Spheric analogs of fullerenes

Michel DEZA and Mathieu DUTOUR SIKIRIC

Ecole Normale Superieure, Paris, and Rudjer Boskovic Institute, Zagreb

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Overview

I. 8 families of standard $(\{a, b\}, k)$-spheres

(R, k)-spheres: curvature $C_i = 2k - i(k-2)$ of i-gons

- Fix $R \subset \mathbb{N}$, an (R, k)-sphere is a k-regular, $k \geq 3$, map on $\mathbb{S}^2$ whose faces are i-gons, $i \in R$. Let $m = \min$ and $M = \max_{i \in R}$.
- Let v, e and $f = \sum_i p_i$ be the numbers of vertices, edges and faces of S, where p_i is the number of i-gonal faces. Clearly, $kv = 2e = \sum_i ip_i$ and the *Euler formula* $v - e + f = 2$ become $4k = \sum_i p_i C_i$, where $C_i = 2k - i(k-2)$ is curvature of i-gons.
- i-gon is *elliptic*, *parabolic*, *hyperbolic* if $i < \frac{2k}{k-2}, = \frac{2k}{k-2}, > \frac{2k}{k-2}$, i.e., $C_i > 0, = 0, < 0$, i.e., $\frac{1}{k} + \frac{1}{i} > \frac{1}{2}, = \frac{1}{2}, < \frac{1}{2}$.
- So, $m < \frac{2k}{k-2}$. For $m \geq 3$, it implies $3 \leq m, k \leq 5$, i.e. 5 Platonic pairs of parameters $(m, k) = (3,3), (4,3), (3,4), (5,3), (3,5)$.
- If $M < \frac{2k}{k-2}$ ($\min_{i \in R} C_i > 0$), then $M \leq 5$, $k = 3$ or $M \leq 3$, $k \in \{4, 5\}$ So, for $m \geq 3$, they are only Octahedron, Icosahedron and 11 $(\{3,4,5\}, 3)$-spheres: 8 dual *deltahedra*, Cube and its truncations on 1 or 2 opposite vertices (*Durer octahedron*).

Standard (R, k)-spheres

- An (R, k)-sphere is standard if $M = \frac{2k}{k-2}$, i.e. $\min_{i \in R} C_i = 0$. So, $(M, k) = (6, 3), (4, 4), (3, 6)$ (Euclidean parameter pairs). Exclusion of hyperbolic faces simplifies enumeration, while the number p_M of parabolic faces not being restricted, there is an infinity of such (R, k)-spheres.
- The number of such v-vertex (R, k)-spheres with $|R| = 2$ increases polynomially with v; their set is countable. Such spheres admit parametrization and description in terms of rings of (*Gaussian* if $k = 4$ and *Eisenstein* if $k = 3, 6$) *integers*. All 8 series of such spheres will be considered in detail.
- Remaining (R, k)-spheres (with $M > \frac{2k}{k-2}$) not admit above, in general. The number of such v-vertex $(\{3, 4\}, 5)$-spheres grows at least exponentially with v; their set is a continuum.

8 families of standard $(\{a,b\},k)$-spheres

- An $(\{a,b\},k)$-sphere is an (R,k)-sphere with $R=\{a,b\}$, $1 \le a < b$. It has $v=\frac{1}{k}(ap_a+bp_b)$ vertices.
- Such standard sphere has $b=\frac{2k}{k-2}$; so, $(b,k)=$ $(6,3)$, $(4,4)$, $(3,6)$ and Euler formula become

$$\begin{array}{rcl} 12=\sum_i(6-i)p_i & \text{if} & k=3 \\ 8=\sum_i(4-i)p_i & \text{if} & k=4 \\ 6=\sum_i(3-i)p_i & \text{if} & k=6 \end{array}$$

- Further, $p_a=\frac{2b}{b-a}$ and all possible (a,p_a) are: $(5,12)$, $(4,6)$, $(3,4)$, $(2,3)$ for $(b,k)=(6,3)$; $(3,8)$, $(2,4)$ for $(b,k)=(4,4)$; $(2,6)$, $(1,3)$ for $(b,k)=(3,6)$.
- Those 8 families can be seen as spheric analogs of the regular plane partitions $\{6^3\}$, $\{4^4\}$, $\{3^6\}$ with p_a a-gonal "defects", disclinations added to get the curvature of the sphere $\mathbb{S}^2$.

8 families: existence criterions

Grűnbaum-Motzkin, 1963: criterion for $k=3 \leq a$; Grűnbaum, 1967: for $(\{3,4\},4)$-spheres; Grűnbaum-Zaks, 1974: for other cases.

k	(a,b)	smallest one	it exists if and only if	p_a	v
3	$(5,6)$	Dodecahedron	$p_6 \neq 1$	12	$20+2p_6$
3	$(4,6)$	Cube	$p_6 \neq 1$	6	$8+2p_6$
4	$(3,4)$	Octahedron	$p_4 \neq 1$	8	$6+p_4$
6	$(2,3)$	$6 \times K_2$	p_3 is even	6	$2+\frac{p_3}{2}$
3	$(3,6)$	Tetrahedron	p_6 is even	4	$4+2p_6$
4	$(2,4)$	$4 \times K_2$	p_4 is even	4	$2+p_4$
3	$(2,6)$	$3 \times K_2$	$p_6=(k^2+kl+l^2)-1$	3	$2+2p_6$
6	$(1,3)$	Trifolium	$p_3=2(k^2+kl+l^2)-1$	3	$\frac{1+p_3}{2}$
5	$(3,4)$	Icosahedron	$p_4 \neq 1$	$2p_4+20$	$2p_4+12$

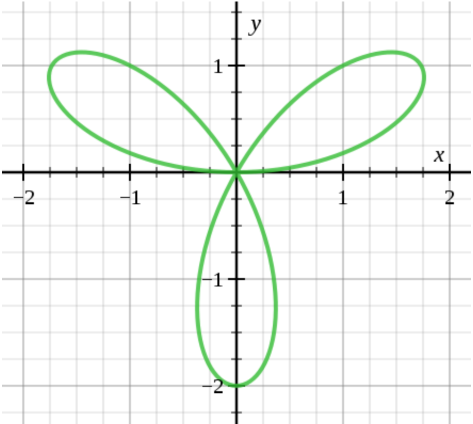
$(\{3,6\},3)$- (Grűnbaum-Motzkin, 1963) and $(\{2,4\},4)$-spheres (Deza-Shtogrin, 2003) admit a simple 2-parametric description.

8 families of standard $(\{a, b\}, k)$-spheres

- Let us denote $(\{a, b\}, k)$-sphere with v vertices by $\{a, b\}_v$.
- $(\{5, 6\}, 3)$- and $(\{4, 6\}, 3)$-spheres are (geometric) fullerenes and *boron nitrides*. $\{5, 6\}_{60}(I_h)$: a new *carbon allotrope* C_{60}. $\{5, 6\}_{620}(I) = GC_{5,1}(\{5, 6\}_{20}) \approx$ *Callaway golf ball* $\{5, 6\}_{660}$.
- $(\{a, b\}, 4)$-spheres are minimal projections of alternating links, whose components are their *central circuits* (those going only ahead) and crossings are the verices.
- By smallest member Dodecahedron $\{5, 6\}_{20}$, Cube $\{4, 6\}_8$, Tetrahedron $\{3, 6\}_4$, Octahedron $\{3, 4\}_6$ and $3 \times K_2$ $\{2, 6\}_2$, $4 \times K_2$ $\{2, 4\}_2$, $6 \times K_2$ $\{2, 3\}_2$, Trifolium $\{1, 3\}_1$, we call eight families: dodecahedrites, cubites, tetrahedrites, octahedrites and 3-bundelites, 4-bundelites, 6-bundelites, trifoliumites.
- b-icosahedrites $((\{3, b\}, 5)$-spheres) are not standard if $b > 3$, $p_b > 0$ since $p_3 = p_b(3b - 10) + 20$ and b-gons are hyperbolic.

Digression on Rose of Three Petals

- The polar equation of the rose (or *rhodonea*) is $r{=}\cos(n\theta)$. $\{1,3\}_1$ models its case $n{=}3$: *quartic* (algebraic of degree 4) plane curve Trifolium $(x^2{+}y^2)^2{=}x(x^2{-}3y^2)$ shown below.
- It models also sextic $(x^2{+}y^2)^3{=}2x(x^2{-}3y^2)$ or $r^3{=}2\cos(3\theta)$: Kiepert curve $d(x,A)d(x,B)d(x,C){=}1$ for reg. triangle ABC



Generation of standard $(\{a,b\},k)$-spheres

- $(\{2,3\},6)$-spheres, except $2\times K_2$ and $2\times K_3$, are the duals of $(\{3,4,5,6\},3)$-spheres with six new vertices put on edge(s). Exp: $(\{5,6\},3)$-spheres with 5-gons organized in six pairs.
- $(\{1,3\},6)$-spheres, except $\{1,3\}_1$ and $\{1,3\}_3$, are as above but with 3 edges changed into 2-gons enclosing one 1-gon.
- $(\{2,6\},3)$-spheres are given by the *Goldberg-Coxeter construction* from Bundle$_3 = 3\times K_2$ $\{2,6\}_2$.
- $(\{1,3\},6)$-spheres come by the *Goldberg-Coxeter construction* (extended below on 6-regular spheres) from Trifolium $\{1,3\}_1$.

Computer generation of the families

Main technique: exhaustive search. Sometimes, speedup by proving that a group of faces cannot be completed to the desired graph.

- The program CPF by Brinkmann-Delgado-Dress-Harmuth, 1997 generates 3-regular plane graphs with specified p-vector.
- ENU by Brinkmann-Harmuth-Heidemeier, 2003 and Heidemeier, 1998 does the same for 4-regular plane graphs. Dutour adapted ENU to deal with 2-gonal faces also.
- CGF by Harmuth generates 3-regular orientable maps with specified genus and p-vector.
- Plantri by Brinkmann-McKay deals with general graphs.
- The package CaGe by Brinkmann-Delgado-Dress-Harmuth, 1997 is used for plane graph drawings.
- The package PlanGraph by Dutour, 2002 is used for handling planar graphs in general.

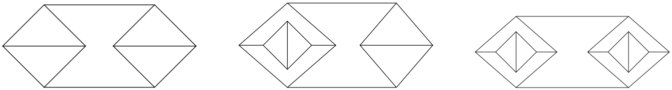
II. Connectedness of $(\{a, b\}, k)$-spheres

Polyhedra and planar graphs

- A graph is called k-connected if after removing any set of $k-1$ vertices it remains connected.
- The skeleton of a polytope P is the graph $G(P)$ formed by its vertices, with two vertices adjacent if they generate a face.
- Steinitz Theorem: a graph is the skeleton of a polyhedron (3-polytope) if and only if it is planar and 3-connected.
- A polyhedron is usually represented by the *Schlegel diagram* of its skeleton, the program used for this is CaGe.
- The dual graph G^* of a plane graph G is the plane graph formed by the faces of G, with two faces adjacent if they share an edge. The skeletons of dual polyhedra are dual.

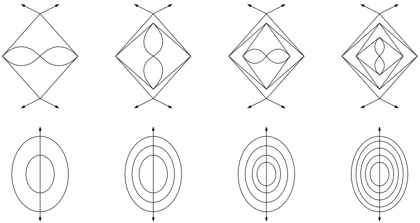
3-connectedness of $(\{a, b\}, 3)$-spheres

- Any $(\{a, b\}, k)$-sphere is 2-connected. But some infinite series of $(\{1, 2, 3\}, 6)$-spheres with $(p_1, p_2){=}(2, 2)$ are *not*.
- Any $(\{a, 6\}, 3)$-sphere is 3-connected if $a = 4, 5$ and not if $a = 2$ (one can delete two vertices adjacent to a 2-gon).
- Except the following series, $(\{3, 6\}, 3)$-spheres (moreover, all $(\{3, 4, 5, 6\}, 3)$-spheres) are 3-connected.



3-connectedness of $(\{a, b\}, 6)$- and $(\{a, b\}, 4)$-spheres

- Any $(\{a, b\}, 6)$-sphere is 3-connected, except $(\{2, 3\}, 6)$- ones which are duals of only 2-connected $(\{3, 6\}, 3)$-spheres, with six vertices of degree 2 added on edges.
- Any $(\{a, b\}, 4)$-sphere is 3-connected, except the following series of $(\{2, 4\}, 4)$-spheres.



REMARK. $\{2, 4\}_v(D_{2d}, D_{2h})$ are *k-inflations* of above. D_4, D_{4h} are $GC_{k,l}(4 \times K_2)$. Remaining D_2: 2 complex or 3 natural parameters.

Hamiltonicity of $(\{a, b\}, k)$-spheres

- Grűnbaum-Zaks, 1974: all $(\{1, 3\}, 6)$- and $(\{2, 4\}, 4)$-spheres are Hamiltonian, but $(\{2, 6\}, 3)$- with $v \equiv 0 \pmod 4$ are not
- Goodey, 1977: $(\{3, 6\}, 3)$- and $(\{4, 6\}, 3)$- are Hamiltonian.
- Conjecture: an Hamiltonian circuit exists in all other cases.

To check hamiltonicity of a $(\{a, b\}, k)$-map on the projective plane $\mathbb{P}^2$, the following theorem (Thomas-Yu, 1994) could help: every 4-connected graph on $\mathbb{P}^2$ has a *contractible* (i.e. being a boundary of 2-cell) Hamiltonian circuit.

