{(x,y)\in V^2} B(x,y) \quad : \ B \in \mathbb{R}^{V\times V},\ B \succeq 0, \\ \sum_{x\in V} B(x,x) = 1, \\ B(x,y) = 0 \quad xy \in E \Big\}$$

- If S is an independent set of G, $B_S(x,y) := \mathbf{1}_S(x)\mathbf{1}_S(y)/|S|$ satisfies the constraints of the above SDP. Thus $|S| \le \vartheta(G)$.

Semidefinite programs

- Primal program:

$$\gamma := \inf \{ \langle A_0, Z \rangle : \quad Z \succeq 0, \\ \langle A_i, Z \rangle = b_i, \quad i = 1, \ldots, m \}$$

where A_i are real symmetric matrices of some size n.

- Dual program:

$$\gamma^* := \sup \{ \quad b_1 x_1 + \cdots + b_m x_m : \\ A_0 - x_1 A_1 - \cdots - x_m A_m \succeq 0 \}$$

Semidefinite programs

- Linear programs (LP) occur when the matrices A_i are diagonal.
- In general, $\gamma^* \leq \gamma$: if x is dual feasible and Z is primal feasible,

$$\sum_{i=1}^{m} b_i x_i \leq \langle A_0, Z \rangle.$$

- Under some mild conditions, $\gamma = \gamma^*$, i.e. there is no duality gap.
- In this case, interior point methods lead to algorithms that allow to approximate γ to an arbitrary precision in polynomial time. Good free solvers are available (NEOS)!

$\vartheta(\mathbb{R}^n)$

- A generalization of $\vartheta(G)$ to the unit distance graph

$$V = \mathbb{R}^n \qquad E = \{(x, y) : \|x - y\| = 1\}$$

  after *Fourier analysis, linear programming, and densities of distance avoiding sets in R^n*, F. M. de Oliveira Filho, F. Vallentin, JEMS 2010.

- Turns out to be explicitely computable.
- Recall when G finite:

$$\vartheta(G) = \max\Big\{ \sum_{(x,y)\in V^2} B(x,y) \quad : \ B \in \mathbb{R}^{V\times V},\ B \succeq 0, \\ \sum_{x\in V} B(x,x) = 1, \\ B(x,y) = 0 \quad xy \in E \Big\}$$

$\vartheta(\mathbb{R}^n)$

- Over $\mathbb{R}^n$: take $B(x, y)$ continuous, positive definite, i.e. for all k, for all $x_1, \ldots, x_k \in \mathbb{R}^n$, $\big(B(x_i, x_j)\big)_{1 \leq i,j \leq k} \succeq 0$.
- Assume B is translation invariant: $B(x, y) = f(x - y)$ (the graph itself is invariant by translation).
- Replace $\sum_{(x,y) \in V^2} B(x, y)$ by

$$\delta(f) := \limsup_{r \to +\infty} \frac{1}{\mathrm{vol}(B_n(r))} \int_{B_n(r)} f(z)dz.$$

$\vartheta(\mathbb{R}^n)$

- Leads to:

$$\vartheta(\mathbb{R}^n) \quad := \sup\Big\{\delta(f): \quad \begin{array}{l} f \in \mathcal{C}_b(\mathbb{R}^n),\ f \succeq 0 \\ f(0) = 1, \\ f(x) = 0 \quad \|x\| = 1 \end{array}\Big\}$$

Theorem

$$m_1(\mathbb{R}^n) \leq \vartheta(\mathbb{R}^n)$$

- Bochner characterization of positive definite functions:

$$f \in \mathcal{C}(\mathbb{R}^n), f \succeq 0 \Longleftrightarrow f(x) = \int_{\mathbb{R}^n} e^{ix\cdot y} d\mu(y),\ \mu \geq 0.$$

$\vartheta(\mathbb{R}^n)$

- f can be assumed to be radial i.e. invariant under $O(\mathbb{R}^n)$:

$$f(x) = \int_0^{+\infty} \Omega_n(t\|x\|)d\alpha(t),\ \alpha \geq 0.$$

where

$$\Omega_n(t) = \Gamma(n/2)(2/t)^{(n/2-1)} J_{n/2-1}(t).$$

- Leads to:

