Linear Complementarity, Unique-Sink Orientations, Oriented Matroids

Jan Foniok

[Komei Fukuda, Bernd Gärtner, Lorenz Klaus, Hans-Jakob Lüthi, Markus Sprecher]

Queen's UNIVERSITY

Conference on Discrete Geometry and Optimization
20 September 2011

Outline:

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A linear program…

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

… and its dual

$$\begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & A^T y \leq c \\ & y \geq 0 \end{aligned}$$

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with slack variables

$$\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ -b \end{pmatrix},$$

$$\begin{pmatrix} u \\ v \end{pmatrix} \geq 0, \quad \begin{pmatrix} x \\ y \end{pmatrix} \geq 0 \quad \text{and} \quad \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

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can be written as a *Linear Complementarity Problem*

Find w, z such that

$$w - Mz = q$$

$$w \geq 0, \quad z \geq 0 \quad \text{and} \quad w^T z = 0$$

Linear Complementarity Problem (LCP)

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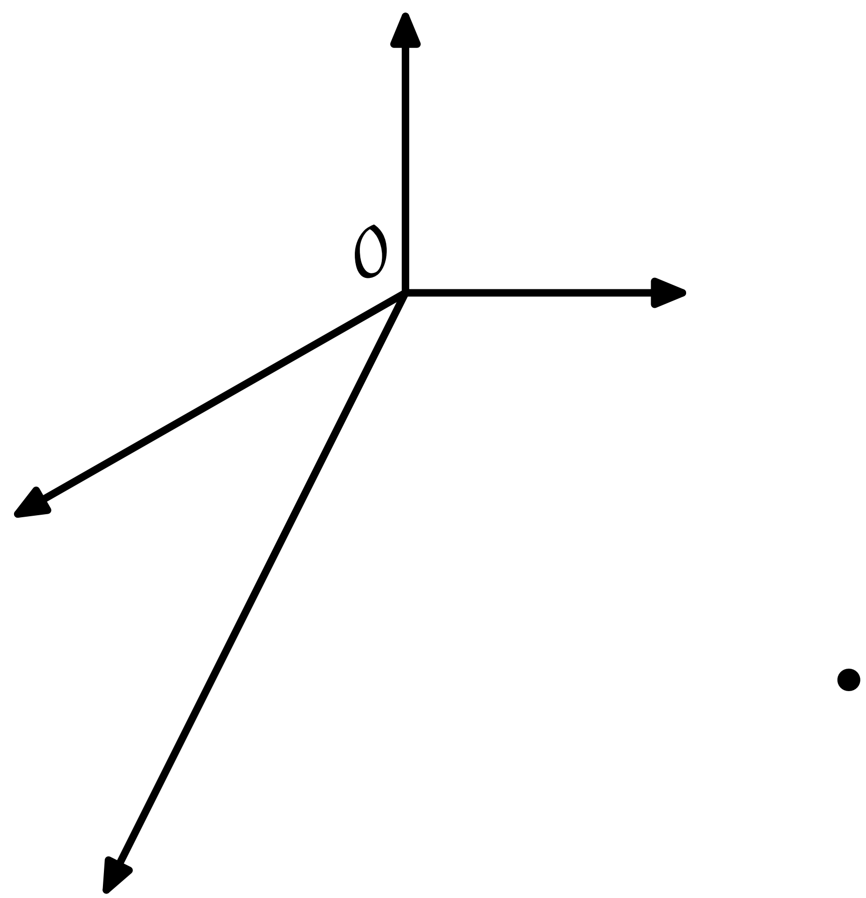
Sources

- linear programming
- quadratic programming
- two player games
- free boundary problems
- optimal stopping
- portfolio optimization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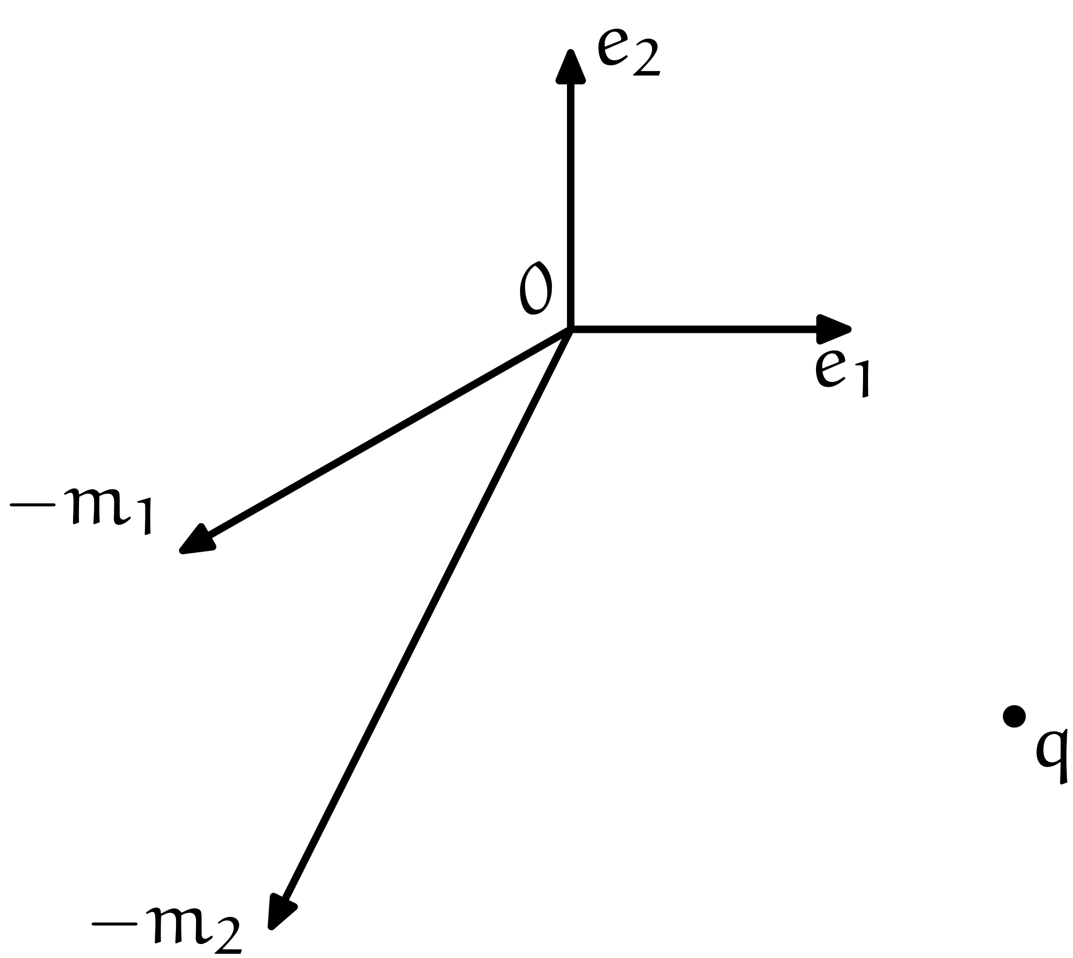
e_2
0
e_1
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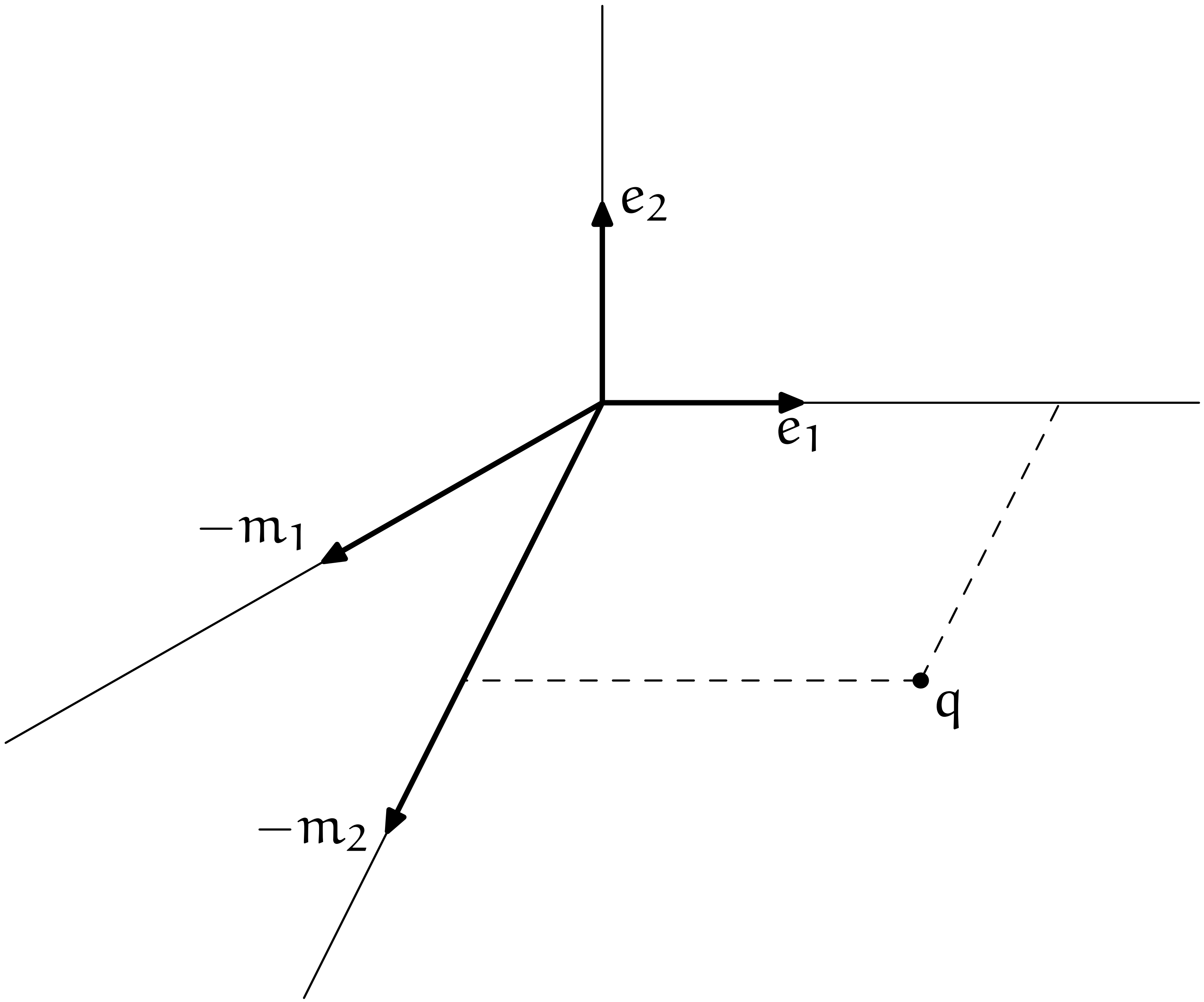
q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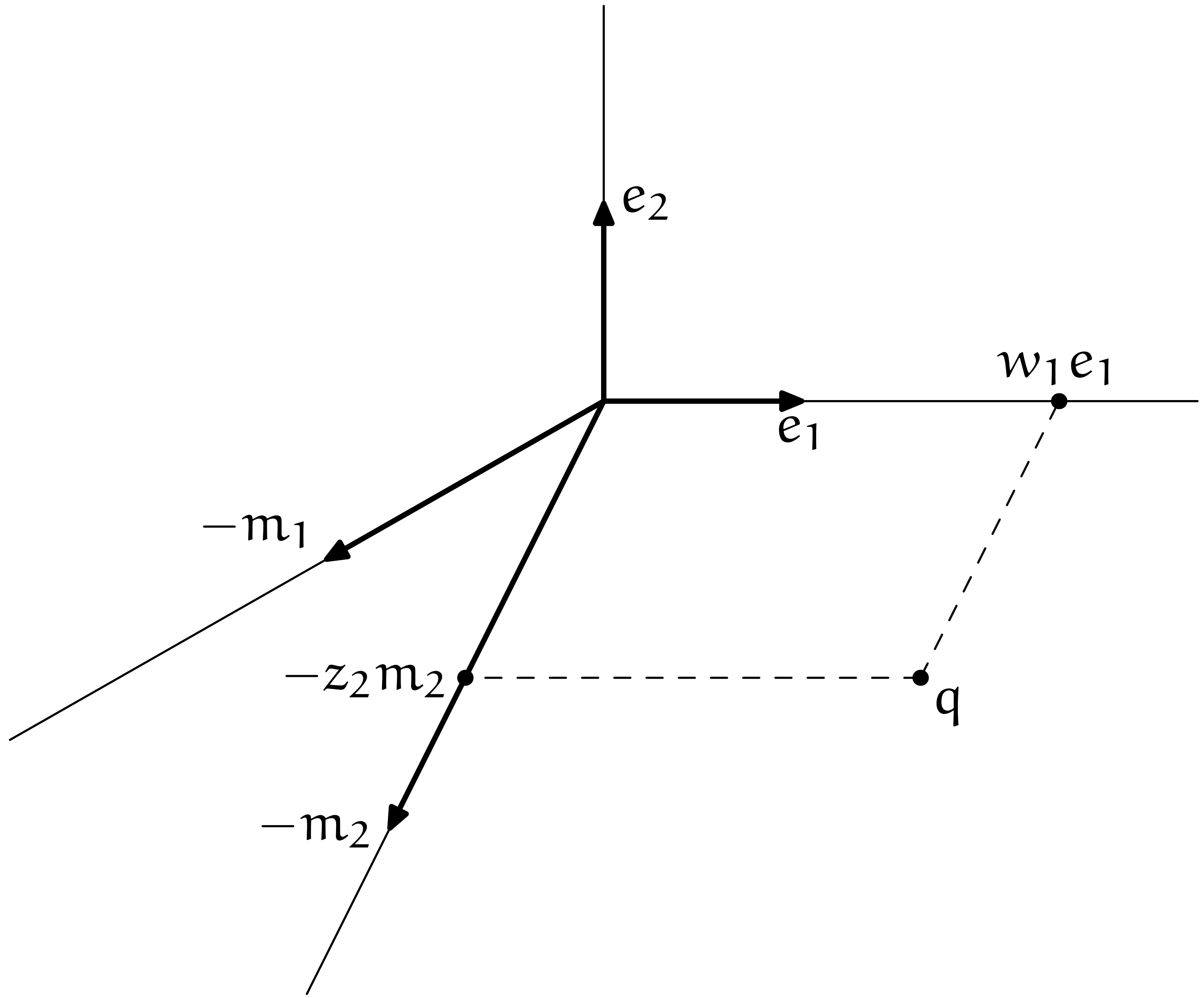