II'. $(\{a, b\}, k)$-spheres with small p_b: listings

Listing of $(\{a, b\}, k)$-spheres with $p_b \leq 3 \leq a \leq b$

- Remind: $(a, k)=(3,3), (4,3), (3,4), (5,3)$ or $(3,5)$ if $a \geq 3$.
- The only $(\{a, b\}, k)$-spheres with $p_b \leq 1$ are 5 Platonic (a^k): Tetrahedron, $Prism_4$, $APrism_3$, snub $Prism_5$, snub $APrism_3$.
- There exists unique 3-connected $(\{a, b\}, k)$-sphere with $p_b=2$ for $(\{4, b\}, 3)$-, $(\{3, b\}, 4)$-, $(\{5, b\}, 3)$-, $(\{3, b\}, 5)$-: $Prism_b$ D_{bh}, $APrism_b$ D_{bd}, snub $Prism_b$ or snub $APrism_b$ D_{bd} each (2 b-gons separated by 2 b-rings of 5-gons or 3b-rings of 3-gons). Doubled b-gon D_{bh} is such $(\{2, b\}, 4)$-sphere.
- Also, for any $(a, k)=(3,3), (3,4), (4,3), (3,5), (5,3)$, there is unique only 2-connected such sphere (D_{2h}) iff $b \equiv 0 \,(mod\, a)$.

Listing of $(\{a,b\},k)$-spheres with $p_b \leq 3 \leq a \leq b$

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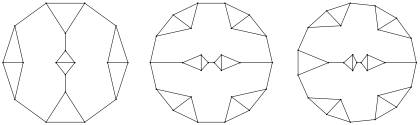
- There exists unique 3-connected $(\{a,b\},k)$-sphere with p_b=2 for $(\{4,b\},3)$-, $(\{3,b\},4)$-, $(\{5,b\},3)$-, $(\{3,b\},5)$-: $Prism_b$ D_{bh}, $APrism_b$ D_{bd}, snub $Prism_b$ or snub $APrism_b$ D_{bd} each (2 b-gons separated by 2 b-rings of 5-gons or $3b$-rings of 3-gons). Doubled b-gon D_{bh} is such $(\{2,b\},4)$-sphere.
- Also, for any $(a,k)=(3,3),(3,4),(4,3),(3,5),(5,3)$, there is unique only 2-connected such sphere (D_{2h}) iff $b \equiv 0 \,(mod\, a)$.
- $(\{a,b\},k)$-sphere with $p_b = 3$ exists if and only if $b \equiv 2, a, 2a-2 \,(mod\, 2a)$ and $b \equiv 4, 6 \,(mod\, 10)$ if a=5.
- Such sphere has symmetry $\neq D_{3h}$ iff $b \equiv a \,(mod\, 2a)$. Such sphere is not unique iff $b \equiv a \,(mod\, 2a)$ and $(a,k) \neq (3,3)$.
- Pictures illustrating all 5 cases with p_b=3 follow; removing central line on them illustrate the cases with p_b=2.

$(\{3, b\}, 3)$-spheres with $p_b = 3$

Such sphere exists iff $b \equiv 2, 3, 4 \,(mod\, 6)$. For b=4+6m, 2+6m, 3+6m, it come from $Prism_3$, $3K_2$, Tetrahedron K_4 by adding m K_4-e's on 3 edges creating symmetry D_{3h}, D_{3h} and resp. C_{3v}. It is 3-connected only for b=4: $Prism_3$.

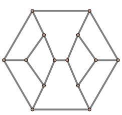


b=4+6   2+12   3+12, C_{3v}

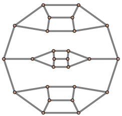
Removing central line gives $(\{3, b{=}3m\}, 3)$-spheres with p_b=2.

$(\{4, b\}, 3)$-spheres with $p_b = 3$

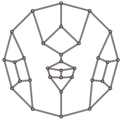
Such sphere exists iff $b \equiv 2, 4, 6\,(mod\,8)$. For b=6+8m, 2+8m, 4+8m, it come from 4-triakon $Prism_3$, $3K_2$, Cube K_2^3 (two) by adding m K_2^3-e's on 3 edges creating D_{3h}, D_{3h} and resp. C_{3v}, C_{2v}. It is 3-connected only for b=6: 4-triakon $Prism_3$ below.



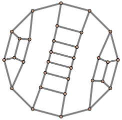
b=6



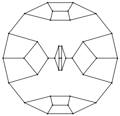
2+8



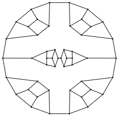
4+8, C_{3v}



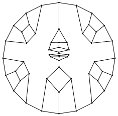
4+8, C_{2v}



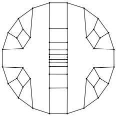
6+8



2+16



20, C_{3v}

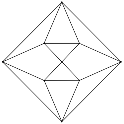


20, C_{2v}

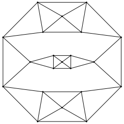
$(\{3, b\}, 4)$-spheres with $p_b = 3$

Such sphere exists iff $b \equiv 2, 3, 4\ (mod\, 6)$. For b=4+6m, 2+6m, 3+6m, it come from 9-vertex $(\{3, b\}, 4)$-sphere, $3K_2$, Octahedron $K_{2,2,2}$ (three) by adding m vertex-split $K_{2,2,2}$'s on 3 edges creating symmetry D_{3h}, D_{3h} and resp. C_{3v}, C_s, C_s.

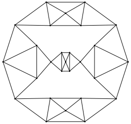
It is 3-connected iff the symmetry is not C_s.



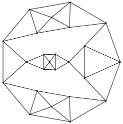
b=4



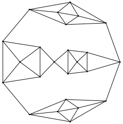
2+6



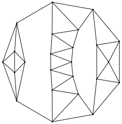
4+6



3+6, C_{3v}



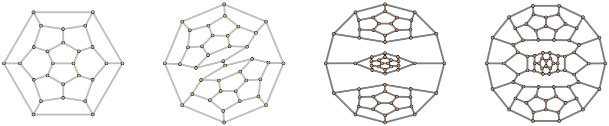
3+6, C_s



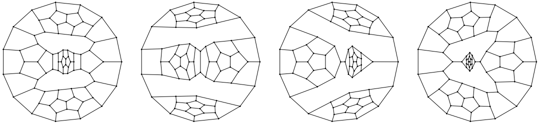
3+6, C_s

$(\{5, b\}, 3)$-spheres with $p_b = 3$

Such sphere exists iff $b \equiv 2, 4, 5, 6, 8 \, (mod \, 10)$. For b=4+10m, 6+10m, 8+10m 2+10m, 5+10m, it come from $14, 26, 38$-vertex $(\{5, b\}, 3)$-spheres with b=4, 6, 8, $3K_2$ and Dodecahedron (five) by adding m $(5, 3)$-polycycles C_1 on 3 edges creating symmetry D_{3h}, D_{3h}, D_{3h}, D_{3h} and resp. two C_{3v}, three C_s.



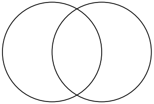
b=6 8 2+10 4+10



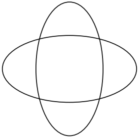
5+10, C_{3v} 15, C_s 15, C_{3v} 15, C_s

III. 8 standard families: 4 smallest members

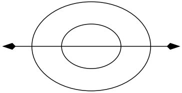
First four $(\{2,4\},4)$- and $(\{3,4\},4)$-spheres



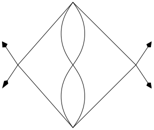
D_{4h} 2^2_1 (2^2)



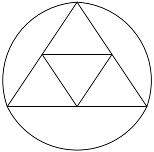
D_{4h} 4^2_1 (4^2)



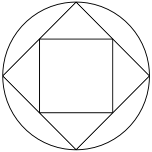
D_{2h} 2×2^2_1 $(2^2, 4)$



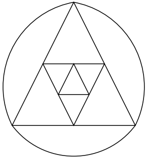
D_{2d} 6^2_2 (6^2)



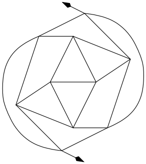
O_h 6^3_2 (4^3)
Borr. rings



D_{4d} 8_{18} (16)



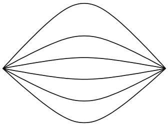
D_{3h} 9_{40} (18)
(Herschel)*



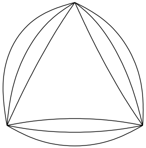
D_2 10^2_{56}
$(6; 14)$

Above links/knots are given in Rolfsen, 1976 and 1990 notation.
Herschel graph: the smallest non-Hamiltonian polyhedral graph.

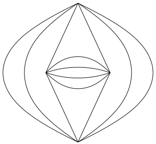
First four $(\{2,3\},6)$- and $(\{1,3\},6)$-spheres



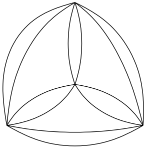
D_{6h} (2^3)



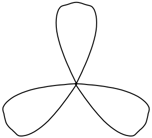
D_{3h} $(3;6)$



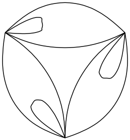
D_{2d} $(2^2;8)$



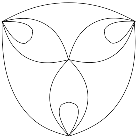
T_d (3^4)



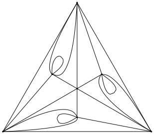
C_{3v} (3)



C_{3h} $(3;6)$



C_{3v} (6^2)

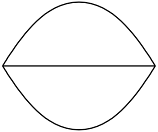


C_3 (21)

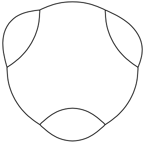
Grűnbaum-Zaks, 1974: $\{1,3\}_v$ exists iff $v = k^2 + kl + l^2$ for integers $0 \leq l \leq k$. We show that the number of $\{1,3\}_v$'s is the number of such representations of v, i.e. found $GC_{k,l}(\{1,3\}_1)$.

First four ({2, 6}, 3)- and ({3, 6}, 3)-spheres

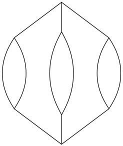
Number of $(\{2,6\}_v$'s is nr. of representations $v{=}2(k^2+kl+l^2)$, $0 \le l \le k$ $(GC_{k,l}(\{2,6\}_2))$. It become 2 for $v{=}7^2{=}5^2{+}15{+}3^2$.



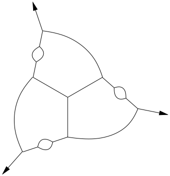
D_{3h} (6)



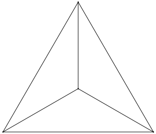
D_{3h} (6^3)



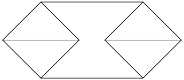
D_{3h} (12^2)



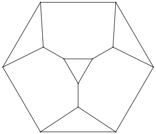
D_3 (42)



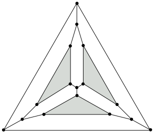
T_d (4^3)



D_{2h} $(8^2, 4^2)$

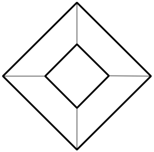


T_d (12^3)

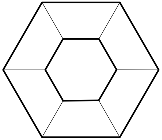


T_d (8^6)

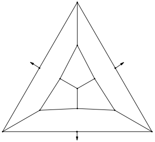
First four $(\{4, 6\}, 3)$- and $(\{5, 6\}, 3)$-spheres



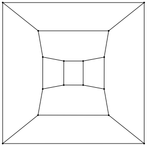
O_h (6^4)



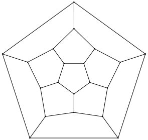
D_{6h} (18^2)



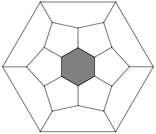
D_{3h} $(6^2; 30)$



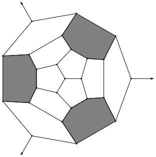
D_{2d} (24^2)



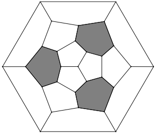
I_h (10^6)



D_{6d} $(12; 60)$



D_{3h} $(12^3; 42)$



T_d (12^7)

IV. Symmetry groups of $(\{a,b\},k)$-spheres

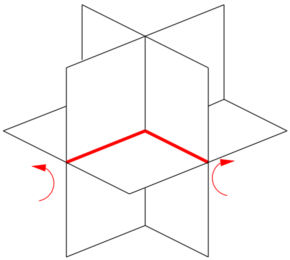
Finite isometry groups

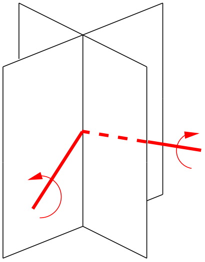
All finite groups of isometries of 3-space $\mathbb{E}^3$ are classified.
In Schoenflies notations, they are:

- C_1 is the trivial group
- C_s is the group generated by a plane reflexion
- $C_i = \{I_3, -I_3\}$ is the inversion group
- C_m is the group generated by a rotation of order m of axis Δ
- C_{mv} ($\simeq$ dihedral group) is the group generated by C_m and m reflexion containing Δ
- $C_{mh} = C_m \times C_s$ is the group generated by C_m and the symmetry by the plane orthogonal to Δ
- S_{2m} is the group of order $2m$ generated by an antirotation, i.e. commuting composition of a rotation and a plane symmetry

Finite isometry groups D_m, D_{mh}, D_{md}

- D_m ($\simeq$ dihedral group) is the group generated of C_m and m rotations of order 2 with axis orthogonal to Δ
- D_{mh} is the group generated by D_m and a plane symmetry orthogonal to Δ
- D_{md} is the group generated by D_m and m symmetry planes containing Δ and which does not contain axis of order 2





Remaining 7 finite isometry groups

- $I_h = H_3$ is the group of isometries of Dodecahedron;
  $I_h \simeq Alt_5 \times C_2$
- $I \simeq Alt_5$ is the group of rotations of Dodecahedron
- $O_h = B_3$ is the group of isometries of Cube
- $O \simeq Sym(4)$ is the group of rotations of Cube
- $T_d = A_3 \simeq Sym(4)$ is the group of isometries of Tetrahedron
- $T \simeq Alt(4)$ is the group of rotations of Tetrahedron
- $T_h = T \cup -T$

While (point group) $Isom(P) \subset Aut(G(P))$ (combinatorial group), Mani, 1971: for any 3-polytope P, there is a map-isomorphic 3-polytope P' (so, with the same skeleton $G(P') = G(P)$), such that the group $Isom(P')$ of its isometries is isomorphic to $Aut(G)$.