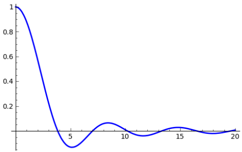
$$\vartheta(\mathbb{R}^n) \quad := \sup \Big\{ \alpha(0) : \quad \alpha \geq 0, \quad \int_0^{+\infty} d\alpha(t) = 1, \quad \int_0^{+\infty} \Omega_n(t) d\alpha(t) = 0 \Big\}$$

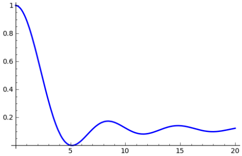
$\vartheta(\mathbb{R}^n)$

- The dual program:

$$\vartheta(\mathbb{R}^n) = \inf\{z_0 : \quad z_0 + z_1 \geq 1, \quad z_0 + z_1\Omega_n(t) \geq 0 \quad \text{for all } t > 0\}$$

- For $n = 4$, graphs of $\Omega_4(t)$ and of the optimal function $f_4^*(t) = z_0^* + z_1^*\Omega_4(t)$:





The minimum of $\Omega_n(t)$ is reached at $j_{n/2,1}$ the first zero of $J_{n/2}$.

$\vartheta(\mathbb{R}^n)$

- We obtain

$$f_n^*(t) = \frac{\Omega_n(t) - \Omega_n(j_{n/2,1})}{1 - \Omega_n(j_{n/2,1})} \qquad \vartheta(\mathbb{R}^n) = \frac{-\Omega_n(j_{n/2,1})}{1 - \Omega_n(j_{n/2,1})}.$$

- The resulting bound

$$m_1(\mathbb{R}^n) \leq \vartheta(\mathbb{R}^n) = \frac{-\Omega_n(j_{n/2,1})}{1 - \Omega_n(j_{n/2,1})}$$

decreases exponentially but not as fast as Frankl Wilson Raigorodskii bound (1.165^{-n} instead of 1.239^{-n}).

$\vartheta_G(\mathbb{R}^n)$

- To summarize, we have seen two essentially different bounds:

$$m_1(\mathbb{R}^n) \leq \begin{cases} \frac{\alpha(G)}{|V|} \text{ with FW graphs and lin. alg. bound for } \alpha(G) \\ \vartheta(\mathbb{R}^n) \text{ encodes } \vartheta(G) \text{ for every } G \hookrightarrow \mathbb{R}^n? \end{cases}$$

- It is possible to combine the two methods.
- $G \hookrightarrow \mathbb{R}^n$, for $x_i \in V$, let $r_i := \|x_i\|$.

$$\vartheta_G(\mathbb{R}^n) := \inf\{z_0 + z_2 \tfrac{\alpha(G)}{|V|} : \quad \begin{array}{l} z_2 \geq 0 \\ z_0 + z_1 + z_2 \geq 1 \\ z_0 + z_1 \Omega_n(t) + z_2 (\frac{1}{|V|} \sum_{i=1}^{|V|} \Omega_n(r_i t)) \geq 0 \\ \text{for all } t > 0\}. \end{array}$$

$\vartheta_G(\mathbb{R}^n)$

Theorem

$$m_1(\mathbb{R}^n) \leq \vartheta_G(\mathbb{R}^n) \leq \vartheta(\mathbb{R}^n)$$

- $\vartheta_G(\mathbb{R}^n) \leq \vartheta(\mathbb{R}^n)$ is obvious: take $z_2 = 0$.
- Sketch proof of $m_1(\mathbb{R}^n) \leq \vartheta_G(\mathbb{R}^n)$: let S a measurable set avoiding 1. Let
$$f_S(x) := \frac{\delta(\mathbf{1}_{S-x}\mathbf{1}_S)}{\delta(S)}.$$
f_S is continuous bounded, $f_S \succeq 0$, $f_S(0) = 1$, $f_S(x) = 0$ if $\|x\| = 1$. Moreover $\delta(f_S) = \delta(S)$.

$\vartheta_G(\mathbb{R}^n)$

- If $V = \{x_1, \dots, x_M\}$, for all $y \in \mathbb{R}^n$,

$$\sum_{i=1}^{M} \mathbf{1}_{S-x_i}(y) \leq \alpha(G).$$

- Leads to the extra condition:

$$\sum_{i=1}^{M} f_S(x_i) \leq \alpha(G).$$

- Design a linear program, apply Bochner theorem, symmetrize by $O(\mathbb{R}^n)$, take the dual.