e_2
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e_1
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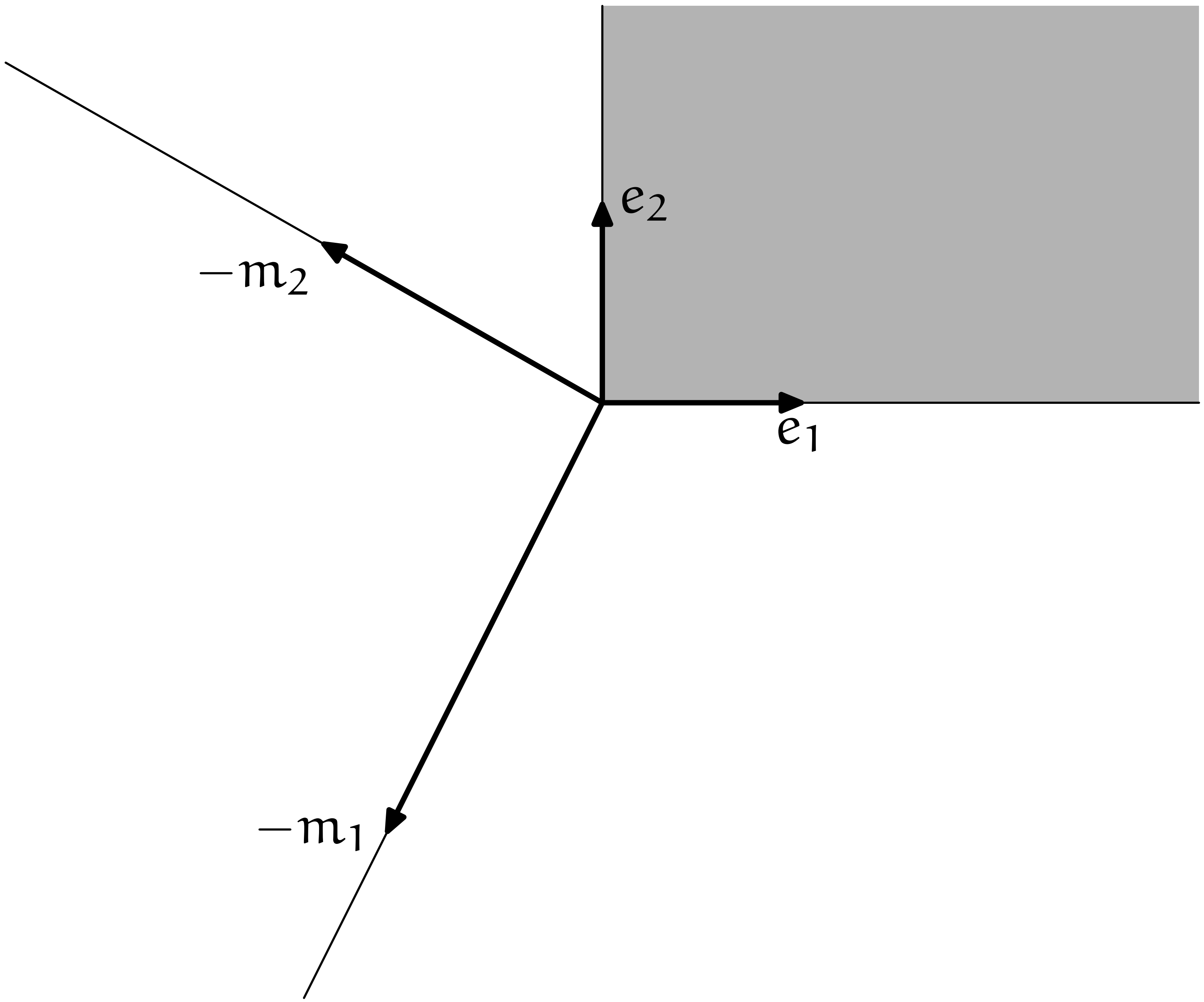
$-z_2 m_2$
q
$-m_2$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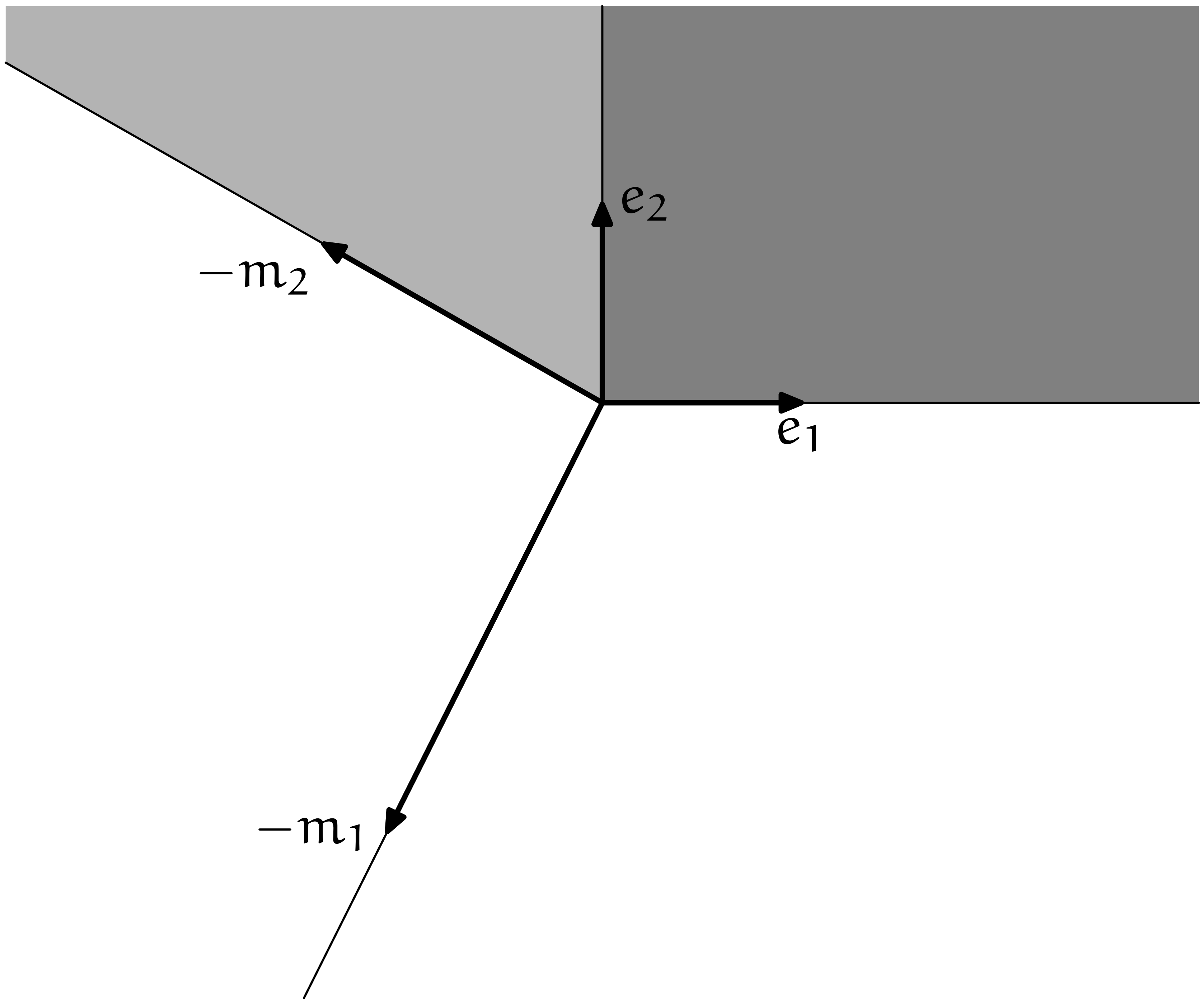
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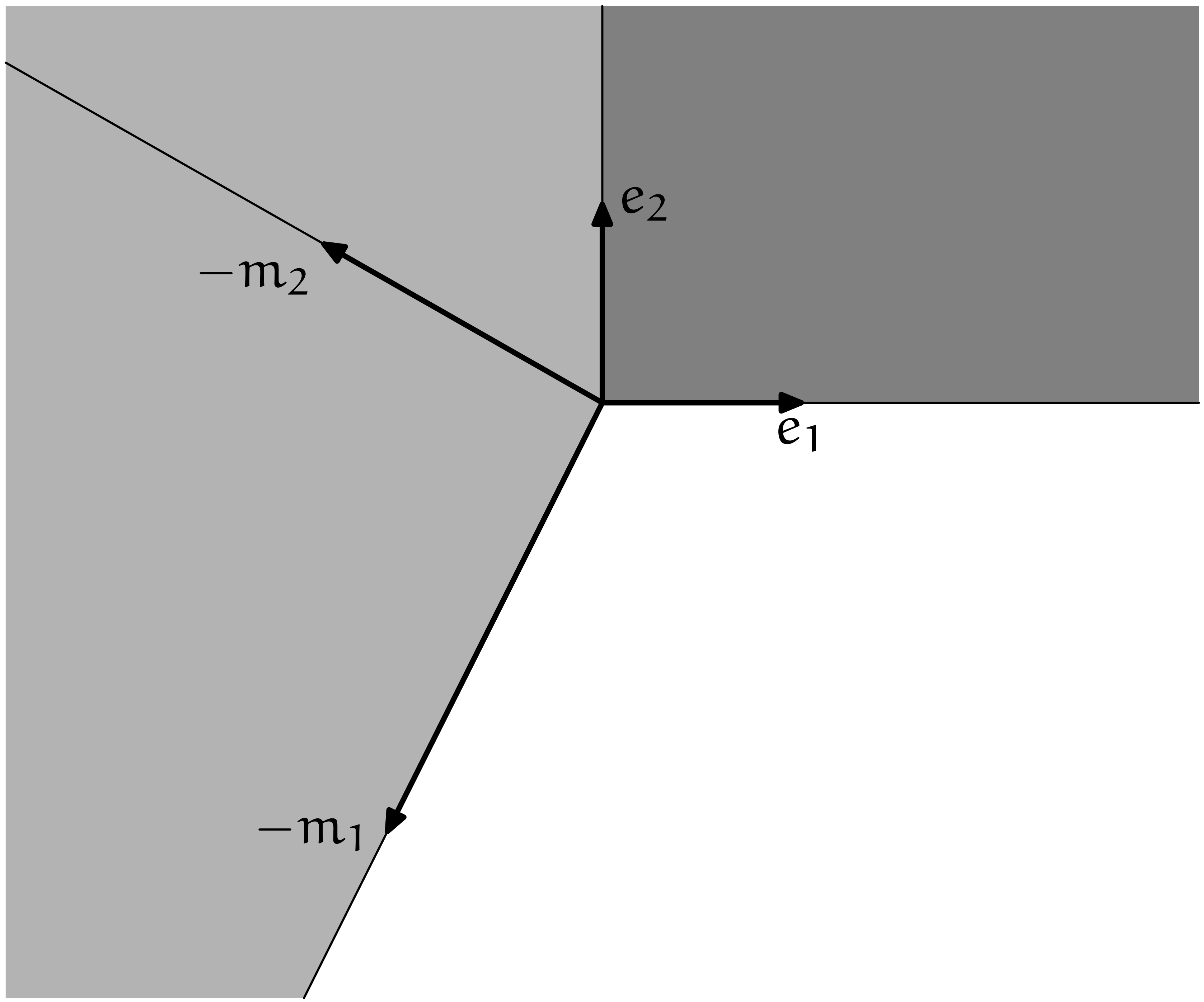
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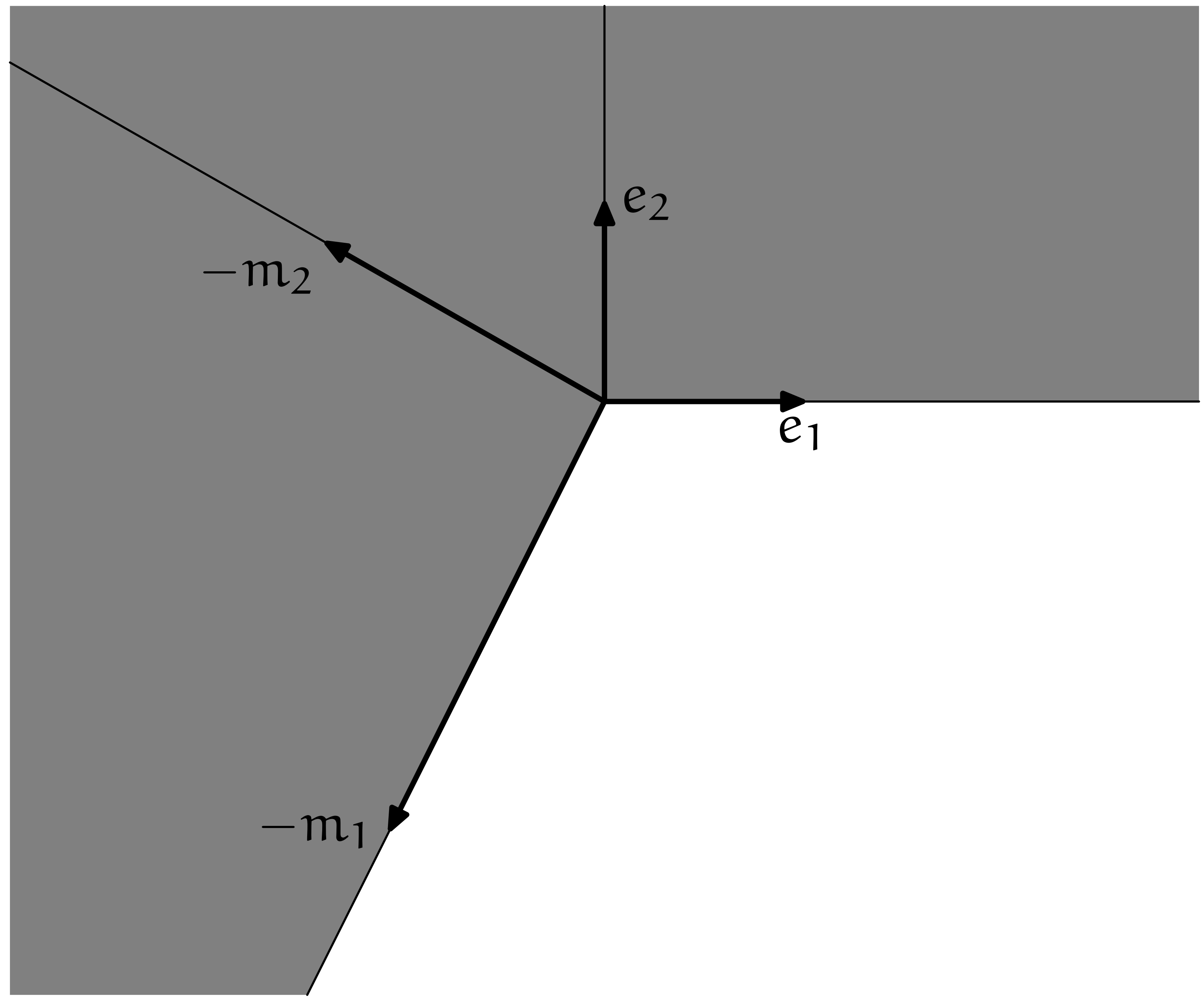