8 families: symmetry groups

- 28 for $\{5, 6\}_v$: C_1, C_s, C_i; C_2, C_{2v}, C_{2h}, S_4; C_3, C_{3v}, C_{3h}, S_6; D_2, D_{2h}, D_{2d}; D_3, D_{3h}, D_{3d}; D_5, D_{5h}, D_{5d}; D_6, D_{6h}, D_{6d}; T, T_d, T_h; I, I_h (Fowler-Manolopoulos, 1995)
- 16 for $\{4, 6\}_v$: C_1, C_s, C_i; C_2, C_{2v}, C_{2h}; D_2, D_{2h}, D_{2d}; D_3, D_{3h}, D_{3d}; D_6, D_{6h}; O, O_h (Deza-Dutour, 2005)
- 5 for $\{3, 6\}_v$: D_2, D_{2h}, D_{2d}; T, T_d (Fowler-Cremona,1997)
- 2 for $\{2, 6\}_v$: D_3, D_{3h} (Grűnbaum-Zaks, 1974)
- 18 for $\{3, 4\}_v$: C_1, C_s, C_i; C_2, C_{2v}, C_{2h}, S_4; D_2, D_{2h}, D_{2d}; D_3, D_{3h}, D_{3d}; D_4, D_{4h}, D_{4d}; O, O_h (Deza-Dutour-Shtogrin, 2003)
- 5 for $\{2, 4\}_v$: D_2, D_{2h}, D_{2d}; D_4, D_{4h}, all in $[D_2, D_{4h}]$ (same)
- 3 for $\{1, 3\}_v$: C_3, C_{3v}, C_{3h} (Deza-Dutour, 2010)
- 22 for $\{2, 3\}_v$: C_1, C_s, C_i; C_2, C_{2v}, C_{2h}, S_4; C_3, C_{3v}, C_{3h}, S_6; D_2, D_{2h}, D_{2d}; D_3, D_{3h}, D_{3d}; D_6, D_{6h}; T, T_d, T_h (same)

1. 38 for icosahedrites $(\{3, 4\}, 5)$- (same, 2011).

8 families: Goldberg-Coxeter construction $GC_{k,l}(.)$

With $\mathbf{T}=\{T, T_d, T_h\}$, $\mathbf{O}=\{O, O_h\}$, $\mathbf{I}=\{I, I_h\}$, $\mathbf{C_1}=\{C_1, C_s, C_i\}$, $\mathbf{C_m}=\{C_m, C_{mv}, C_{mh}, S_{2m}\}$, $\mathbf{D_m}=\{D_m, D_{mh}, D_{md}\}$, we get

- for $(\{5,6\},3)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_5}$, $\mathbf{D_6}$, $\mathbf{T}$, $\mathbf{I}$
- for $(\{2,3\},6)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\{D_6, D_{6h}\}$, $\mathbf{T}$
- for $(\{4,6\},3)$-: $\mathbf{C_1}$, $\mathbf{C_2}\backslash S_4$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\{D_6, D_{6h}\}$, $\mathbf{O}$
- for $(\{3,4\},4)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_4}$, $\mathbf{O}$
- for $(\{3,6\},3$-: $\mathbf{D_2}$, $\{T, T_d\}$ $\{D_3, D_{3h}\}$
- for $(\{2,4\},4)$-: $\mathbf{D_2}$, $\{D_4, D_{4h}\}$
- for $(\{2,6\},3)$-: $\{D_3, D_{3h}\}$
- for $(\{1,3\},6)$-: $\mathbf{C_3}\backslash S_6=\{C_3, C_{3v}, C_{3h}\}$

1. if $(\{3,4\},5)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{C_4}$, $\mathbf{C_5}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_4}$, $\mathbf{D_5}$, $\mathbf{T}$, $\mathbf{O}$, $\mathbf{I}$.

8 families: Goldberg-Coxeter construction $GC_{k,l}(.)$

With **T**=$\{T, T_d, T_h\}$, **O**=$\{O, O_h\}$, **I**=$\{I, I_h\}$, $\mathbf{C_1}$=$\{C_1, C_s, C_i\}$, $\mathbf{C_m}$=$\{C_m, C_{mv}, C_{mh}, S_{2m}\}$, $\mathbf{D_m}$=$\{D_m, D_{mh}, D_{md}\}$, we get

- for $(\{5, 6\}, 3)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_5}$, $\mathbf{D_6}$, **T**, **I**
- for $(\{2, 3\}, 6)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\{D_6, D_{6h}\}$, **T**
- for $(\{4, 6\}, 3)$-: $\mathbf{C_1}$, $\mathbf{C_2}\backslash S_4$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\{D_6, D_{6h}\}$, **O**
- for $(\{3, 4\}, 4)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_4}$, **O**
- for $(\{3, 6\}, 3$-: $\mathbf{D_2}$, $\{T, T_d\}$ $\{D_3, D_{3h}\}$
- for $(\{2, 4\}, 4)$-: $\mathbf{D_2}$, $\{D_4, D_{4h}\}$
- for $(\{2, 6\}, 3)$-: $\{D_3, D_{3h}\}$
- for $(\{1, 3\}, 6)$-: $\mathbf{C_3}\backslash S_6$=$\{C_3, C_{3v}, C_{3h}\}$

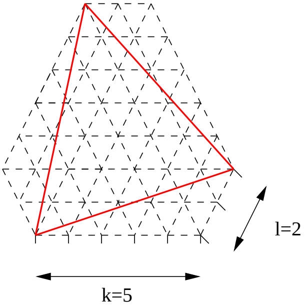
1. if $(\{3, 4\}, 5)$-: $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{C_4}$, $\mathbf{C_5}$, $\mathbf{D_2}$, $\mathbf{D_3}$, $\mathbf{D_4}$, $\mathbf{D_5}$, **T**, **O**, **I**.

Spheres of blue symmetry are $GC_{k,l}$ from 1st such; so, given by one complex (Gaussian for k=4, Eisenstein for k=3, 6) parameter. Goldberg, 1937 and Coxeter, 1971: $\{5, 6\}_v(I, I_h)$, $\{4, 6\}_v(O, O_h)$, $\{3, 6\}_v(T, T_d)$. Dutour-Deza, 2004 and 2010: for other cases.

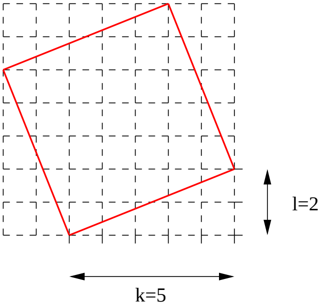
V. Goldberg-Coxeter construction

Goldberg-Coxeter construction $GC_{k,l}(.)$

- Take a 3- or 4-regular plane graph G. The faces of dual graph G^* are triangles or squares, respectively.
- Break each face into pieces according to parameter (k, l). *Master polygons* below have area $\mathcal{A}(k^2+kl+l^2)$ or $\mathcal{A}(k^2+l^2)$, where $\mathcal{A}$ is the area of a small polygon.



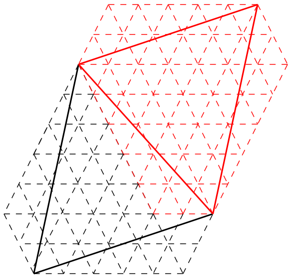
3−valent case



4−valent case

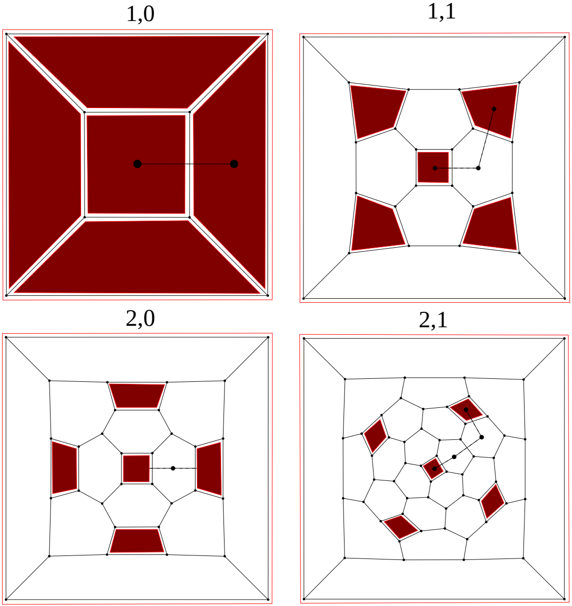
Gluing the pieces together in a coherent way

- Gluing the pieces so that, say, 2 non-triangles, coming from subdivision of neighboring triangles, form a small triangle, we obtain another triangulation or quadrangulation of the plane.



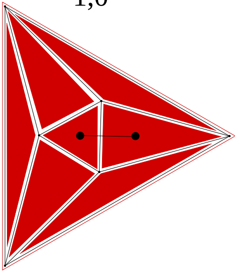
- The dual is a 3- or 4-regular plane graph, denoted $GC_{k,l}(G)$; we call it Goldberg-Coxeter construction.
- It works for any 3- or 4-regular map on oriented surface.

$GC_{k,l}(Cube)$ for $(k, l) = (1, 0), (1, 1), (2, 0), (2, 1)$

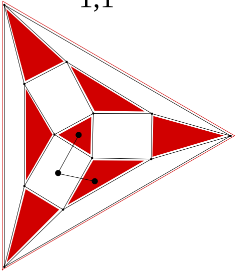


Goldberg-Coxeter construction from Octahedron

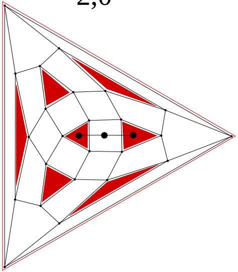
1,0



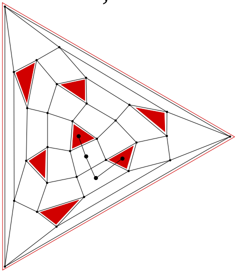
1,1



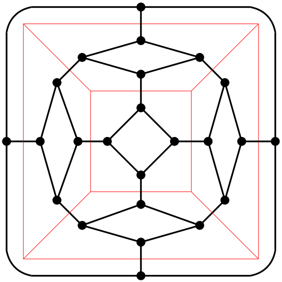
2,0



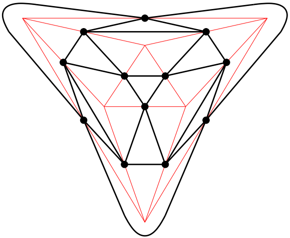
2,1



The case $(k, l) = (1, 1)$



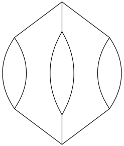
3-regular case
$GC_{1,1}$ is called leapfrog
($\frac{1}{3}$-truncation of the dual)
truncated Octahedron



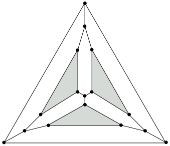
4-regular case
$GC_{1,1}$ is called medial
($\frac{1}{2}$-truncation)
Cuboctahedron

The case $(k,l)=(k,0)$ of $GC_{k,l}(G)$: k-inflation

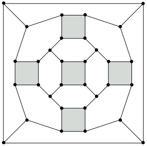
Chamfering (*quadrupling*) $GC_{2,0}(G)$ of 8 1st $(\{a,b\},k)$-spheres, (a,b)=$(2,6),(3,6),(4,6),(5,6)$ and $(2,4),(3,4),(1,3),(2,3)$, are:



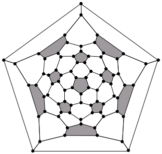
D_{3h} (12^2)



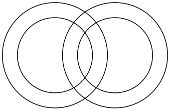
T_d (8^6)



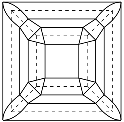
O_h (12^8)



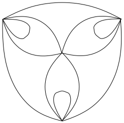
I_h (20^{12})



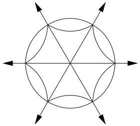
D_{4h} (4^4)



O_h (8^6)



C_{3v} (6^2)

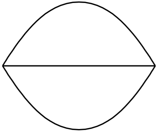


D_{6h} $(4^3,6^2)$

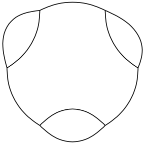
For 4-regular G, $GC_{2k^2,0}(G)=GC_{k,k}(GC_{k,k}(G))$ by $(k+ki)^2=2k^2i$.

First four $GC_{k,l}(3 \times K_2)$ and $GC_{k,l}(4 \times K_2)$

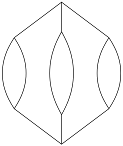
All $(\{2, 6\}, 3)$-spheres are $G_{k,l}(3 \times K_2)$: D_{3h}, D_{3h}, D_3 if $l=0$, k, else.