$\vartheta_G(\mathbb{R}^n)$

- Bad knews: cannot be solved explicitly (we don't know how to)
- Challenge: to compute good feasible functions.
- First method: to sample an interval $[0, M]$, solve a finite LP, then adjust the optimal solution (OV, $G =$ simplex).

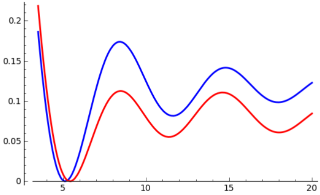


Figure: $f_4^*(t)$ (blue) and $f_{4,G}^*(t)$(red) for $G =$ simplex

$\vartheta_G(\mathbb{R}^n)$

- Observation: the optimal has a zero at $y > j_{n/2,1}$.
- Idea: to parametrize $f = z_0 + z_1\Omega_n(t) + z_2\Omega_n(rt)$ with y: $f(y) = f'(y) = 0$, $f(0) = 1$ determines f.
- We solve for:

$$\begin{cases} z_0 + z_1 + z_2 = 1 \\ z_0 + z_1\Omega_n(y) + z_2\Omega_n(ry) = 0 \\ z_1\Omega_n'(y) + rz_2\Omega_n'(ry) = 0 \end{cases}$$

- Then, starting with $y = j_{n/2,1}$, we move y to the right until $f_y(t) := z_0(y) + z_1(y)\Omega_n(t) + z_2(y)\Omega_n(rt)$ takes negative values.

Numerical improvements

n	LP upper bound for $m_1(\mathbb{R}^n)$ without graphs	LP lower bound for $\chi_m(\mathbb{R}^n)$ without graphs	LP upper bound for $m_1(\mathbb{R}^n)$ with simplex	LP lower bound for $\chi_m(\mathbb{R}^n)$ with simplex	LP upper bound for $m_1(\mathbb{R}^n)$ with FW graph	LP lower bound for $\chi_m(\mathbb{R}^n)$ with FW graph
15	0.00404638	248	0.00359372	279	0.00349172	287
16	0.00314283	319	0.00282332	248	0.00253343	395
17	0.00245212	408	0.00223324	448	0.00188025	532
18	0.00192105	521	0.00177663	563	0.00143383	698
19	0.00151057	663	0.00141992	705	0.00102386	977
20	0.001191806	840	0.00113876	879	0.000729883	1371
21	0.0009432098	1061	0.00091531	1093	0.000524659	1907
22	0.000748582	1336	0.00073636	1359	0.000392892	2546
23	0.000595665	1679	0.00059204	1690	0.000295352	3386
24	0.000475128	2105	0.00047489	2106	0.000225128	4442
25	0.0003798295	2633			0.000173756	5756
26	0.000304278	3287			0.000135634	7373
27	0.000244227	4095			0.000103665	9647
28	0.000196383	5093			0.0000725347	13787
32	0.00008342574	11987			0.00003061037	32669
36	0.000036212868	27615			0.000010504745	95196
44	0.000007168656	139497			0.0000013007413	768793
52	0.0000014908331	670766			0.00000016991978	5885131

Questions, comments

- Exponential behavior of $\vartheta_{FW}(\mathbb{R}^n)$?
- Further improvements for small dimensions: change the graph, consider several graphs. For $n = 2$, several triangles lead to 0.268412 (OV); several Moser spindles to 0.262387 (F. Oliveira 2011).
- Can we reach $m_1(\mathbb{R}^2) < 0.25$? (conjectured by Erdös; would give another proof of $\chi_m(\mathbb{R}^2) \geq 5$).
- Applies to other spaces, expecially to $m(S^{n-1}, \theta)$. The theta approach was developed (B. Nebe Oliveira Vallentin 2009).
- In turn, a bound for $m_1(S(0, r))$ can replace a finite graph G in $\vartheta_G(\mathbb{R}^n)$.
- The Lovász theta method was successfuly adapted to $\mathbb{R}^n$. What about the linear algebra method (Gil Kalai) ?