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Computational complexity [Chung, 1989]

It is NP-complete to decide whether a solution exists.

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Proof.

Reduction from the equality-constrained *knapsack problem*: Given a set $A = \{a_1, a_2, \ldots, a_n\}$ of positive integers and an integer b, decide whether there is a subset of A that sums to b.

The problem is equivalent to the following LCP: $w, z \geq 0$, $w^T z = 0$,

$$w_i + z_i = a_i \text{ for all } i = 1, \ldots, n,$$

$$w_{n+1} + z_{n+1} = b - \sum_{i=1}^{n} z_i,$$

$$w_{n+2} + z_{n+2} = -b + \sum_{i=1}^{n} z_i.$$

□

An important special case:

P-matrix: all principal minors positive

P-LCP: an LCP with a P-matrix

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Complexity:

- unlikely to be NP-hard
- in the class PPAD; not known to be PPAD-complete
- no polynomial algorithm known

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Theorem [Megiddo, 1988]

Consider the following problem:

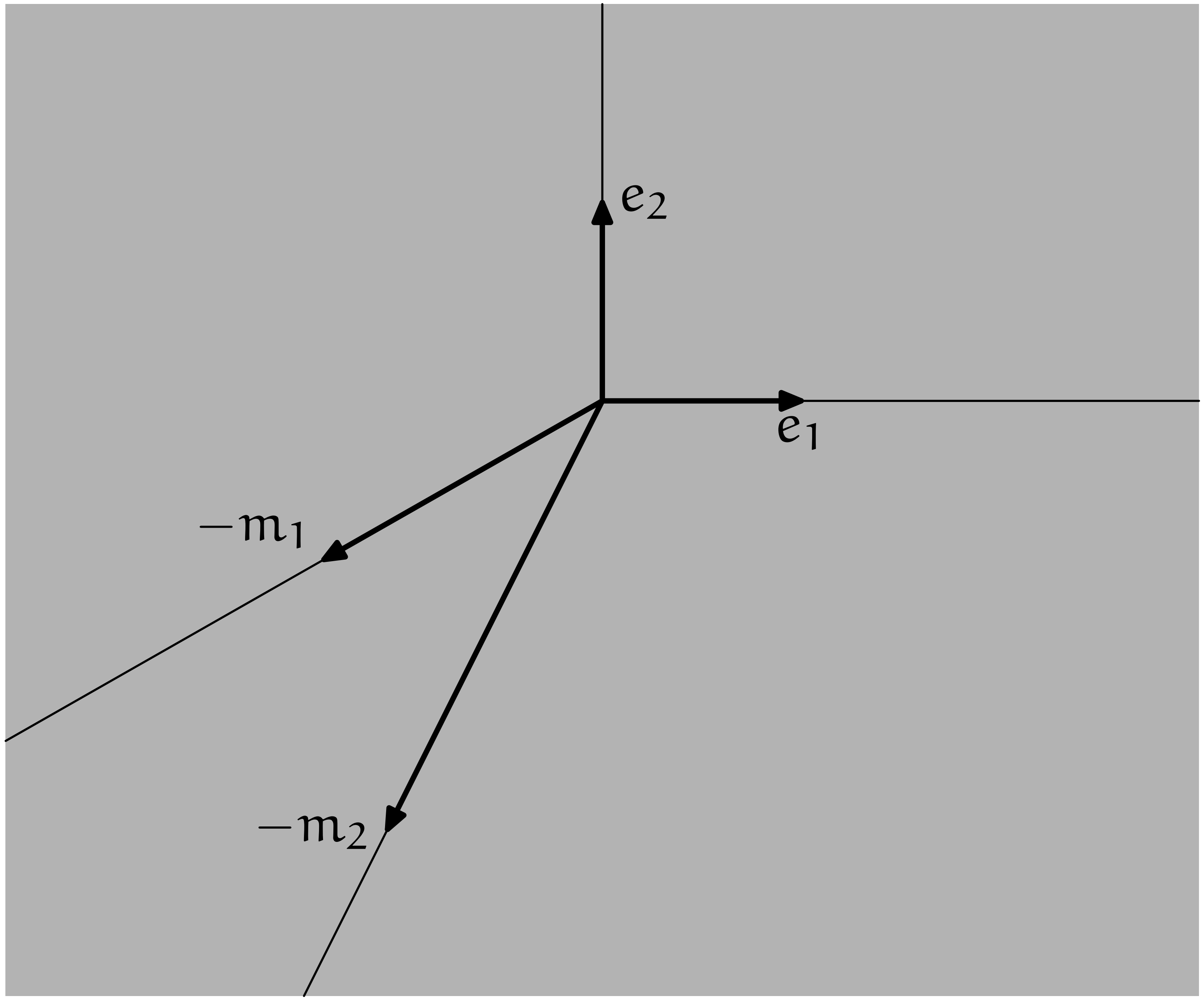
- *Given* M *and* q*, either find a solution* (w, z) *to* $\mathrm{LCP}(M, q)$*, or exhibit a non-positive principal minor of* M*.*

If this problem is NP-hard, then NP = co-NP.