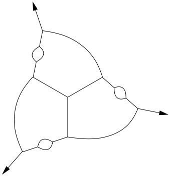
D_{3h} $3 \times K_2$



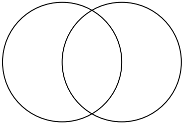
D_{3h} leapfrog



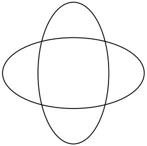
D_{3h} $G_{2,0}$



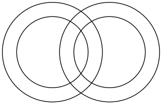
D_3 $G_{2,1}$



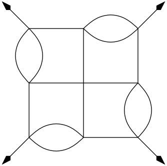
D_{4h} $4 \times K_2$



D_{4h} medial

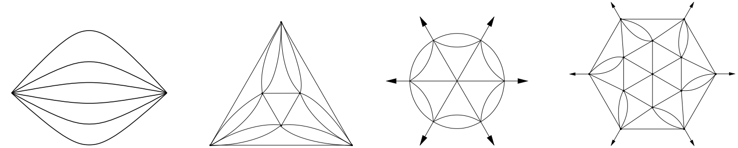


D_{4h} $G_{2,0}$

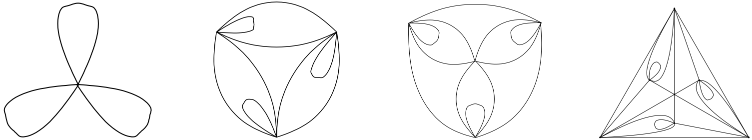


D_4 $G_{2,1}$

First four $GC_{k,l}(6\times K_2)$ and $GC_{k,l}(Trifolium)$



D_{6h} | D_{3d} $G_{1,1}$ | D_{6h} $G_{2,0}$ | D_6 $G_{2,1}$



C_{3v} | C_{3h} $G_{1,1}$ | C_{3v} $G_{2,0}$ | C_3 $G_{2,1}$

All $(\{2,3\},6)$-spheres are $G_{k,l}(6\times K_2)$: C_{3v}, C_{3h}, C_3 if $l{=}0, k$, else.

Plane tilings $\{4^4\}$, $\{3^6\}$ and complex rings $\mathbb{Z}[i]$, $\mathbb{Z}[w]$

- The vertices of regular plane tilings $\{4^4\}$ and $\{3^6\}$ form each, convenient algebraic structures: lattice and ring. Path-metrics of those graphs are l_1- 4-*metric* and *hexagonal* 6-*metric*.
- $\{4^4\}$: square lattice $\mathbb{Z}^2$ and ring $\mathbb{Z}[i]=\{z=k+li : k, l \in \mathbb{Z}\}$ of Gaussian integers with norm $N(z)=z\overline{z}=k^2+l^2=||(k, l)||^2$.
- $\{3^6\}$: hexagonal lattice $A^2=\{x \in \mathbb{Z}^3 : x_0+x_1+x_2=0\}$ and ring $\mathbb{Z}[w]=\{z=k+lw : k, l \in \mathbb{Z}\}$, where $w=e^{i\frac{\pi}{3}}=\frac{1}{2}(1+i\sqrt{3})$, of Eisenstein integers with norm $N(z)=z\overline{z}=k^2+kl+l^2=\frac{1}{2}||x||^2$ We identify points $x=(x_0, x_1, x_2) \in A^2$ with $x_0+x_1 w \in \mathbb{Z}[w]$.

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- A natural number $n=\prod_i p_i^{\alpha_i}$ is of form $n=k^2+l^2$ if and only if any α_i is even, whenever $p_i \equiv 3(mod\, 4)$ (*Fermat Theorem*). It is of form $n=k^2+kl+l^2$ if and only if $p_i \equiv 2 \pmod 3$.
- The first cases of non-unicity with $gcd(k,l)=gcd(k_1,l_1)=1$ are $91=9^2+9+1^2=6^2+30+5^2$ and $65=8^2+1^2=7^2+4^2$. The first cases with $l=0$ are $7^2=5^2+15+3^2$ and $5^2=4^2+3^2$.

The bilattice of vertices of hexagonal plane tiling $\{6^3\}$

- We identify the *hexagonal lattice* A^2 (or *equilateral triangular lattice* of the vertices of the *regular plane tiling* $\{3^6\}$) with *Eisenstein ring* (of Eisenstein integers) $\mathbb{Z}[w]$.
- The hexagon centers of $\{6^3\}$ form $\{3^6\}$. Also, with vertices of $\{6^3\}$, they form $\{3^6\}$, rotated by 90° and scaled by $\frac{1}{3}\sqrt{3}$.
- The complex coordinates of vertices of $\{6^3\}$ are given by vectors $v_1{=}1$ and $v_2{=}w$. The lattice $L{=}\mathbb{Z}v_1{+}\mathbb{Z}v_2$ is $\mathbb{Z}[w]$.
- The vertices of $\{6^3\}$ form bilattice $L_1 \cup L_2$, where the bipartite complements, $L_1{=}(1{+}w)L$ and $L_2{=}1{+}(1{+}w)L$, are stable under multiplication. Using this,

$GC_{k,l}(G)$ for 6-regular graph G can be defined similarly to 3- and 4-regular case, but only for $k + lw \in L_2$, i.e. $k \equiv l \pm 1\,(mod\,3)$.

Ring formalism

$\mathbb{Z}[i]$ (Gaussian integers) and $\mathbb{Z}[\omega]$ (Eisenstein integers) are *unique factorization* rings

Dictionary

	3-regular G	4-regular G	6-regular G
the ring	Eisenstein $\mathbb{Z}[\omega]$	Gaussian $\mathbb{Z}[i]$	Eisenstein $\mathbb{Z}[\omega]$
Euler formula	$\sum_i (6-i)p_i{=}12$	$\sum_i (4-i)p_i{=}8$	$\sum_i (3-i)p_i{=}6$
curvature 0	hexagons	squares	triangles
ZC-circuits	zigzags	central circuits	both
$GC_{11}(G)$	leapfrog graph	medial graph	or. tripling

Goldberg-Coxeter operation in ring terms

- Associate $z{=}k{+}lw$ (Eisenstein) or $z{=}k{+}li$ (Gaussian integer) to the pair (k,l) in 3-,6- or 4-regular case. Operation $GC_z(G)$ correspond to scalar multiplication by $z{=}k{+}lw$ or $k{+}li$.
- Writing $GC_z(G)$, instead of $GC_{k,l}(G)$, one has:

$$GC_z(GC_{z'}(G)) \quad = \quad GC_{zz'}(G)$$

- If G has v vertices, then $GC_{k,l}(G)$ has $vN(z)$ vertices, i.e., $v(k^2{+}l^2)$ in 4-regular and $v(k^2{+}kl{+}l^2)$ in 3- or 6-reg. case.

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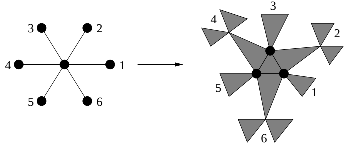
- $GC_z(G)$ has all rotational symmetries of G in 3- and 4-regular case, and all symmetries if $l=0,k$ in general case.
- $GC_z(G)=GC_{\overline{z}}(\overline{G})$ where $\overline{G}$ differs by a plane symmetry only from G. So, if G has a symmetry plane, we reduce to $0\le l\le k$; otherwise, graphs $GC_{k,l}(G)$ and $GC_{l,k}(G)$ are not isomorphic.

$GC_{k,l}(G)$ for 6-regular plane graph G and *any* k, l

- Bipartition of G^* gives vertex 2-coloring, say, red/blue of G.
- Truncation $Tr(G)$ of $\{1, 2, 3\}_v$ is a 3-regular $\{2, 4, 6\}_{6v}$.
- Coloring white vertices of G gives face 3-coloring of $Tr(G)$. White faces in $Tr(G)$ correspond to such in $GC_{k,l}(Tr(G))$.
- For $k \equiv l \pm 1\,(mod\,3)$, i.e. $k + lw \in L_2$, define $GC_{k,l}(G)$ as $GC_{k.l}(Tr(G))$ with all white faces shrinked.
- If $k \equiv l\,((mod\,3)$, faces of $Tr(G)$ are white in $GC_{k,l}(Tr(G))$. Among 3 faces around each vertex, one is white. Coloring other red gives unique 3-coloring of $GC_{k,l}(Tr(G))$. Define $GC_{k,l}(G)$ as pair G_1, G_2 with $Tr(G_1){=}Tr(G_2){=}GC_{k,l}(Tr(G))$ obtained from it by shrinking all red or blue faces.
- $GC_{1,0}(G) = G$ and $GC_{1,1}(G)$ is oriented tripling.

Oriented tripling $GC_{1,1}(G)$ of 6-regular plane graph G

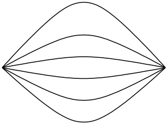
- Let C_1, C_2 be bipartite classes of G^*. For each C_i, oriented tripling $GC_{1,1}(G)$ is 6-regular plane graph $Or_{C_i}(G)$ coming by each vertex of $G \to 3$ vertices and 4 3-gonal faces of $Or_{C_i}(G)$. Symmetries of $Or_{C_i}(G)$ are symmetries of G preserving C_i.
- Orient edges of C_i clockwise. Select 3 of 6 neighbors of each vertex v: $\{2,4,6\}$ are those with directed edge going to v; for $\{1,5,5\}$, edges go to them.



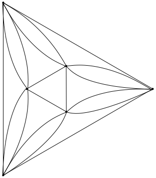
- Any $z{=}k{+}lw{\neq}0$ with $k{\equiv}l \pmod 3$ can be written as $(1{+}w)^s(k'{+}l'w)w$, where $s{\geq}0$ and $k'{\equiv}l' \pm 1 \pmod 3$. So, it holds reduction $GC_{k,l}(G){=}G_{k',l'}(Or^s(G))$.

Examples of oriented tripling $GC_{1,1}(G)$

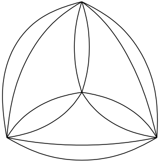
Below: $\{2,3\}_2$ and $\{2,3\}_4$ have *unique* oriented tripling.



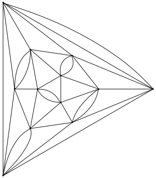
2 D_{6h}



6 D_{3d}



4 T_d



12 T_h

Examples of oriented tripling $GC_{1,1}(G)$

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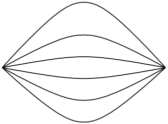
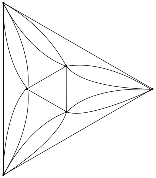
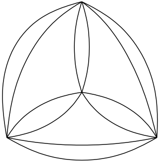
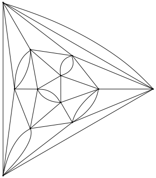
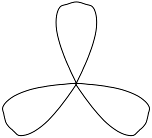
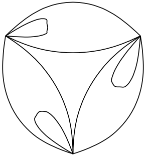

2 D_{6h}

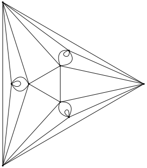

6 D_{3d}

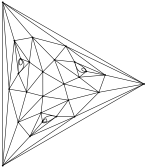

4 T_d

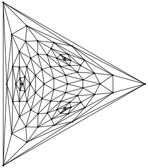

12 T_h


1 C_{3v}


3 C_{3h}


9 C_{3v}

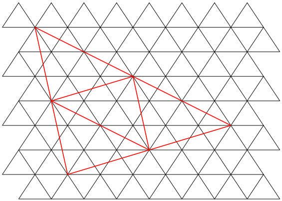

27 C_{3h}


81 C_{3v}

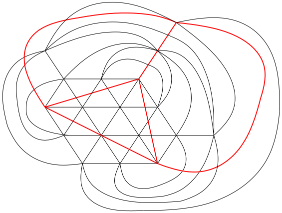
Above: first 4 *consecutive* oriented triplings of the Trifolium.

VI. Parameterizing $(\{a, b\}, k)$-spheres

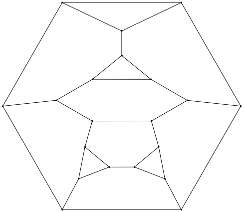
Example: construction of the $(\{3, 6\}, 3)$-spheres in $Z[\omega]$



In the central triangle ABC, let A be the origin of the complex plane



The corresponding triangulation



All $(\{3, 6\}, 3)$-spheres come this way; two complex parameters in $Z[\omega]$ defined by the points B and C

Parameterizing $(\{a, b\}, k)$-spheres

Thurston, 1998 implies: $(\{a, b\}, k)$-spheres have p_a-2 parameters and the number of v-vertex ones is $O(v^{m-1})$ if m=p_a-2 > 2.
Idea: since b-gons are of zero curvature, it suffices to give relative positions of a-gons having curvature $2k - a(k-2) > 0$.
At most $p_a - 1$ vectors will do, since one position can be taken 0.
But once $p_a - 1$ a-gons are specified, the last one is constrained.
The number of m-parametrized spheres with at most v vertices is $O(v^m)$ by direct integration. The number of such v-vertex spheres is $O(v^{m-1})$ if $m > 1$, by a *Tauberian* theorem.

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- Goldberg, 1937: $\{a, 6\}_v$ (highest 2 symmetries): 1 parameter
- Fowler and al., 1988: $\{5, 6\}_v$ (D_5, D_6 or T): 2 parameters.
- Grűnbaum-Motzkin, 1963: $\{3, 6\}_v$: 2 parameters.
- Deza-Shtogrin, 2003: $\{2, 4\}_v$; 2 parameters.
- Thurston, 1998: $\{5, 6\}_v$: 10 (again complex) parameters.
  Graver, 1999: $\{5, 6\}_v$: 20 integer parameters.
- Rivin, 1994: parameter desciption by dihedral angles.

Parameterizing (R,k)-spheres without hyperbolic faces

Thurston, 1998 parametrized (dually, as triangulations) such $(R,3)$-spheres, i.e. 19 series of $(\{3,4,5,6\},3)$-spheres.
In general, such (R,k)-spheres are given by $m{=}\sum_{3\leq i<\frac{2k}{k-2}} p_i - 2$ complex parameters $z_1,\dots,z_m$.
The number of vertices is expressed as a non-degenerate Hermitian form $q{=}q(z_1,\dots,z_m)$ of signature $(1,m-1)$.
Let H^m be the cone of $z{=}(z_1,\dots,z_m)\in\mathbb{C}^m$ with $q(z)>0$.
Given (R,k)-sphere is described by different parameter sets; let $M{=}M(\{p_3,\dots,p_m\},k)$ be the discrete linear group preserving q.
For $k{=}3$, the quotient $H^m/(\mathbb{R}_{>0}\times M)$ is of finite covolume (Thurston, 1998, actually, 1993). Sah, 1994 deduced from it that the number of corresponding spheres grows as $O(v^{m-1})$.
Dutour partially generalized above for other k and surface maps.

8 families: number of complex parameters by groups

- $\{5,6\}_v$ $\mathbf{C_1}(10)$, $\mathbf{C_2}(6)$, $\mathbf{C_3}(4)$, $\mathbf{D_2}(4)$, $\mathbf{D_3}(3)$, $\mathbf{D_5}(2)$, $\mathbf{D_6}(2)$, $\mathbf{T}(2)$, $\{I, I_h\}(1)$
- $\{4,6\}_v$ $\mathbf{C_1}(4)$, $\mathbf{C_2}\backslash S_4(3)$, $\mathbf{D_2}(2)$, $\mathbf{D_3}(2)$, $\{D_6, D_{6h}\}(1)$, $\{O, O_h\}(1)$
- $\{3,4\}_v$ $\mathbf{C_1}(6)$, $\mathbf{C_2}(4)$, $\mathbf{D_2}(3)$, $\mathbf{D_3}(2)$, $\mathbf{D_4}(2)$, $\{O, O_h\}(1)$
- $\{2,3\}_v$ $\mathbf{C_1}(4)$, $\mathbf{C_2}(3?)$, $\mathbf{C_3}(3?)$, $\mathbf{D_2}(2?)$, $\mathbf{D_3}(2?)$, $\mathbf{T}(1)$, $\{D_6, D_{6h}\}(1)$
- $\{3,6\}_v$ $\mathbf{D_2}(2)$, $\{T, T_d\}(1)$
- $\{2,4\}_v$ $\mathbf{D_2}(2)$, $\{D_4, D_{4h}\}(1)$
- $\{2,6\}_v$ $\{D_3, D_{3h}\}(1)$
- $\{1,3\}_v$ $\{C_3, C_{3v}, C_{3h}\}(1)$

Thurston, 1998 implies: $(\{a, b\}, k)$-spheres have p_a-2 parameters and the number of v-vertex ones is $O(v^{m-1})$ if $m{=}p_a\text{-}2 > 1$.

Number of complex parameters

$\{5,6\}_v$

Group	#param.
$\mathbf{C_1}$	10
$\mathbf{C_2}$	6
$\mathbf{C_3}, \mathbf{D_2}$	4
$\mathbf{D_3}$	3
$\mathbf{D_5}, \mathbf{D_6}, \mathbf{T}$	2
$\mathbf{I}$	1

$\{3,4\}_v$

Group	#param.
$\mathbf{C_1}$	6
$\mathbf{C_2}$	4
$\mathbf{D_2}$	3
$\mathbf{D_3}, \mathbf{D_4}$	2
$\mathbf{O}$	1

$\{4,6\}_v$

Group	#param.
$\mathbf{C_1}$	4
$\mathbf{C_2}$	3
$\mathbf{D_2}, \mathbf{D_3}$	2
$\mathbf{D_6}, \mathbf{O}$	1

$\{2,3\}_v$

Group	#param.
C_1	4
$\mathbf{C_2}, \mathbf{C_3}$	3?
$\mathbf{D_2}, \mathbf{D_3}$	2?
$\mathbf{D_6}, \mathbf{T}$	1

$\{3,6\}_v$- and $\{2,4\}_v$: 2 complex parameters but 3 natural ones will do: *pseudoroad* length, number of circumscribing *railroads*, *shift*.

VII. Railroads and tight $(\{a, b\}, k)$-spheres

ZC-circuits

- The edges of any plane graph are doubly covered by zigzags (Petri or left-right paths), i.e., circuits such that any two but not three consecutive edges bound the same face.
- The edges of any *Eulerian* (i.e., even-valent) plane graph are partitioned by its central circuits (those going straight ahead).
- A ZC-circuit means *zigzag* or *central circuit* as needed.
  CC- or Z-vector enumerate lengths of above circuits.

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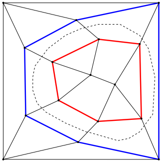
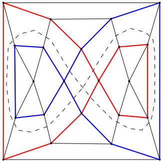
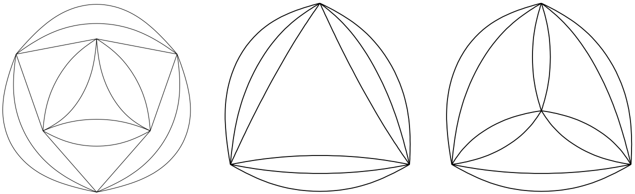
- A ZC-circuit means *zigzag* or *central circuit* as needed. CC- or Z-vector enumerate lengths of above circuits.
- A railroad in a 3-, 4- or 6-regular plane graph is a circuit of 6-, 4- or 3-gons, each adjacent to neighbors on opposite edges. Any railroad is bound by two "parallel" ZC-circuits. It (any if 4-, simple if 3- or 6-regular) can be collapsed into 1 ZC-circuit.



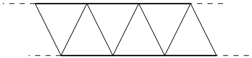


Railroad in a 6-regular sphere: examples

APrism$_3$ with 2 base 3-gons doubled is the $\{2, 3\}_6$ (D_{3d}) with CC-vector $(3^2, 4^3)$, all five central circuits are simple.
Base 3-gons are separated by a simple railroad R of six 3-gons, bounded by two parallel central 3-circuits around them. Collapsing R into one 3-circuit gives the $\{2, 3\}_3$ (D_{3h}) with CC-vector $(3; 6)$.



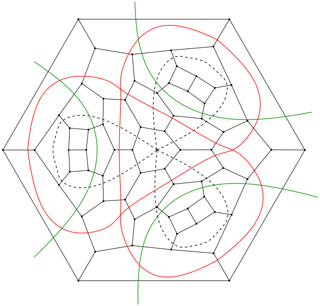
D_{3d} $(3^2, 4^3)$ D_{3h} $(3; 6)$ T_d (3^4)



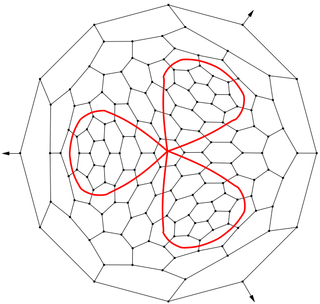
Above $\{2, 3\}_4$ (T_d) has no railroads but it is not strictly tight, i.e. no any central circut is adjacent to a non-3-gon *on each side*.

Railroads flower: Trifolium $\{1, 3\}_1$

Railroads can be simple or self-intersect, including triply if $k = 3$.
First such Dutour $(\{a, b\}, k)$-spheres for $(a, b) = (4, 6), (5, 6)$ are:



$\{4, 6\}_{66}(D_{3h})$ twice



$\{5, 6\}_{172}(C_{3v})$

Which plane curves with at most triple self-intersectionss come so?

Number of ZC-circuits in tight $(\{a, b\}, k)$-sphere

- Call an $(\{a, b\}, k)$-sphere tight if it has no railroads.

- ≤ 15 for $\{5, 6\}_v$ Dutour, 2004
- ≤ 9 for $\{4, 6\}_v$ and $\{2, 3\}_v$ Deza-Dutour, 2005 and 2010
- ≤ 3 for $\{2, 6\}_v$ and $\{1, 3\}_v$ same
- ≤ 6 for $\{3, 4\}_v$ Deza-Shtogrin, 2003
- Any $\{3, 6\}_v$ has ≥ 3 zigzags with equality iff it is tight. All $\{3, 6\}_v$ are tight iff $\frac{v}{4}$ is prime and none iff it is even.
- Any $\{2, 4\}_v$ has ≥ 2 central circuits with equality iff it is tight. There is a tight one for any even v.

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- Any $\{2,4\}_v$ has ≥ 2 central circuits with equality iff it is tight. There is a tight one for any even v.

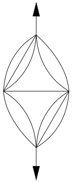
First tight ones with max. of ZC-circuits are $GC_{21}(\{a,b\}_{\min})$: $\{5,6\}_{140}(I)$, $\{4,6\}_{56}(O)$, $\{2,6\}_{14}(D_3)$, $\{3,4\}_{30}(0)$; $\{2,3\}_{44}(D_{3h})$ and $\{a,b\}_{\min}$: $\{3,6\}_4(T_d)$, $\{2,4\}_2(D_{4h})$. Besides $\{2,3\}_{44}(D_{3h})$, ZC-circuits are: $(28^{15}),(21^8),(14^3),(10^6),(4^3),(2^2)$, all simple.

Maximal number M_v of central circuits in *any* $\{2,3\}_v$

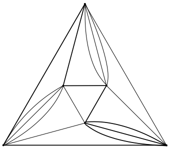
- $M_v = \frac{v}{2}+1$, $\frac{v}{2}+2$ for $v \equiv 0,2 \pmod 4$. It is realized by the series of symmetry D_{2d} with CC-vector $2^{\frac{v}{2}}, 2v_{0,v}$ and of symmetry D_{2h} with CC-vector $2^{\frac{v}{2}}, v^2_{0,\frac{v-2}{4}}$ if $v \equiv 0,2 \pmod 4$.
- For odd v, M_v is $\lfloor\frac{v}{3}\rfloor+3$ if $v \equiv 2,4,6 \pmod 9$ and $\lfloor\frac{v}{3}\rfloor+1$, otherwise. Define t_v by $\frac{v-t_v}{3} = \lfloor\frac{v}{3}\rfloor$. M_v is realized by the series of symmetry C_{3v} if $v \equiv 1 \pmod 3$ and D_{3h}, otherwise. CC-vector is $3^{\lfloor\frac{v}{3}\rfloor}, (2\lfloor\frac{v}{3}\rfloor + t_v)^3_{0,\lfloor\frac{v-2t_v}{9}\rfloor}$ if $v \equiv 2,4,6 \pmod 9$ and $3^{\lfloor\frac{v}{3}\rfloor}, (2v+t_v)_{0,v+2t_v}$, otherwise.

Smallest CC-knotted or Z-knotted $\{2, 3\}_v$

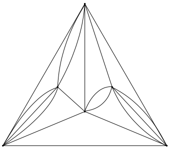
- The minimal number of central circuits or zigzags, 1, have CC-knotted and Z-knotted $\{2, 3\}_v$. They correspond to plane curves with only triple self-intersection points. For $v \leq 16$, there are $1, 2, 4, 7, 9, 12$ Z-knotted if $v = 3, 7, 9, 11, 13, 15$ and $1, 2, 2, 4, 11, 9, 1, 19$ CC-knotted if $v = 4, 6, 8, 10, 12, 14, 15, 16$.
- Conjecture (holds if $v \leq 54$): any Z-knotted $\{2, 3\}_v$ has odd v and a CC-knotted $\{2, 3\}_v$ is Z-knotted if and only if v is odd.



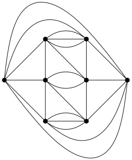
4 D_2



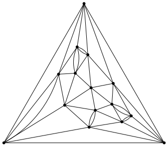
6 D_3



6 C_2



8 D_2



15 C_1

VIII. Tight pure $(\{a, b\}, k)$-spheres

Tight $(\{a, b\}, k)$-spheres with only simple ZC-circuits

- Call $(\{a, b\}, k)$-sphere pure if any of its ZC-circuits is *simple*, i.e. has no self-intersections. Such ZC-circuit can be seen as a Jordan curve, i.e. a plane curve which is topologically equivalent to (a homeomorphic image of) the unit circle.
- Any $(\{3, 6\}, 3)$- or $(\{2, 4\}, 4)$-sphere is pure. They are tight if and only if have three or, respectively, two ZC-circuits.
- Any ZC-circuit of $\{2, 6\}_v$ or $\{1, 3\}_v$ self-intersects.

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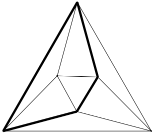
- Any ZC-circuit of $\{2, 6\}_v$ or $\{1, 3\}_v$ self-intersects.