Why are P-matrices interesting?

- LCP(M, q) has a unique solution for every vector q

  [Samelson, Thrall, Wesler 1958; Ingleton 1966]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  [Samelson, Thrall, Wesler 1958; Ingleton 1966]
- "nice" geometric properties
- unresolved complexity status
  - not NP-hard (?), PPAD (?)
  - squeezed between tractable **positive definite** matrices and NP-hard **P_0-matrices**
  - no polynomial algorithm known...
- actually arise in applications

Algorithms for LCPs

interior point: [Kojima, Megiddo, Mizuno, Noma, Wright, Ye, Yoshise, Zhang, ...]

- relax the condition $w^{\mathsf{T}}z = 0$
- minimize $w^{\mathsf{T}}z$ instead
- in some cases polynomial (e.g., convex)

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pivoting: [Lemke 1970, & many others since]

- works with complementary or almost complementary bases
- needs a pivot rule
- *can be purely combinatorial*

The issue of degeneracy

- $\mathrm{LCP}(M, q)$ is **degenerate** if q can be expressed as a linear combination of some $n-1$ columns of $(\, I \; -M \,)$
- for practical purposes, it may be a problem
- for theory, we always *assume* that our LCP is *non-degenerate*
- non-degeneracy may be achieved by a symbolic perturbation of q

The combinatorics of LCPs

$$q = w - Mz$$
$$w^T z = 0$$

The hard part: determine whether $w_i = 0$ or $z_i = 0$ for each i.
The rest is a system of linear equations.

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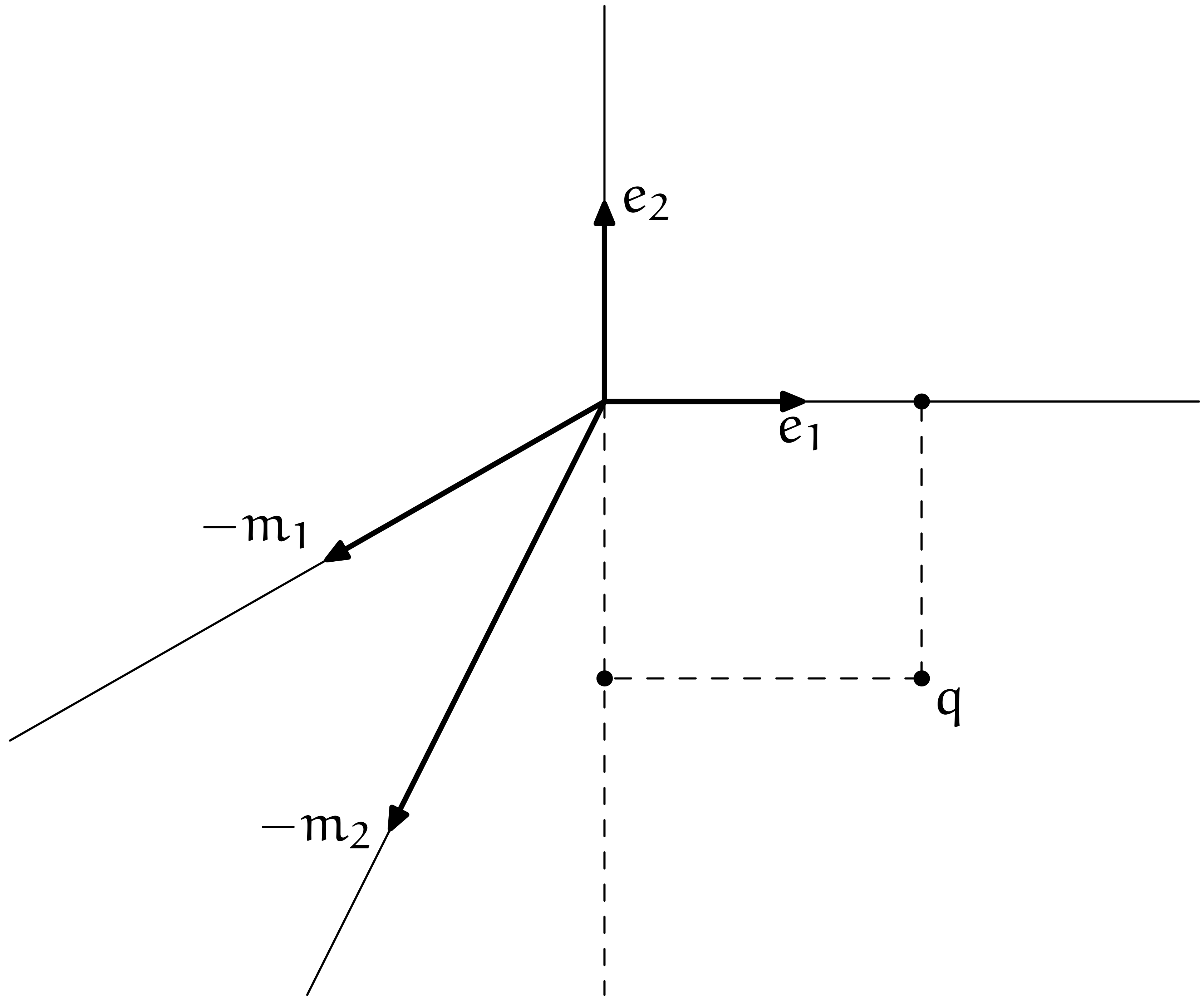
Simple principal pivoting methods

- start with an arbitrary *complementary basis*
- if not feasible, do a *principal pivot*:
  - insert a (negative) variable into the basis (*pivot rule!*)
  - remove the complementary variable from the basis
- repeat until solution is reached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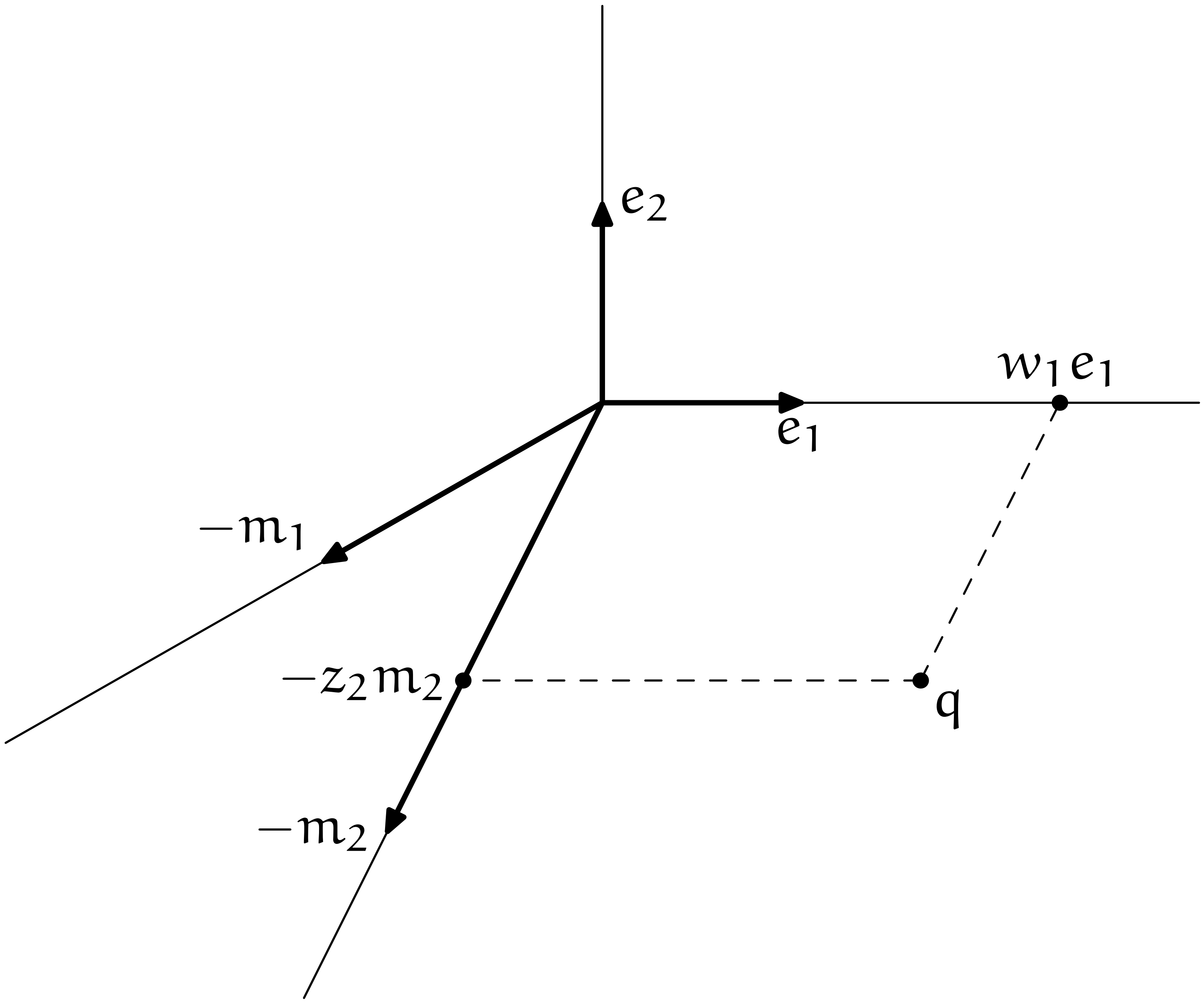