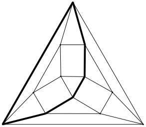
The number of tight pure $(\{a, b\}, k)$-spheres is:

- 9? for $\{5, 6\}_v$ computer-checked for $v \leq 300$ by Brinkmann
- 2 for $\{4, 6\}_v$ Deza-Dutour, 2005
- 8 for $\{3, 4\}_v$ same
- 5 for $\{2, 3\}_v$ same, 2010

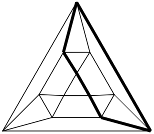
All tight ({3, 4}, 4)-spheres with only simple central circuits



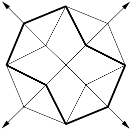
6 O_h (4^3)
Octahedron



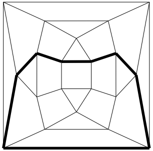
12 O_h (6^4)
$GC_{11}(Oct.)$



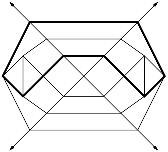
12 D_{3h} (6^4)



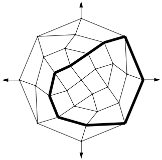
14 D_{4h}
$(6^2, 8^2)$



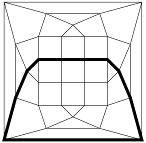
20 D_{2d} (8^5)



22 D_{2h}
$(8^3, 10^2)$



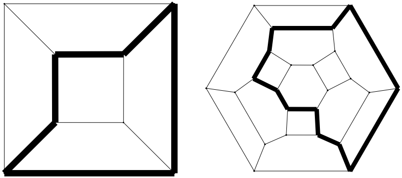
30 O (10^6)
$GC_{21}(Oct.)$



32 D_{4h}
$(10^4, 12^2)$

All tight $(\{4,6\},3)$-spheres with only simple zigzags

There are exactly two such spheres: Cube and its leapfrog $GC_{11}(Cube)$, truncated Octahedron.



6 O_h (6^4) **24** O_h (10^6)

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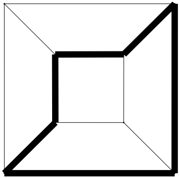
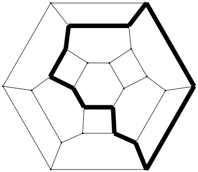


6 O_h (6^4)

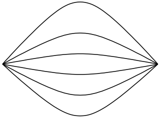


24 O_h (10^6)

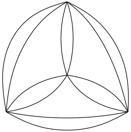
Proof is based on a) The size of intersection of two simple zigzags in any $(\{4,6\},3)$-sphere is $0,2,4$ or 6 and
b) Tight $(\{4,6\},3)$-sphere has at most 9 zigzags.

For $(\{2,3\},6)$-spheres, a) holds also, implying a similar result.

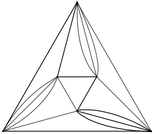
Tight $(\{2,3\},6)$-spheres with only simple ZC-circuits



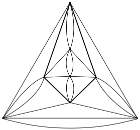
2 D_{6h} (2^3) (6^2)



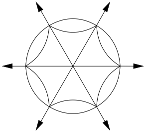
4 T_d (3^4) (6^4)



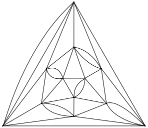
6 D_3 no $(12, 8^3)$



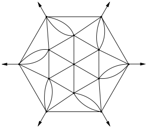
8 D_{2d} $(5^4, 4)$ no



D_{6h} $(4^3, 6^2)$ (8^6) no



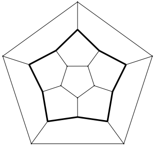
12 T_h (6^6) (12^6)



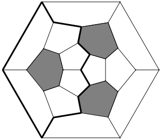
14 D_6 no (14^6)

All CC-pure, tight: Nrs. 1,2,4,5,6 (Nrs. 3,7 are not CC-pure).
All Z-pure, tight: Nrs. 1,2,3,6,7 (4 is not Z-pure, 5 is not Z-tight).
1st, 3rd are strictly CC-, Z-tight: all ZC-circuits sides touch 2-gons

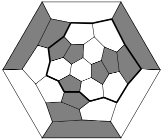
7 tight $(\{5,6\},3)$-spheres with only simple zigzags



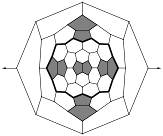
20 I_h (10^6)



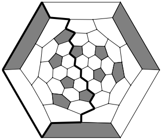
28 T_d (12^7)



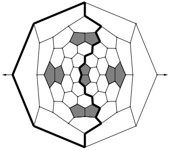
48 D_3 (16^9)



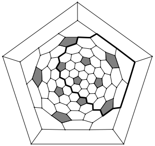
76 D_{2d} $(22^4, 20^7)$



88 T (22^{12})



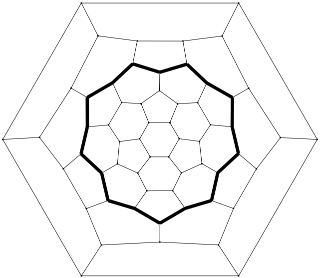
92 T_h $(24^6, 22^6)$



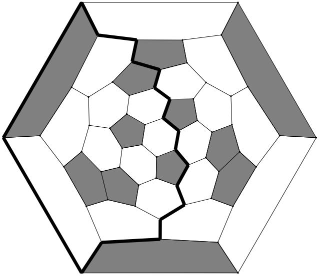
140 I, (28^{15})

The zigzags of $1,2,3,5,7$th above and next two form 7 Grűnbaum arrangements of Jordan curves, i.e. any two intersect in 2 points.
The groups of $1,5,7$th and $\{5,6\}_{60}(I_h)$ are zigzag-transitive.

Two other such $(\{5,6\},3)$-spheres



60 I_h (18^{10})



60 D_3 (18^{10})

This pair was first answer on a question in Grűnbaum, 1967, 2003 *Convex Polytopes* about existence of *simple* polyhedra with the same p-vector but different zigzags. The groups of above $\{5,6\}_{60}$ have, acting on zigzags, 1 and 3 orbits, respectively.

IX. Other fullerene analogs: c-disks $(\{a, b, c_1\}, k)$

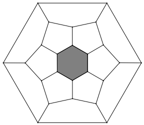
Other fullerene-like spheres with hyperbolic faces

Related non-standard (R, k)-spheres with $\frac{1}{k} + \frac{1}{\max_{i \in R} i} < \frac{1}{2}$ are:

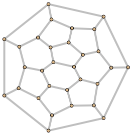
- G-fulleroids (Deza-Delgado, 2000; Jendrol-Trenkler, 2001 and Kardos, 2007): $(\{5, b\}, 3)$-spheres with $b \geq 7$ and symmetry G.
- b-Icosahedrites: $(\{3, b\}, 5)$-spheres. So, they have $p_3 = (3b\text{-}10)p_b + 20$ 3-gons and $v = 2(b\text{-}3)p_b + 12$ vertices.
- Haeckel, 1887: $(\{5, 6, c\}, 3)$-spheres with $c = 7, 8$ representing skeletons of radiolarian zooplankton Aulonia hexagona.
- $(\{a, b, c\}, k)$-disk is an $(\{a, b, c\}, k)$-sphere with $p_c = 1$; so, its $v = \frac{2}{k-2}(p_a\text{-}1 + p_b) = \frac{2}{2k - a(k-2)}(a + c + p_b(b\text{-}a))$ and (setting $b' = \frac{2k}{k-2}$) $p_a = \frac{b'+c}{b'-a} + p_b \frac{b-b'}{b'-a}$. So, $p_a = \frac{b+c}{b-a}$ if $b = b'$ (8 families).
- Fullerene c-disk is the case $(a, b, c; k) = (5, 6, c; 3)$ of above. So, they have $p_5 = c + 6$ and $v = 2(p_6 + c + 5)$ vertices.

Minimal fullerene $((\{5,6\},3))$ c-disks

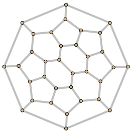
If c=3, 4, 5, it is 1-vertex-, 1-edge-truncated, usual Dodecahedron.
Their number is 2, 3, 10 and p_6=c-3 if c=9, 10, 11; else, 1 with min $p_6(c)$=3, 2, 0, 1, 3, 4, 5, 6 (tube C_s/C_2) if c=3, 4, 5, 6, 7, 8, 12, ≥ 13.



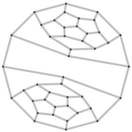
6 24 D_{6d}



7 30 C_s

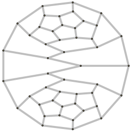


8 34 C_{2v}

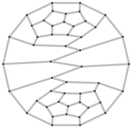


12 44 C_{2v}

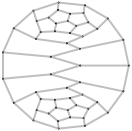
Conjecture: $(\{5,6,c_1\},3)$ with $c \geq 13$ exists iff v is even $\geq 2c+22$.



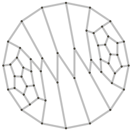
13 48 C_s



14 50 C_2



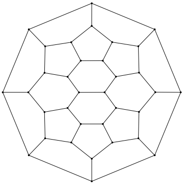
15 52 C_s



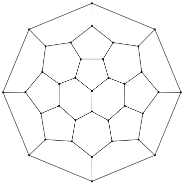
16 54 C_2

Symmetries of fullerene c-disks $(\{5, 6, c_1\}, 3)$, $c \geq 3$

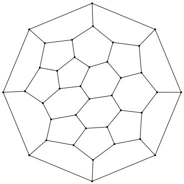
- Their groups: C_m, C_{mv} with $m \equiv 0 (mod\ c)$ (since any symmetry should stabilize unique c-gonal face) and $m \in \{1, 2, 3, 5, 6\}$ since the axis pass by a vertex, edge or face.
- The minimal such 3-connected 8- and 9-disks are given below.



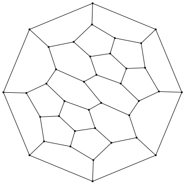
8 34 C_{2v}



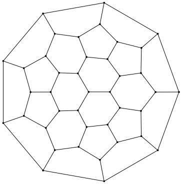
8 36 C_s



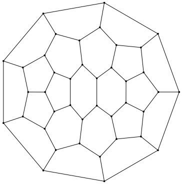
8 38 C_1



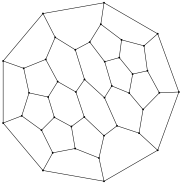
8 38 C_2



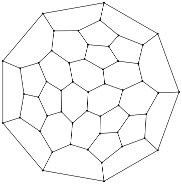
9 40 C_{3v}



9 40 C_s



9 42 C_1

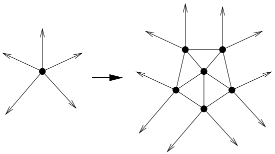


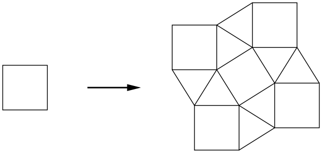
9 52 C_3

X. Icosahedrites: $(\{3, 4\}, 5)$-spheres

Icosahedrites, i.e., $(\{3,4\},5)$-spheres

- They have $p_3 = 2p_b + 20$ and $v = 2p_b + 12$ vertices.
- Their number is $1, 0, 1, 1, 5, 12, 63, 246, 1395, 7668, 45460$ for $v = 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32$. It grows at least exponentially with v. So, there is a continuum of icosahedrites, while 8 standard families are countable.
- p_a is fixed in for standard $(\{a,b\},k)$-spheres permitting Goldberg-Coxeter construction and parametrization of graphs which imply the polynomial growth of their number. It does not happen for icosahedrites; no parametrization for them.

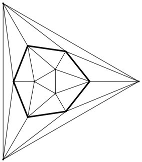




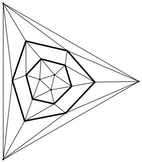
A-operation keeps symmetries; B-operation: only rotational ones.

Proof for the number of icosahedrites

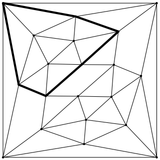
A weak zigzag ia a left/right, but never extreme, edge-circuit.
If a v-vertex icosahedrite has a simple weak zigzag of length 6, a $(v+6)$-vertex one come by inserting a corona (6-ring of three 4-gons alternated by three pairs of adjacent 3-gons) instead of it.
But such spheres exist for $v=18$, 20, 22; so, for $v \equiv 0, 2, 4 (mod\, 6)$.
There are two options of inserting corona; so, the number of v-vertex icosahedrites grows at least exponentially.



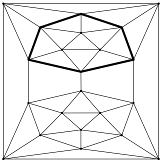
12 I_h
$wZ=6^{10}$
$Z=10^6$



18 D_3
$6^2, 8^3; 54_{0,9}$
$90_{27,18}$



20 D_{2d}
$6^4, 20^2; 18^2_{0,3}$
$10^4; 30^2_{3,0}$

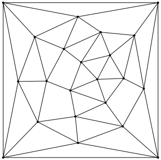


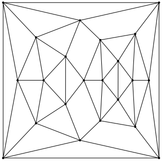
22 D_{5h}
$6^{10}; 50_{15,0}$
$10^2; 90_{15,20}$

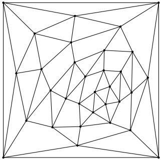
An usual (strong) zigzag is a left/right, both extreme, edge-circuit.

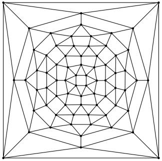
38 symmetry groups of icosahedrites

- Agregating $\mathbf{C_1}$=$\{C_1, C_s, C_i\}$, $\mathbf{C_m}$=$\{C_m, C_{mv}, C_{mh}, S_{2m}\}$, $\mathbf{D_m}$=$\{D_m, D_{mh}, D_{md}\}$, $\mathbf{T}$=$\{T, T_d, T_h\}$, $\mathbf{O}$=$\{O, O_h\}$, $\mathbf{I}$=$\{I, I_h\}$, all 38 symmetries of $(\{3,4\},5)$-spheres are:
  $\mathbf{C_1}$, $\mathbf{C_m}$, $\mathbf{D_m}$ for $2 \le m \le 5$ and $\mathbf{T}$, $\mathbf{O}$, $\mathbf{I}$.
- Any group appear an infinite number of times since one gets an infinity by applying A-operation iteratively.
- Group limitations came from k-fold axis only. Is it occurs for all $(\{a,b\},k)$-spheres with b-faces of negative curvature?
- Examples (minimal whenever $v \le 32$) are given below:


22 C_1

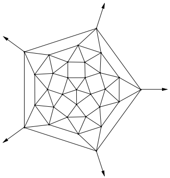

22 C_s


32 C_i

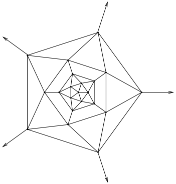

72 O_h

Minimal $(\{3,4\},5)$-spheres of 5-fold symmetry

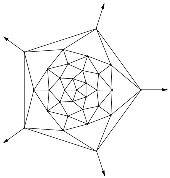
It exists iff $p_4 \equiv 0\,(mod\,5)$, i.e., $v = 2p_4 + 12 \equiv 2\,(mod\,10)$.