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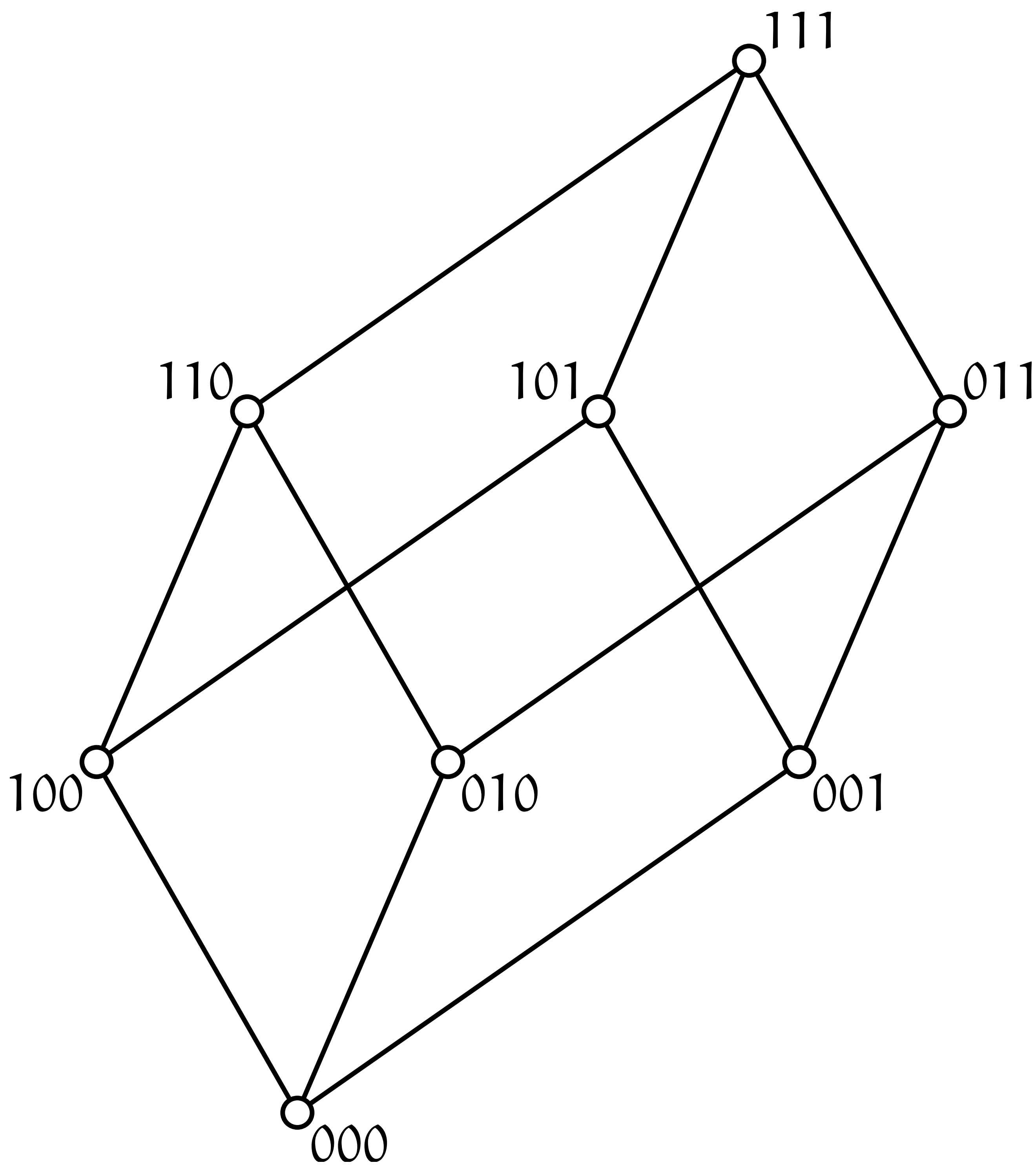
Unique-sink orientations



Unique-sink orientation — USO

an oriented graph with

- $V = \{0,1\}^n$
- $u \sim v$ iff in Hamming distance 1

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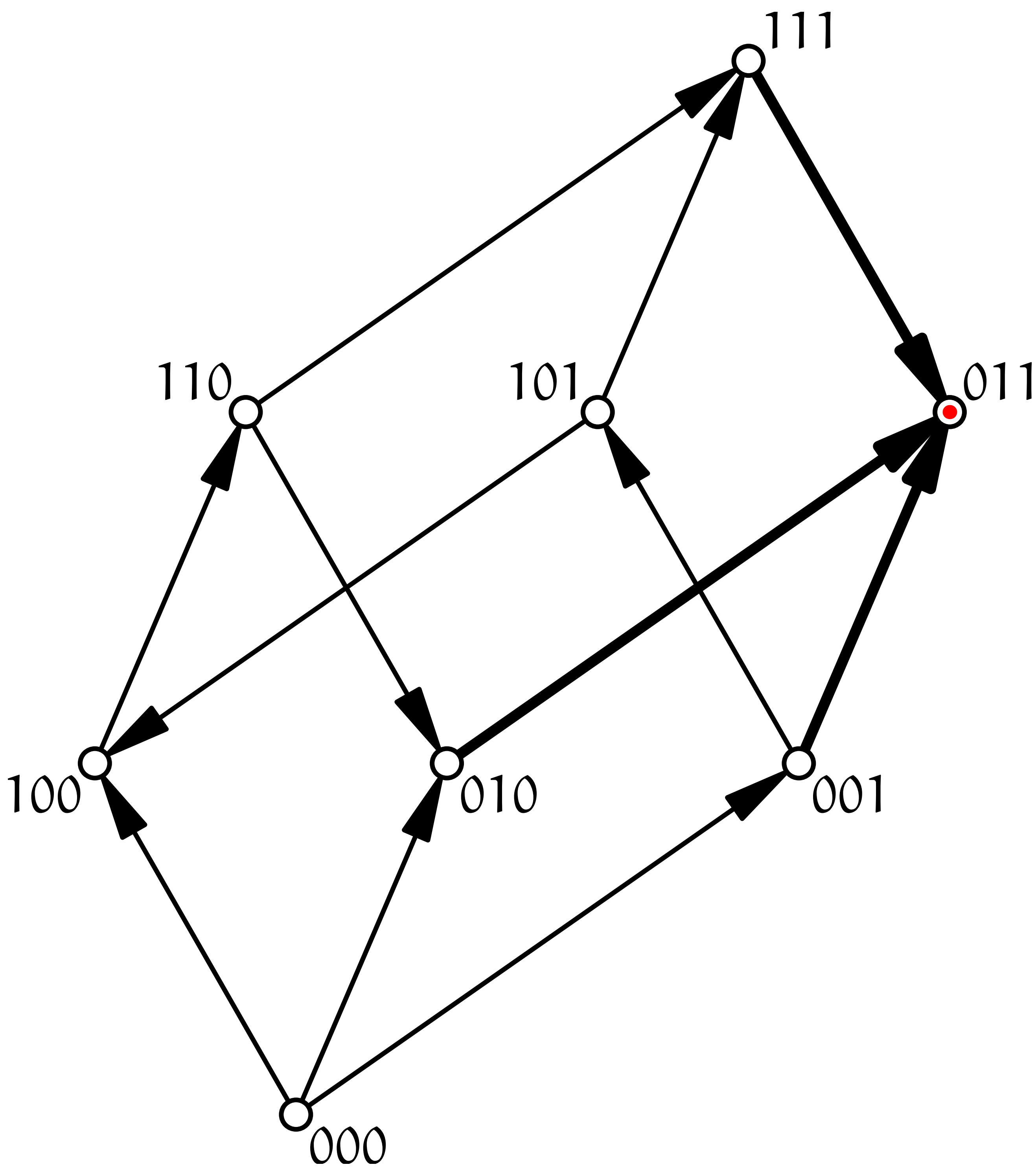

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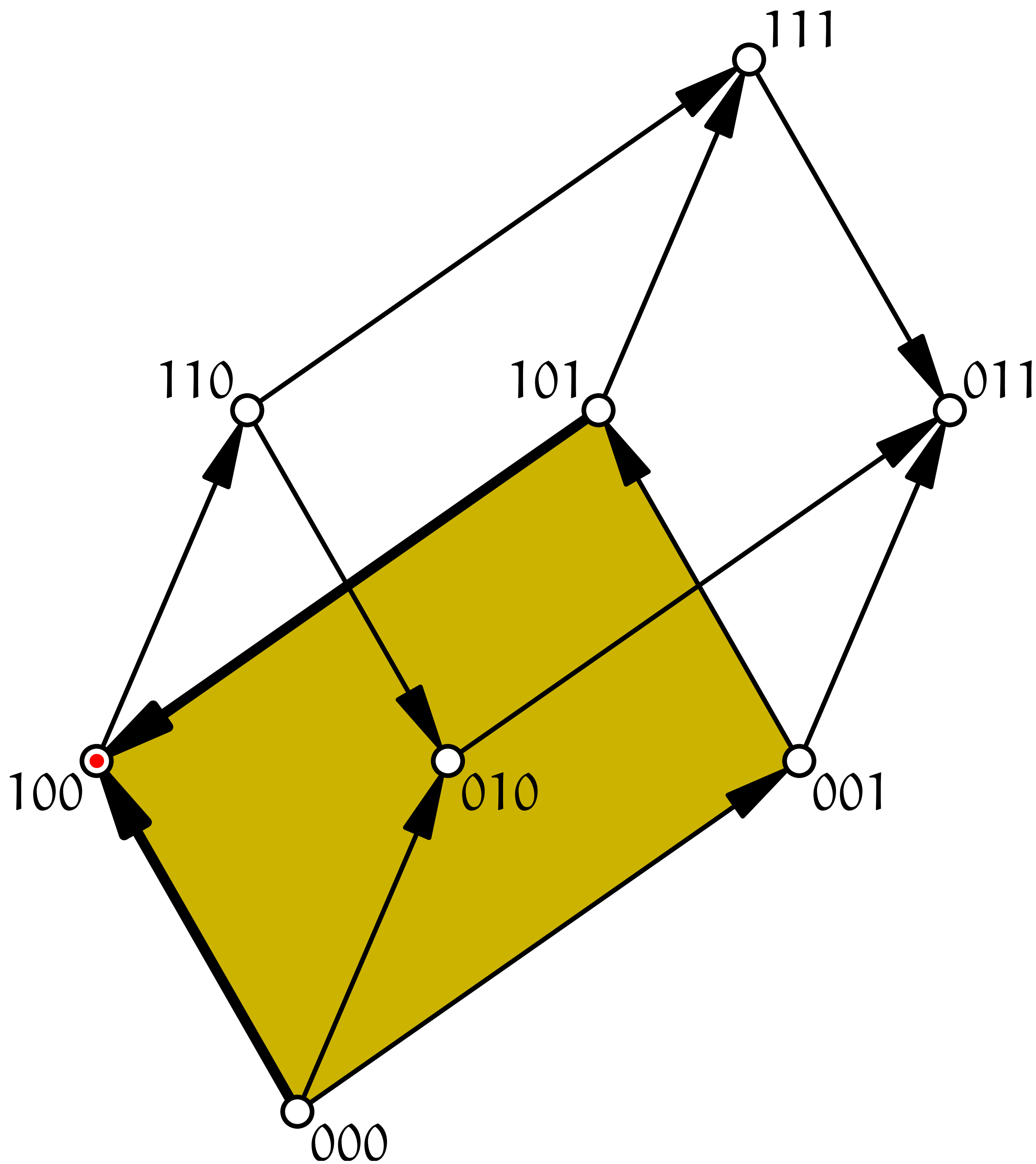