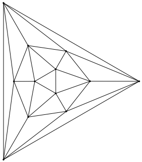
32 D_5



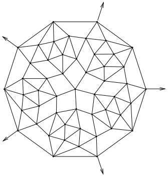
22 D_{5h}



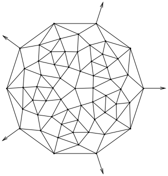
32 D_{5d}



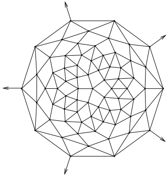
12 I_h



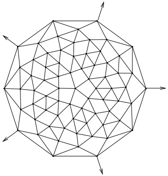
52 C_5



62 C_{5h}



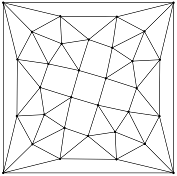
72 C_{5v}



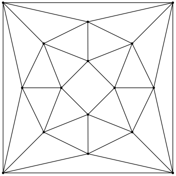
72 S_{10}

Minimal ($\{3,4\}, 5$)-spheres of 4-fold symmetry

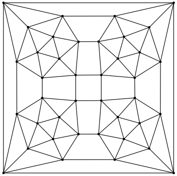
It exists iff $p_4 \equiv 2\,(mod\,4)$, i.e., $v = 2p_4 + 12 \equiv 0\,(mod\,8)$.



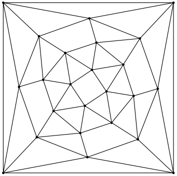
32 D_4



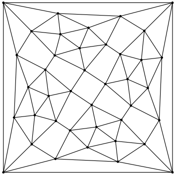
16 D_{4d}



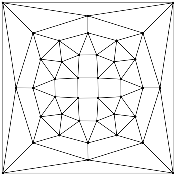
40 D_{4h}



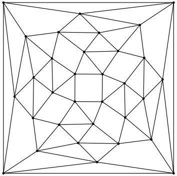
24 O



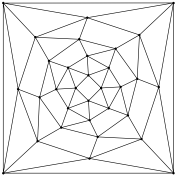
40 C_4



40 C_{4v}



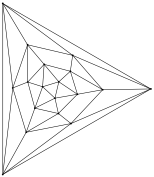
32 C_{4h}



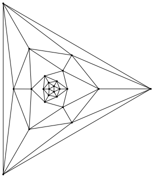
32 S_8

Minimal $(\{3, 4\}, 5)$-spheres of 3-fold symmetry

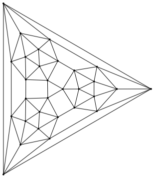
It exists iff $p_4 \equiv 0\,(mod\,3)$, i.e., $v = 2p_4 + 12 \equiv 0\,(mod\,6)$.



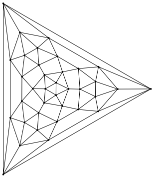
18 D_3



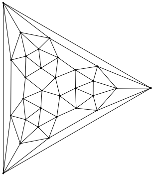
24 D_{3d}



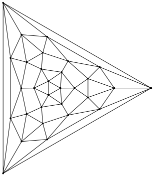
30 D_{3h}



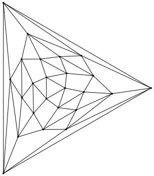
36 T_d



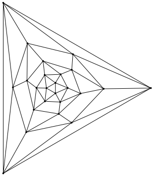
30 C_3



30 C_{3v}

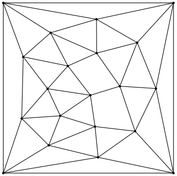


24 C_{3h}

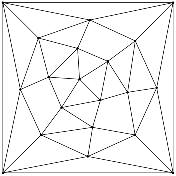


24 S_6

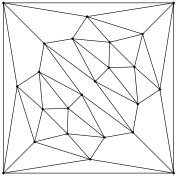
Minimal $(\{3, 4\}, 5)$-spheres of 2-fold symmetry



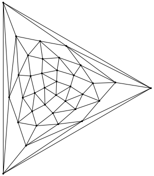
20 D_2



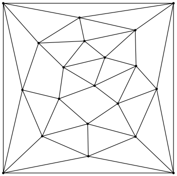
20 D_{2d}



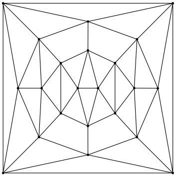
24 D_{2h}



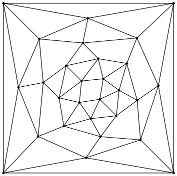
36 T_h



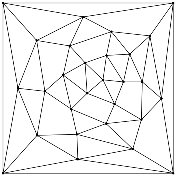
20 C_2



22 C_{2v}



28 C_{2h}



28 S_4

XI. Standard $(\{a, b\}, k)$-maps on surfaces

Standard (R, k)-maps

- Given $R \subset \mathbb{N}$ and a surface $\mathbb{F}^2$, an (R, k)-$\mathbb{F}^2$ is a k-regular map M on surface $\mathbb{F}^2$ whose faces have gonalities $i \in R$.
- Euler characteristic $\chi(M)$ is $v - e + f$, where v, e and $f = \sum_i p_i$ are the numbers of vertices, edges and faces of M.
- Since $kv{=}2e{=}\sum_i ip_i$, Euler formula $\chi = v - e + f$ becomes Gauss-Bonnet-like one $2\chi(M)k = \sum_i p_i(2k - i(k-2))$.
- Again, let our maps be standard, i.e., $\min_{i \in R}(\frac{1}{k} + \frac{1}{i}) = \frac{1}{2}$. So, $M{=}max\{i \in R\}{=}\frac{2k}{k-2}$ and $(M, k){=}(6, 3), (4, 4), (3, 6)$.
- There are infnity of standard maps (R, k)-$\mathbb{F}^2$, since the number p_M of parabolic faces is not restricted.
- Also, $\chi \geq 0$ with $\chi = 0$ if and only if $R = \{m\}$. So, $\mathbb{F}^2$ is $\mathbb{S}^2$, $\mathbb{T}^2$, $\mathbb{P}^2$, $\mathbb{K}^2$ with $\chi = 2, 0, 1, 0$, respectively.
- Such $(\{a, b\}, k)$-$\mathbb{F}^2$ map has $b{=}\frac{2k}{k-2}$, $p_a{=}\frac{\chi b}{b-a}$, $v{=}\frac{1}{k}(ap_a{+}bp_b)$ So, $(a{=}b, k){=}(6, 3), (3, 6), (4, 4)$ if $\mathbb{F}^2$ is $\mathbb{T}^2$ or $\mathbb{K}^2$.
- But $\chi{=}\frac{p_3 - 2p_4}{10}$ for icosahedrite maps $(\{3, 4\}, 5)$ (non-standard) So, $\chi{<}0$ is possible and $\chi{=}0$ (i.e., $\mathbb{F}^2{=}\mathbb{T}^2, \mathbb{K}^2$) iff $p_3{=}2p_4$.

Digression on interesting non-standard $(\{5, 6, c\}, 3)$-maps

Such maps, generalizing fullerenes, have $c \geq 7$. Examples are:

- G-fulleroids (Deza-Delgado, 2000; Jendrol-Trenkler, 2001 and Kardos, 2007): $(\{5, b\}, 3)$-spheres with $b \geq 7$ and symmetry G
- Haeckel, 1887: $(\{5, 6, c\}, 3)$-spheres with $c = 7, 8$ representing skeletons of radiolarian zooplankton Aulonia hexagona.
- Azulenoids: $(\{5, 6, 7\}, 3)$-tori; so $g = 1$, $p_5 = p = 7$.
- Schwarzits: $(\{5, 6, c\}, 3)$-maps on minimal surfaces of constant negative curvature ($g \geq 2$) with $c = 7, 8$. Knor-Potocnik-Siran-Skrekovski, 2010: such $(\{6, c\}, 3)$-maps exist for any $g \geq 2$, $p_6 \geq 0$ and $c = 7, 8, 9, 10, 12$. For $c = 7, 8$ such polyhedral maps exist.

The $(\{a, b\}, k)$-maps on torus and Klein bottle

The connected *closed* (compact and without boundary) irreducible surfaces are: sphere $\mathbb{S}^2$, torus $\mathbb{T}^2$ (two orientable), real projective plane $\mathbb{P}^2$ and Klein bottle $\mathbb{K}^2$ with $\chi = 2, 0, 1, 0$, respectively.

The maps $(\{a, b\}, k)$-$\mathbb{T}^2$ and $(\{a, b\}, k)$-$\mathbb{K}^2$ have $a = b = \frac{2k}{k-2}$; so, $(a = b, k)$ should be $(6, 3), (3, 6)$ or $(4, 4)$.

We consider only polyhedral maps, i.e. no loops or multiple edges (1- or 2-gons), and any two faces intersect in edge, point or $\emptyset$ only.

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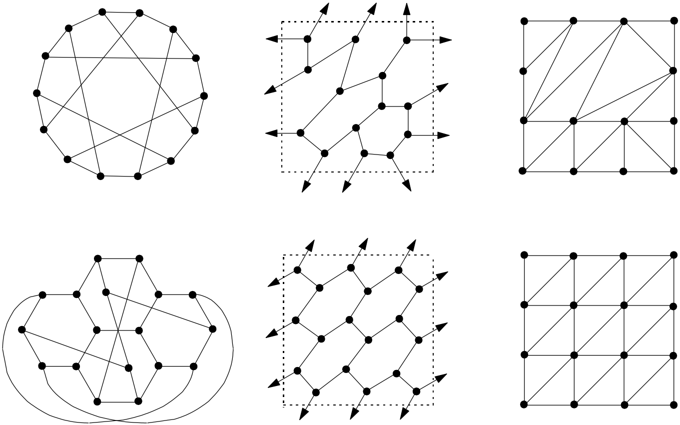
Smallest $\mathbb{T}^2$ and $\mathbb{K}^2$-embeddings for $(a{=}b, k){=}(6, 3), (3, 6), (4, 4)$:
as 6-regular triangulations: K_7 and $K_{3,3,3}$ $(p_3 = 14, 18)$;
as 3-regular polyhexes: Heawood graph (dual K_7) and dual $K_{3,3,3}$;
as 4-regular quadrangulations: K_5 and $K_{2,2,2}$ $(p_4 = 5, 6)$.

K_5 and $K_{2,2,2}$ are also smallest $(\{3, 4\}, 4)$-$\mathbb{P}^2$ and $(\{3, 4\}, 4)$-$\mathbb{S}^2$, while K_4 is the smallest $(\{4, 6\}, 3)$-$\mathbb{P}^2$ and $(\{3, 6\}, 3)$-$\mathbb{S}^2$.

Smallest 3-regular maps on $\mathbb{T}^2$ and $\mathbb{K}^2$: duals K_7, $K_{3,3,3}$



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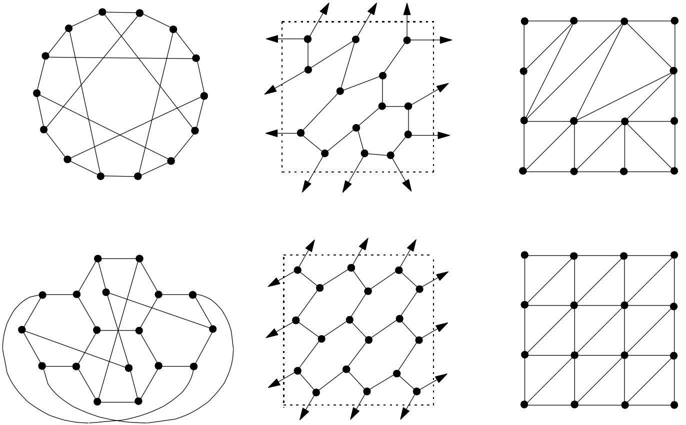


3-regular polyhexes on $\mathbb{T}^2$, cylinder, Möbius surface, $\mathbb{K}^2$ are $\{6^3\}$'s quotients by fixed-point-free group of isometries, generated by: two translations, a transl., a glide reflection, transl. *and* glide reflection.

8 families: symmetry groups with inversion

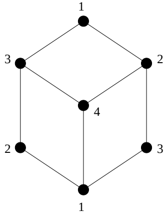
The point symmetry groups with inversion operation are: T_h, O_h, I_h, C_{mh}, D_{mh} with even m and D_{md}, S_{2m} with odd m. So, they are

- 9 for $\{5, 6\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{6h}, S_6, T_h, D_{5d}, I_h
- 7 for $\{2, 3\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{6h}, S_6, T_h
- 6 for $\{4, 6\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{6h}, O_h
- 6 for $\{3, 4\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{4h}, O_h
- 2 for $\{2, 4\}_v$: D_{2h}, D_{4h}
- 1 for $\{3, 6\}_v$: D_{2h}
- 0 for $\{2, 6\}_v$ and $\{1, 3\}_v$
- Cf. 12 for icosahedrites (($\{3, 4\}, 5$)-spheres): C_i, C_{2h}, C_{4h}, D_{2h}, D_{4h}, D_{3d}, D_{5d}, S_6, S_{10}, T_h, O_h, I_h

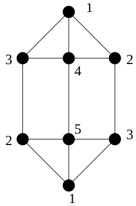
(R, k)-maps on the projective plane are the antipodal quotients of centrosymmetric (R, k)-spheres; so, halving their p-vector and v.