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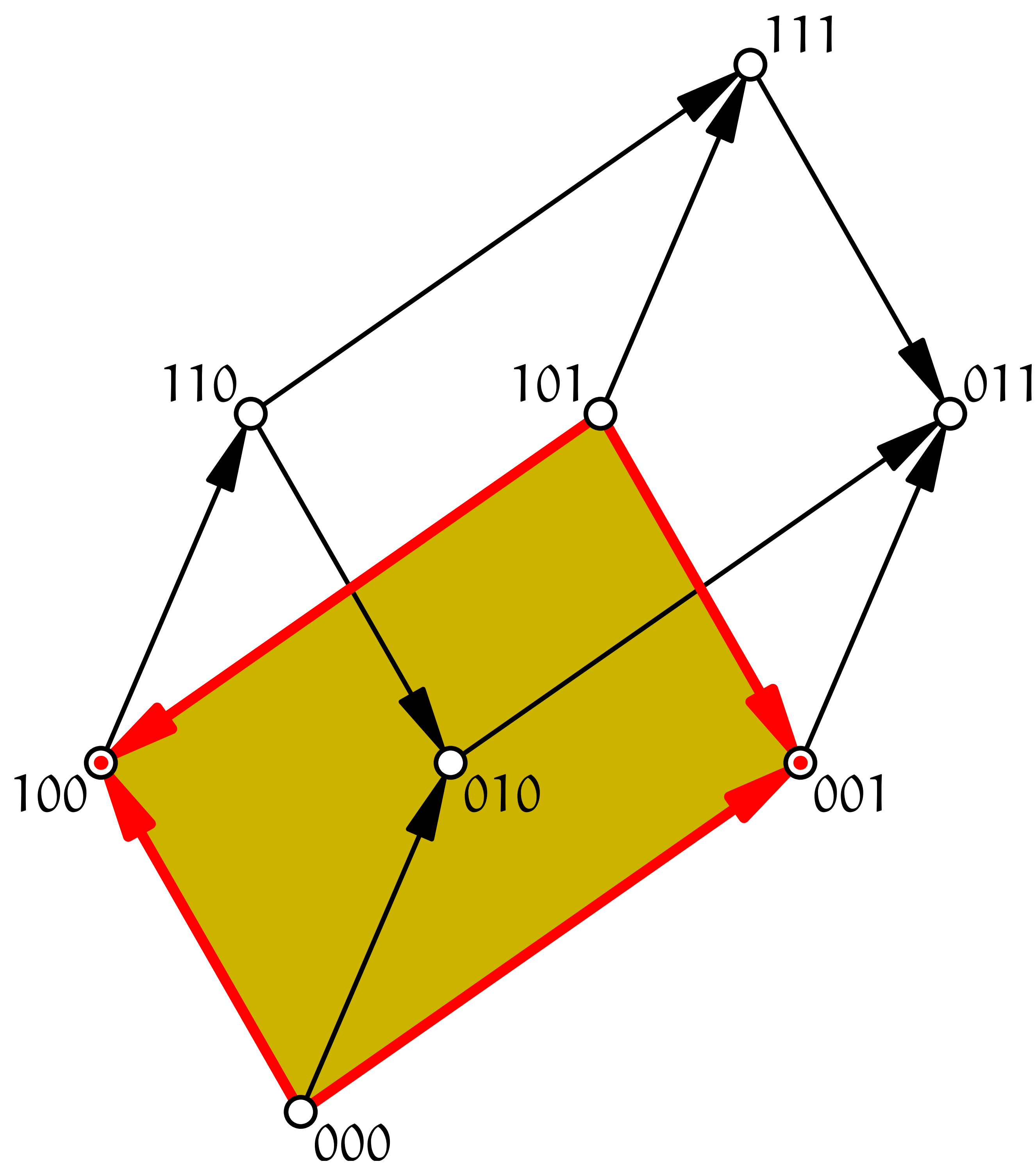

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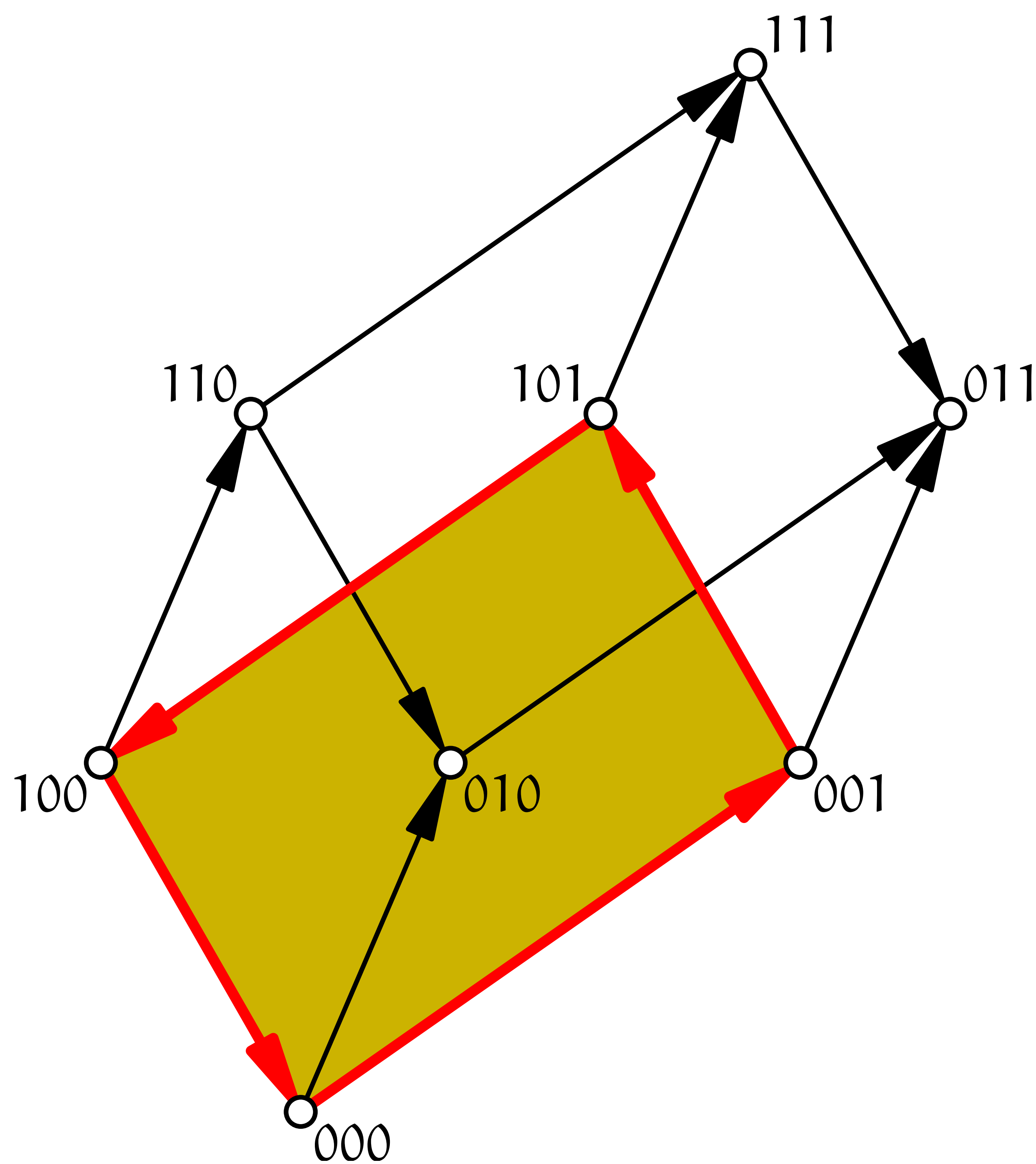

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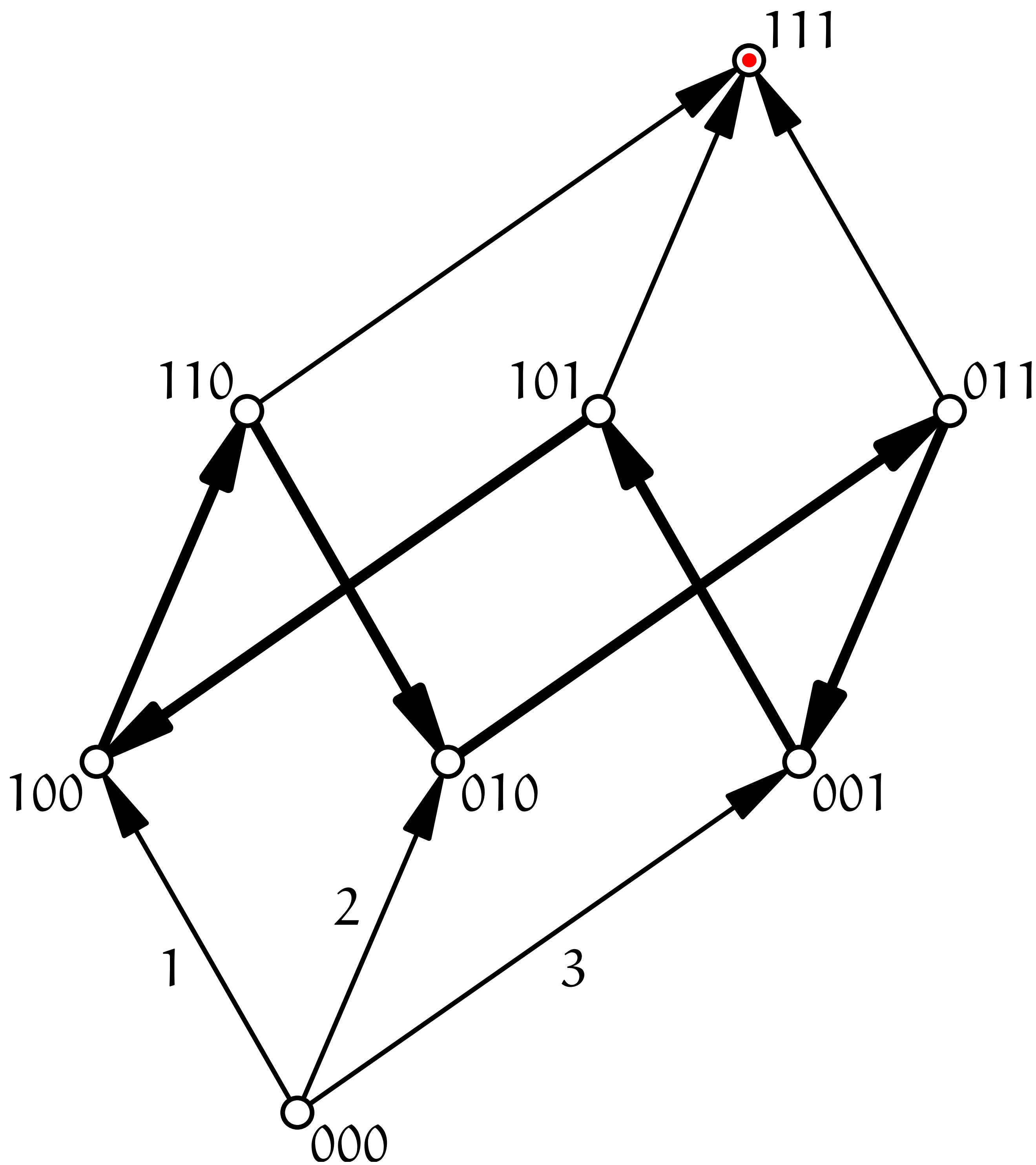

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So the whole cube must have a unique sink, but also proper subcubes, like this square. And not two sinks. And not none. Cycles may occur.

The combinatorics of LCPs

$$q = w - Mz$$
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The hard part: determine whether $w_i = 0$ or $z_i = 0$ for each i.