Smallest $(\{a, b\}, k)$-maps on the projective plane

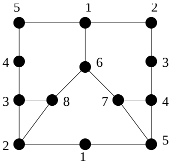
- The smallest ones for $(a, b) = (4, 6), (3, 4), (3, 6), (5, 6)$ are: K_4 (smallest $\mathbb{P}^2$-quadrangulation), K_5, 2-truncated K_4, dual K_6 (Petersen graph), i.e., the antipodal quotients of Cube $\{4, 6\}_8$, $\{3, 4\}_{10}(D_{4h})$, $\{3, 6\}_{16}(D_{2h})$, Dodecahedron $\{5, 6\}_{20}$.
- The smallest ones for $(a, b) = (2, 4), (2, 3)$ are points with 2, 3 loops; smallest without loops are $4 \times K_2$, $6 \times K_2$ but on $\mathbb{P}^2$.



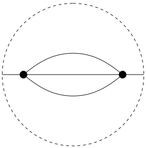
$\{4, 6\}_4$



$\{3, 4\}_5$



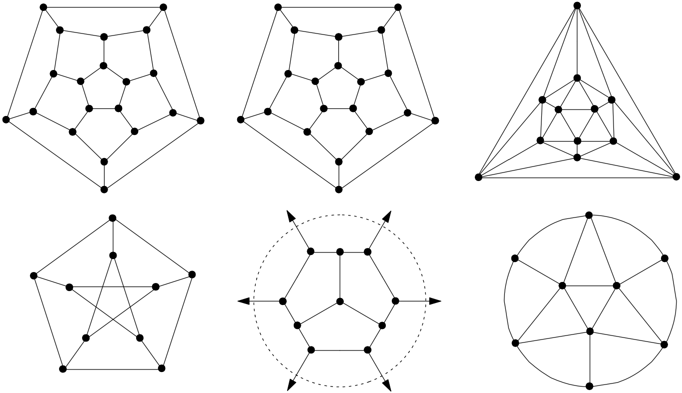
$\{3, 6\}_8$



$\{2, 4\}_2$

Smallest $(\{5, 6\}, 3)$-$\mathbb{P}^2$ and $(\{3, 4\}, 5)$-$\mathbb{P}^2$

The Petersen graph (in positive role) is the smallest $\mathbb{P}^2$-fullerene.
Its $\mathbb{P}^2$-dual, K_6, is the smallest $\mathbb{P}^2$-icosahedrite (half-Icosahedron).
K_6 is also the smallest (with 10 triangles) triangulation of $\mathbb{P}^2$.



6 families on projective plane: parameterizing

- $\{5,6\}_v$: C_i, C_{2h}, D_{2h}, S_6, D_{3d}, D_{6h}, T_h, D_{5d}, I_h
- $\{2,3\}_v$: C_i, C_{2h}, D_{2h}, S_6, D_{3d}, D_{6h}, T_h
- $\{4,6\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{6h}, O_h
- $\{3,4\}_v$: C_i, C_{2h}, D_{2h}, D_{3d}, D_{4h}, O_h
- $\{2,4\}_v$: D_{2h}, D_{4h}
- $\{3,6\}_v$: D_{2h}

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$(\{2,3\},6)$-spheres T_h and D_{6h} are $GC_{k,k}(2 \times \mathit{Tetrahedron})$ and, for $k \equiv 1,2 \pmod 3$, $GC_{k,0}(6 \times K_2)$, respectively. Other spheres of blue symmetry are $GC_{k,l}$ with $l = 0, k$ from the first such sphere.

So, each of 7 blue-symmetric families is described by one natural parameter k and contains $O(\sqrt{v})$ spheres with at most v vertices.

$(\{a, b\}, k)$-maps on Euclidean plane and 3-space

- An $(\{a, b\}, k)$-$\mathbb{E}^2$ is a k-regular tiling of $\mathbb{E}^2$ by a- and b-gons.
- $(\{a, b\}, k)$-$\mathbb{E}^2$ have $p_a \leq \frac{b}{b-a}$ and $p_b = \infty$. It follows from Alexandrov, 1958: any metric on $\mathbb{E}^2$ of non-negative curvature can be realized as a metric of convex surface on $\mathbb{E}^3$. In fact, consider plane metric such that all faces became regular in it. Its curvature is 0 on all interior points (faces, edges) and ≥ 0 on vertices. A convex surface is at most half-$\mathbb{S}^2$.
- There are ∞ of $(\{a, b\}, k)$-$\mathbb{E}^2$ if $2 \leq p_a \leq \frac{b}{b-a}$ and 1 if $p_a = 0, 1$.
- The plane fullerenes (or nanocones) $(\{5, 6\}, k)$-$\mathbb{E}^2$ are classified by Klein and Balaban, 2007: the number of *equivalence* (isomorphism up to a finite induced subgraph) classes is 2,2,2,1 for $p_5 = 2, 3, 4, 5$, respectively.

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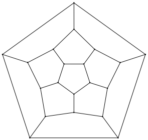
- An $(\{a,b\},k)$-$\mathbb{E}^3$ is a 3-periodic k'-regular face-to-face tiling of the Euclidean 3-space $\mathbb{E}^3$ by $(\{a,b\},k)$-spheres.
- Next, we will mention such tilings by 4 special fullerenes, which are important in Chemistry and Crystallography. Then we consider extension of $(\{a,b\},k)$-maps on manifolds.

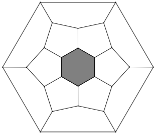
XII. Beyond surfaces

Frank-Kasper ({*a*, *b*}, *k*)-spheres and tilings

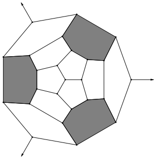
- A ({*a*, *b*}, *k*)-sphere is Frank-Kasper if no *b*-gons are adjacent.
- All cases are: smallest ones in 8 families, 3 ($\{5,6\},3$)-spheres (24-, 26-, 28-vertex fullerenes), ($\{4,6\},3$)-sphere $Prism_6$, 3 ($\{3,4\},4$)-spheres ($APrism_4$, $APrism_3^2$, Cuboctahedron), ($\{2,4\},4$)-sphere doubled square and two ($\{2,3\},6$)-spheres (tripled triangle and doubled Tetrahedron).



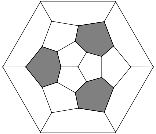
20, I_h



24 D_{6d}



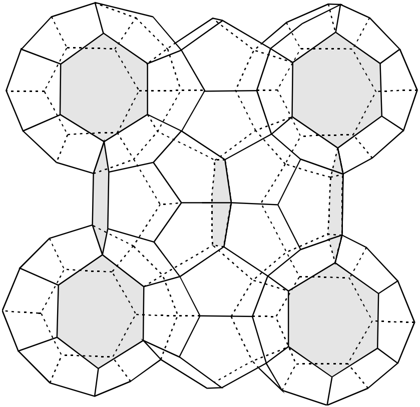
26, D_{3h}



28, T_d

FK space fullerenes

A FK space fullerene is a 3-periodic 4-regular face-to-face tiling of 3-space $\mathbb{E}^3$ by four Frank-Kasper fullerenes $\{5, 6\}_v$.
They appear in crystallography of alloys, clathrate hydrates, zeolites and bubble structures. The most important, A_{15}, is below.



Weaire-Phelan, 1994: best known solution of weak Kelvin problem

Other $\mathbb{E}^3$-tilings by $(\{a, b\}, k)$-spheres

- An $(\{a, b\}, k)$-$\mathbb{E}^3$ is a 3-periodic k'-regular face-to-face $\mathbb{E}^3$-tiling by $(\{a, b\}, k)$-spheres. Some examples follow.
- Deza-Shtogrin, 1999: first known non-FK space fullerene $(\{5,6\},3)$-$\mathbb{E}^3$: 4-regular $\mathbb{E}^3$-tiling by $\{5,6\}_{20}$, $\{5,6\}_{24}$ and its elongation $\simeq \{5,6\}_{36}$ (D_{6h}) in proportion 7:2:1.

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- space cubite $(\{4, 6\}, 3)$-$\mathbb{E}^3$: tiling by $Prism_4$, $Prism_6$ with bipyramidal star. Examples: 5- and 6-regular $\mathbb{E}^3$-tilings by $Prism_6$ and by Cube (Voronoi tilings of lattices $A_2 \times \mathbb{Z}$ and $\mathbb{Z}^3$ with stars $Prism_3^*$ and $\beta_3 = Prism_4^*$), respectively.

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- space octahedrite $(\{3,4\},4)$-$\mathbb{E}^3$: 6-regular (star-Octahedron) tiling by Octahedron, Cuboctahedron in proportion 1:1. It is uniform (vertex-transitive and with Archimedean tiles) and Delaunay tiling of J-complex (mineral **perovskite** structure).
- Cf. $\mathbb{H}^3$-tilings: 6-regular $\{5,3,4\}$ by $\{5,6\}_{20}$, (Löbell, 1931) by $\{5,6\}_{24}$ and 12-reg. $\{5,3,5\}$ by $\{5,6\}_{20}$, $\{4,3,5\}$ by Cube.

Fullerene manifolds

- Given $3 \leq a < b \leq 6$, $\{a, b\}$-manifold is a $(d-1)$-dimensional d-valent compact connected *manifold* (locally homeomorphic to $\mathbb{R}^{d-1}$) whose 2-faces are only a- or b-gonal.
- So, any i-face, $3 \leq i \leq d$, is a polytopal i-$\{a, b\}$-manifold.
- Most interesting case is $(a, b) = (5, 6)$ (fullerene manifold), when $d = 2, 3, 4, 5$ only since (Kalai, 1990) any 5-polytope has a 3- or 4-gonal 2-face.

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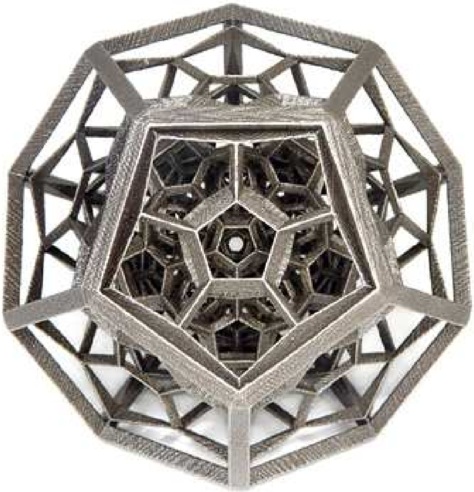
- The smallest polyhex is 6-gon on $\mathbb{T}^2$. The "greatest": $\{633\}$, the convex hull of vertices of $\{63\}$, realized on a horosphere.
- Prominent 4-fullerene (600-vertex on $\mathbb{S}^3$) is 120-cell ($\{533\}$). The "greatest" polypent: $\{5333\}$, tiling of $\mathbb{H}^4$ by 120-cells.

Projection of 120-cell in 3-space (G.Hart)



$\{533\}$: 600 vertices, 120 dodecahedral facets, $|Aut| = 14400$

4- and 5-fullerenes

- All known finite 4-fullerenes are "mutations" of 120-cell by interfering in one of ways to construct it: tubes of 120-cells, coronas, inflation-decoration method, etc.
  Some putative facets: $\simeq \{5,6\}_v(G)$ with (v,G)=$(20,I_h)$, $(24,D_{6h})$, $(26,D_3)$, $(28,T_d)$, $(30,D_{5h})$, $(32,D_{3h})$, $(36,D_{6h})$.
- $(\{5,6\},3)$-$\mathbb{E}^3$: example of interesting infinite 4-fullerenes.

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- $(\{5,6\},3)$-$\mathbb{E}^3$: example of interesting infinite 4-fullerenes.
- All known 5-fullerenes come from $\{5333\}$'s by following ways.
  With 6-gons also: glue two $\{5333\}$'s on some 120-cells and delete their interiors. If it is done on only one 120-cell, it is $\mathbb{R} \times \mathbb{S}^3$ (so, simply-connected).
  Finite compact ones: the quotients of $\{5333\}$ by its symmetry group (partitioned into 120-cells) and gluings of them.

Quotient d-fullerenes

- Selberg, 1960, Borel, 1963: if a discrete group of motions of a symmetric space has a compact fundamental domain, then it has a torsion-free normal subgroup of finite index.
- So, the *quotient* of a d-fullerene by such symmetry group (its points are group orbits) is a finite d-fullerene.

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- Exp 2: Poincaré dodecahedral space: the quotient of 120-cell by I_h ; so, its f-vector is $(5, 10, 6, 1) = \frac{1}{120}$f(120-cell).
- Cf. 6-, 12-regular $\mathbb{H}^3$-tilings $\{5, 3, 4\}$, $\{5, 3, 5\}$ by $\{5, 6\}_{20}$ and 6-regular $\mathbb{H}^3$-tiling by (right-angled) $\{5, 6\}_{24}$.
  Seifert-Weber, 1933 and Löbell, 1931 spaces are quotients of last 2 with f-vectors $(1, 6, p_5{=}6, 1)$, $(24, 72, 48{+}8{=}p_5{+}p_6, 8)$.