Inducing a USO

- a choice of $w_i = 0$ or $z_i = 0$ corresponds to a 0-1-vector
- 0-1-vectors are vertices of a hypercube
- solve equations: negative values → outgoing edges
- *for a P-matrix, this is a USO* [Stickney, Watson, 1978]
- find the sink → found the LCP solution

Goal: Find the sink

Input representation: by the vertex enumeration oracle: ask for the orientation of edges incident with a given vertex

Algorithm efficiency: number of oracle calls as function of dimension

Algorithms

Naive algorithm: check all vertices (2^n queries)

Path-following algorithms: simple principal pivoting

"Random access" algorithms: seesaw

Best general algorithms known to date

	deterministic	randomized
general USOs	1.609^n [Szabó, Welzl]	1.438^n [Szabó, Welzl, Rote]
acyclic USOs		$\exp(2\sqrt{n})$ [Matoušek, Sharir, Welzl, Gärtner]

Some matrix classes

P-matrix: all principal minors positive

K-matrix: P-matrix and all off-diagonal elements ≤ 0

and

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Theorem [F., Fukuda, Gärtner, Lüthi, 2009]

Any *path-following algorithm with* any *starting vertex finds the sink of any K-USO after at most* $2n + 1$ *oracle queries.*

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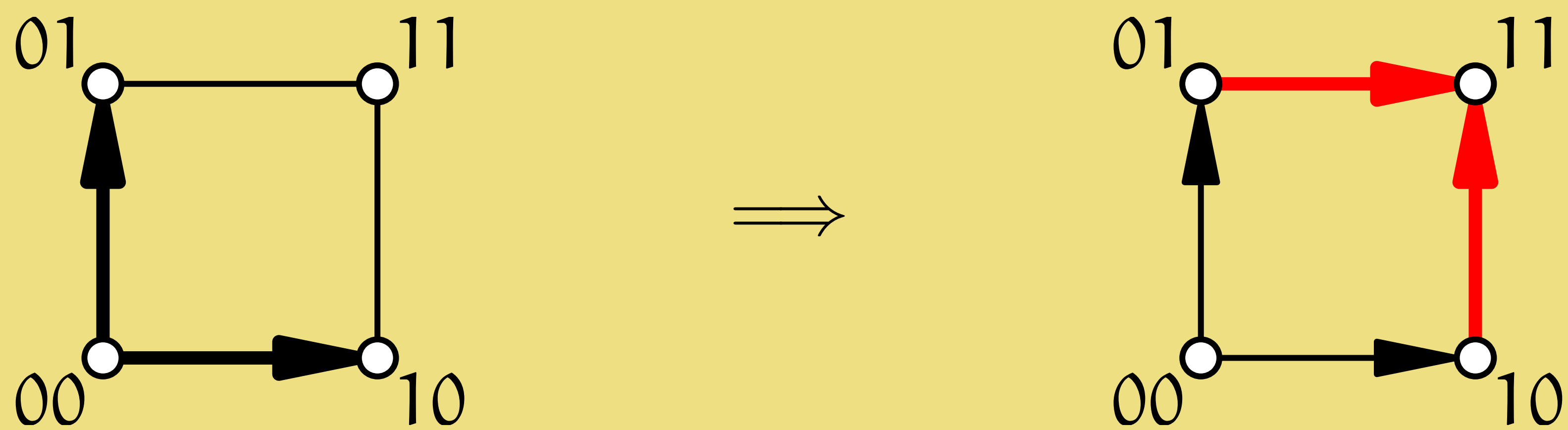
Lemma

In any K-USO:



The proof uses a K-matrix characterization of [Fiedler, Pták, 1962] but can also be done purely combinatorially (coming in a minute).

Does the "Lemma" *characterize* K-USOs?

No. Because:

There are at least $2^{2^{n/\operatorname{poly}(n)}}$ n-dimensional USOs satisfying the "Lemma", but at most $2^{O(n^3)}$ P-USOs.

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Proof of the upper bound [F., Gärtner, Klaus, Sprecher, 2010+].

The orientation is determined by the signs of $2^n \cdot n$ values of polynomials in the entries of M and q. Each of the polynomials has degree at most n.

Theorem [Warren, 1968]

The number of distinct (nowhere-zero) sign patterns of s real polynomials in k variables, each of degree at most d, is at most $(4eds/k)^k$.

□

Counting USOs

class	lower bound	upper bound
all USOs [Matoušek]	$n^{\Omega(2^n)}$	$n^{O(2^n)}$
acyclic USOs [Matoušek]	$2^{2^{n-1}}$	$(n+1)^{2^n}$
satisfying "Lemma"	$2^{2^n/\sqrt{n}}$	
Holt–Klee USOs [Develin]	$2^{2^n/\,\mathrm{poly}(n)}$	
P-USOs	$2^{\Omega(n^3)}$	$2^{O(n^3)}$
K-USOs	$2^{\Omega(n^3)}$	[F., Gärtner, Klaus, Sprecher, 2010+]

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Holt–Klee USOs

In every subcube of dimension d there are d vertex-disjoint directed paths from the (unique) source to the (unique) sink.

Every P-USO is Holt–Klee. [Gärtner, Morris, Rüst, 2008]

Counting USOs

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Lemma

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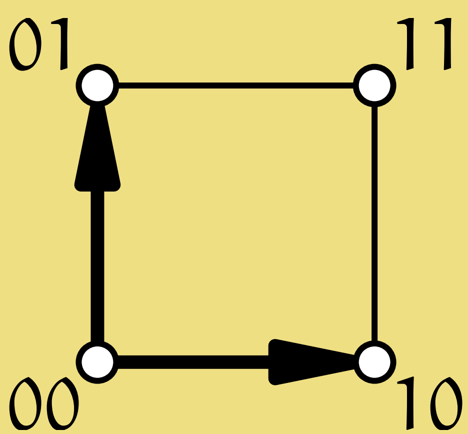
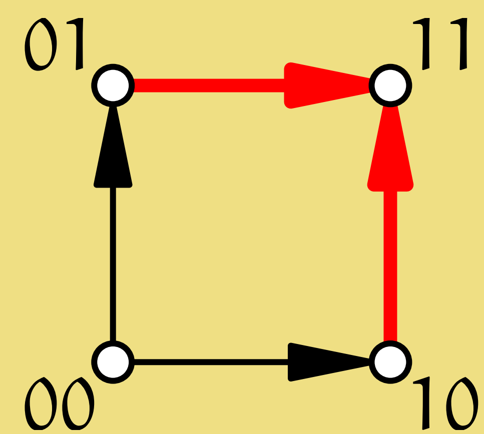


$\Longrightarrow$



Many K-USOs

The K-matrix:

$$M(\beta) = \begin{pmatrix} 1 & -1-\beta_{1,2} & -1-\beta_{1,3} & \dots & -1-\beta_{1,n} \\ 0 & 1 & -1-\beta_{2,3} & \dots & -1-\beta_{2,n} \\ \hdotsfor{5} \\ 0 & 0 & 0 & \dots & -1-\beta_{n-1,n} \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

The right-hand side:

$$q = (-1, 1, -1, \dots, (-1)^n)^T$$

There are $2^{\Omega(n^3)}$ choices for $\beta_{i,j}$, each resulting in a different USO.

Deterministic vs. randomized pivot rules

- There tends to be a “bad example” – a slow P-USO – for any studied deterministic pivot rule.
- Therefore examine *randomized pivot rules*, analyze *expected* running time.

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Some randomized pivot rules

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Some randomized pivot rules

RANDOM EDGE chooses the outgoing edge uniformly at random.

RANDOMIZED MURTY chooses a permutation of the indices uniformly at random at the beginning, then in every pivot step chooses the outgoing edge with the minimum index with respect to this permutation.

Morris's slow example for Random Edge

Consider the LCP(M, q) with n odd,

$$M = \begin{pmatrix} 1 & 2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 & 2 \\ 2 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}, \qquad q = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \\ -1 \end{pmatrix}. \qquad (*)$$

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Theorem [Morris, 2002]

Random Edge takes at least $((n-1)/2)!$ iterations in expectation to solve $(*)$.

Morris's slow example for RANDOM EDGE

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RANDOM EDGE takes at least $((n-1)/2)!$ iterations in expectation to solve $(*)$.

Theorem [F., Fukuda, Gärtner, Lüthi, 2009]

RANDOMIZED MURTY starting in *any* vertex of the cube takes at most $2n^2 - (5n-3)/2$ steps to solve $(*)$.

Oriented matroids

$$\begin{aligned} w - Mz &= q \\ w, z &\geq 0 \\ \\ w^T z &= 0 \end{aligned} \quad \longleftrightarrow \quad \begin{aligned} \begin{bmatrix} I & -M & -q \end{bmatrix} x &= 0 \\ x_i &\geq 0 \quad \forall i \in [2n] \\ x_{2n+1} &> 0 \\ x_i \cdot x_{i+n} &= 0 \quad \forall i \in [n] \end{aligned}$$

Oriented matroids

- $\hat{\mathcal{V}} = \{\operatorname{sgn} x : \begin{bmatrix} I & -M & -q \end{bmatrix} x = 0\}$
- $\operatorname{sgn} x$ is a vector in $\{-, 0, +\}^{2n+1}$ defined as $(\operatorname{sgn} x)_i := \operatorname{sgn} x_i$
- The collection $\hat{\mathcal{V}}$ of sign vectors is the set of vectors of an oriented matroid on $2n + 1$ elements.

What is an *oriented matroid* $\mathcal{M}$?

- a set E of elements
- $\mathcal{V}$, a set of vectors; $\mathcal{V} \subseteq \{-, 0, +\}^E$
  (V1) $0 \in \mathcal{V}$.
  (V2) If $X \in \mathcal{V}$, then $-X \in \mathcal{V}$.
  (V3) If $X, Y \in \mathcal{V}$, then $X \circ Y \in \mathcal{V}$.
  (V4) If $X, Y \in \mathcal{V}$ and $e \in X^+ \cap Y^-$, then there exists $Z \in \mathcal{V}$ with $Z^+ \subseteq X^+ \cup Y^+$, $Z^- \subseteq X^- \cup Y^-$, $Z_e = 0$, and $(\underline{X} \setminus \underline{Y}) \cup (\underline{Y} \setminus \underline{X}) \cup (X^+ \cap Y^+) \cup (X^- \cup Y^-) \subseteq \underline{Z}$.
- $\mathcal{C}$, the set of circuits; these are vectors with minimal support
- a basis is a set of elements that contains the support of no vector

Complementarity in oriented matroids

- the set E of elements has complementary pairs (w_i, z_i)
- matroid and its one-element extension:
  - $\mathcal{V} = \{\operatorname{sgn} x : \begin{bmatrix} I & -M \end{bmatrix} x = 0\}$
  - $\hat{\mathcal{V}} = \{\operatorname{sgn} x : \begin{bmatrix} I & -M & -q \end{bmatrix} x = 0\}$
- the oriented matroid complementarity problem is to find in $\hat{\mathcal{V}}$ a vector like this:

0	0	+	+
+	+	0	

Let $\mathcal{V} = \{\operatorname{sgn} x : \begin{bmatrix} I & -M \end{bmatrix} x = 0\}$ where M is a P-matrix.

Lemma (†) [Todd, 1984]

For every sign vector $X \in \mathcal{V}$ there is a an index i such that $X_{w_i} \cdot X_{z_i} = +$.

Let $\mathcal{V} = \{\operatorname{sgn} x : \begin{bmatrix} I & -M \end{bmatrix} x = 0\}$ where M is a P-matrix.

Lemma (†) [Todd, 1984]

For every sign vector $X \in \mathcal{V}$ there is a an index i such that $X_{w_i} \cdot X_{z_i} = +$.

Let $\mathcal{V} = \{\operatorname{sgn} x : \begin{bmatrix} I & -M \end{bmatrix} x = 0\}$ where M is a K-matrix.

Lemma

For every sign vector $X \in \mathcal{V}$, we have

(i) Lemma (†) holds and

(ii) If $X_Z \geq 0$, then whenever $X_{w_i} = +$, then also $X_{z_i} = +$

$-$	0	$+$	$+$
$+$	0	0	$+$

$-$	$+$	0	$+$
$+$	0	0	$-$

A theorem of Fiedler–Pták

A theorem of Fiedler–Pták

Чехословацкий математический журнал т. 12 (87) 1962, Прага

ON MATRICES WITH NON-POSITIVE OFF-DIAGONAL ELEMENTS AND POSITIVE PRINCIPAL MINORS

MIROSLAV FIEDLER and VLASTIMIL PTÁK, Praha

(Received July 28, 1960)

The authors study a class of matrices which occur frequently in applications to convergence properties of iteration processes in linear algebra and spectral theory of matrices.

(4,3) Theorem. *Let $A \in \mathbf{Z}$. Then the following conditions are equivalent to each other:*

1° *There exists a vector $x \geqq 0$ such that $Ax > 0$;*

2° *there exists a vector $x > 0$ such that $Ax > 0$;*

3° *there exists a diagonal matrix D with positive diagonal elements such that $ADe > 0$ (here e is the vector whose all coordinates are 1);*

4° *there exists a diagonal matrix D with positive diagonal elements such that the matrix $W = AD$ is a matrix with dominant positive principal diagonal;*

5° *for each diagonal matrix R such that $R \geqq A$ the inverse R^{-1} exists and $\sigma(R^{-1}(P - A)) < 1$, where P is the diagonal of A;*

6° *if $B \in \mathbf{Z}$ and $B \geqq A$, then B^{-1} exists;*

7° *each real proper value of A is positive;*

8° *all principal minors of A are positive;*

9° *there exists a strictly increasing sequence $0 \neq M_1 \subset M_2 \subset \ldots \subset M_n = N$ such that the principal minors $\det A(M_i)$ are positive;*

10° *there exists a permutation matrix P such that PAP^{-1} may be written in the form RS where R is a lower triangular matrix with positive diagonal elements such that $R \in \mathbf{Z}$ and S is an upper triangular matrix with positive diagonal elements such that $S \in \mathbf{Z}$;*

11° *the inverse A^{-1} exists and $A^{-1} \geqq 0$;*

12° *the real part of each proper value of A is positive;*

13° *for each vector $x \neq 0$ there exists an index k such that $x_k y_k > 0$ for $y = Ax$.*

Theorem (The combinatorial Fiedler–Pták theorem [F., Fukuda, Klaus, '11])

Let every sign vector $X \in \mathcal{V}$ satisfy:

(ii) If $X_Z \geq 0$, then whenever $X_{w_i} = +$, then also $X_{z_i} = +$

Then the following statements are equivalent:

(a) $\forall X \in \mathcal{V}$ *there is an index* i *such that* $X_{w_i} \cdot X_{z_i} = +$.

(b) $\exists X \in \mathcal{V} : X_Z \geq 0$ *and* $X_W > 0$

(c) $\exists X \in \mathcal{V} : X > 0$

(d) $\forall X \in \mathcal{V} : X_W \geq 0 \implies X_Z \geq 0$

(a)* $\forall Y \in \mathcal{V}^*$ *there is an index* i *such that* $Y_{w_i} \cdot Y_{z_i} = -$.

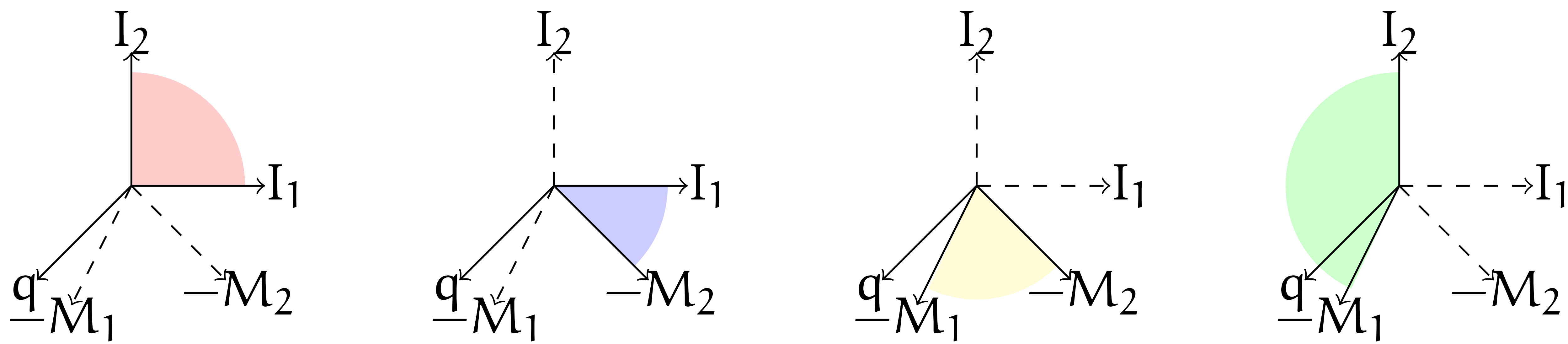
(b)* $\exists Y \in \mathcal{V}^* : Y_W \leq 0$ *and* $Y_Z > 0$

(c)* $\exists Y \in \mathcal{V}^* : Y_W < 0$ *and* $Y_Z > 0$

(d)* $\forall Y \in \mathcal{V}^* : Y_Z \geq 0 \implies Y_W \leq 0$

SimplePrincipalPivoting algorithm

$\text{cone}(\{I_1, I_2\})$ $\text{cone}(\{I_1, -M_2\})$ $\text{cone}(\{-M_1, -M_2\})$ $\text{cone}(\{-M_1, I_2\})$



w_1	w_2	q
−	−	+
0	0	
z_1	z_2	

w_1	w_2	q
−	0	+
0	+	
z_1	z_2	

w_1	w_2	q
0	0	+
+	−	
z_1	z_2	

w_1	w_2	q
0	+	+
+	0	
z_1	z_2	

Pivoting on P-matroids [Todd, 1984]

In every pivot step i*, we have:*

X^i

0	0	−	+
+	−	0	

⟶

X^{i+1}

0	⊕	?	+
?	0	0	

Note that $X^i, X^{i+1} \in \hat{\mathcal{V}} = \{\operatorname{sgn} x : \begin{bmatrix} I & -M & -q \end{bmatrix} x = 0\}$.

Pivoting on K-matroids [F., Fukuda, Klaus, 2011]

SimplePrincipalPivot behaves as follows:

X^i							X^k				
	0	+	0	+	$\longrightarrow$	$\cdots$ $\longrightarrow$		0	?	?	+
	⊕	0	−					⊕	?	?	

We find an upper bound on the number of pivot steps on each complementary pair (w_i, z_i).
The worst case scenario is:

$\cdots$	$\longrightarrow$	0	$\longrightarrow$	⊕	$\longrightarrow$	$\cdots$	$\longrightarrow$	−	$\longrightarrow$	0	$\longrightarrow$	0	$\longrightarrow$
		−		0				0		⊕		⊕	

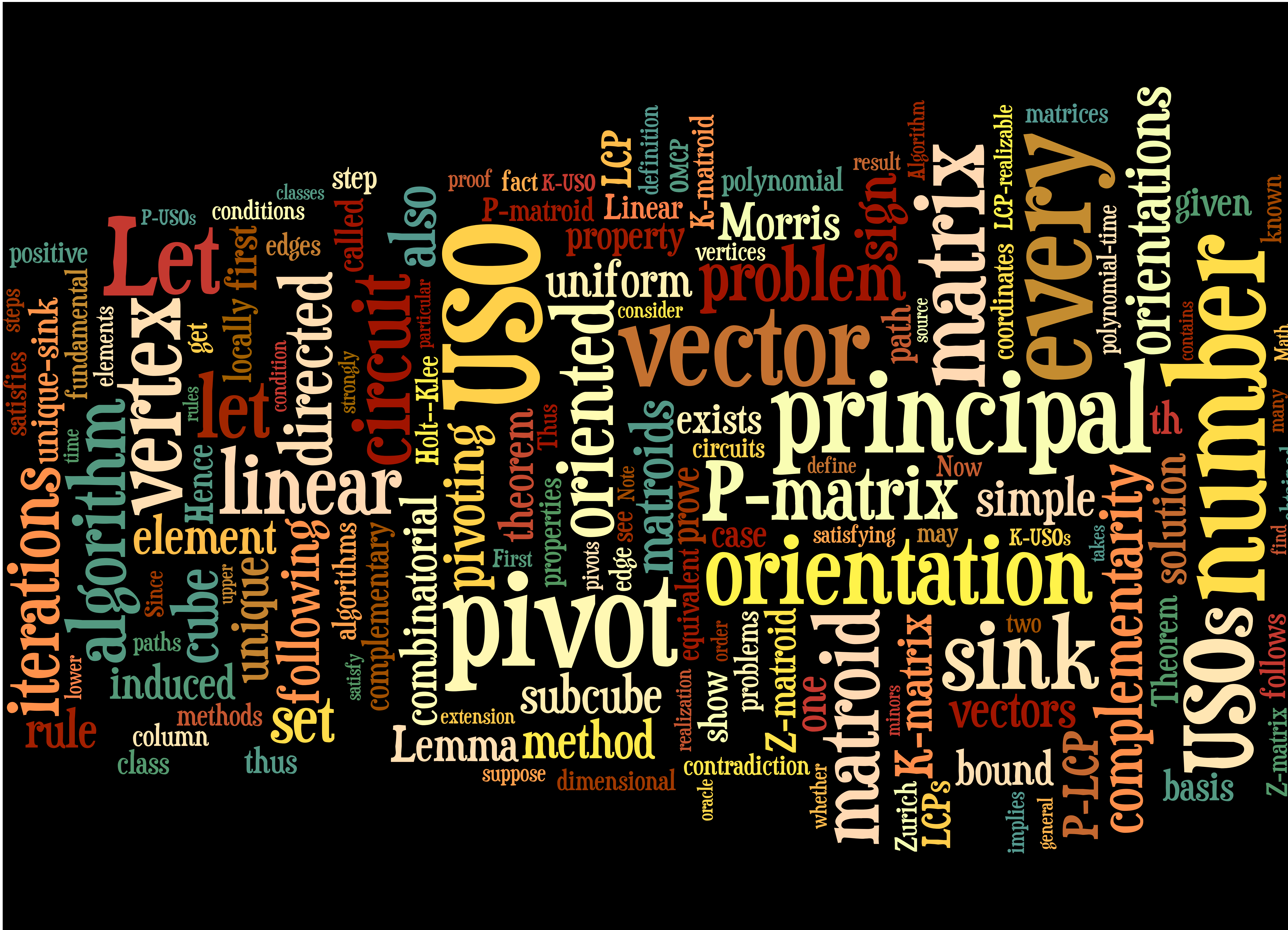
SimplePrincipalPivot needs at most two pivot steps for each complementary pair.

Summary

- LCPs hard in general
- LCPs with P-matrices: much studied, but embarrassingly open complexity status
- unique-sink orientations as tools to study pivoting algorithms
- oriented matroids capture the combinatorial structure (no numbers)
- purely combinatorial proofs possible
- interplay of several areas of mathematics
    - linear algebra & (continuous) geometry
    - discrete geometry
    - algebraic geometry
    - combinatorics & order theory

Some open problems

- **complexity:** Are P-matrix LCPs PPAD-complete?
- a **subexponential algorithm** for general USOs?
- the **Holt–Klee** condition on USOs: find sink in polynomially many steps?
- better **lower bounds** for solving USOs
- identify **new matrix classes** with polynomial LCPs
- **strongly polynomial** algorithm for linear programming ?!? 


positive
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