Universal Rigidity Theory and Semidefinite Programming for Sensor Network Localization

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Joint work with Biswas, So, Alfakih, Taheri, ...

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Sensor Network Localization or Graph Realization

Given a graph $G = (V, E)$ and a set of non–negative weights, $\{d_{ij} : (i,j) \in E\}$ on edges, the goal is to compute a localization of G in the Euclidean space $\mathbf{R}^d$ for a given low dimension d. That is,

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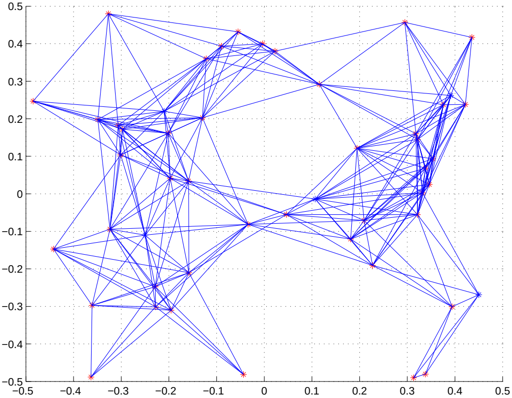
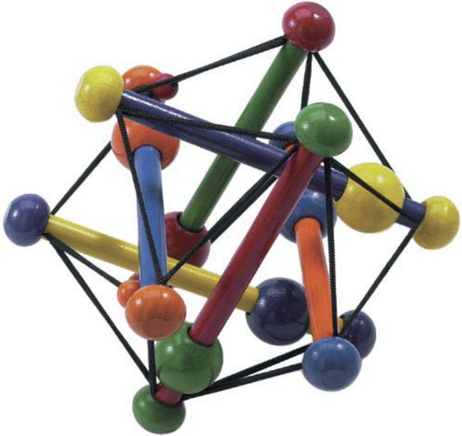
- to position the vertexes of G in $\mathbf{R}^d$ such that
- the Euclidean distance between a pair of adjacent vertexes (i,j) equals to (or bounded by) the prescribed weight $d_{ij} \in E$.

Sometimes, the positions of a few vertexes are known and they are called anchors.

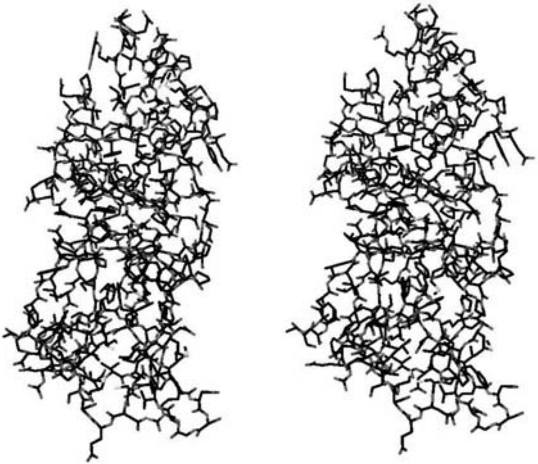
50-vertex 2-D Sensor Network Localization



3-D Tensegrity Network; a Toy Example by Anstreicher



Molecular Conformation: 1F39(1534 atoms) with 85% of distances below 6Å and 10% noises



Quadratic Equality and Inequality Systems

Given network (G, D), find $\mathbf{x}_j \in \mathbf{R}^d$ such that

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Or given anchors $\mathbf{a}_k \in \mathbf{R}^d$, $d_{ij} \in N_x$, and $\hat{d}_{kj} \in N_a$, find $\mathbf{x}_i \in \mathbf{R}^d$ such that

$$\begin{aligned} \|\mathbf{x}_i - \mathbf{x}_j\|^2 \quad (\leq) = (\geq) \quad & d_{ij}^2, \; \forall \, (i,j) \in N_x, \; i < j, \\ \|\mathbf{a}_k - \mathbf{x}_j\|^2 \quad (\leq) = (\geq) \quad & \hat{d}_{kj}^2, \; \forall \, (k,j) \in N_a; \end{aligned}$$

that is, edge (ij) (or (kj)) connects sensors i and j (or anchor k and sensor j) with the Euclidean length equal to d_{ij} (or $\hat{d}_{kj}$).

Key Questions Related to SNL

Consider a bar SNL problem:

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- Is the network partially localizable with a certification?

Rigidity Notions for SNL, Connelly ...

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- Strong Rigidity (SR): let $P = [\mathbf{p}_1, \ldots, \mathbf{p}_n]$ be a localization and $\mathbf{e}$ be the vector of all ones, and extended matrix

$$A = \begin{bmatrix} P \\ e^T \end{bmatrix}.$$

Then, there is a rank $n - d - 1$ and positive semidefinite stress matrix for SNL such that

$$S_{ij} = 0, \ \forall (i,j) \notin E(G), \quad \text{and} \quad AS = \mathbf{0}.$$

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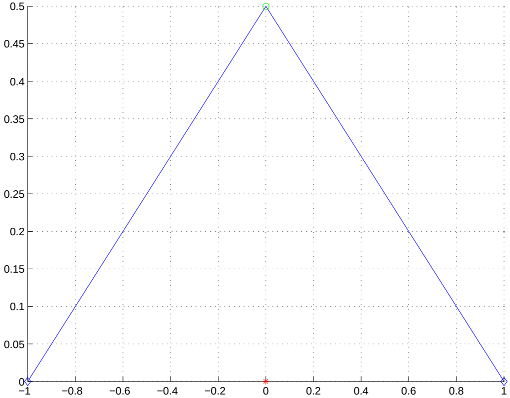
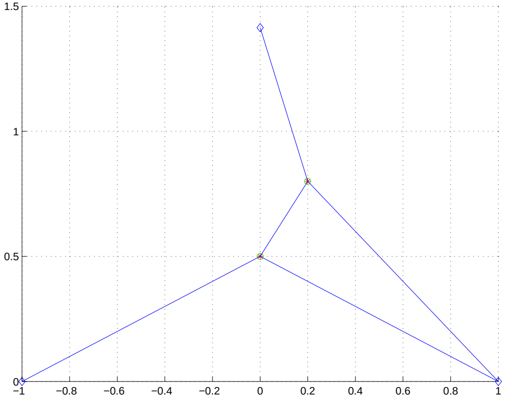
Similar rigidity notions hold for SNL with anchors.

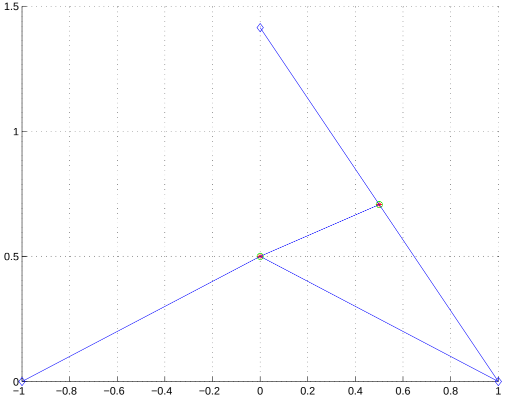
UR (not even GR when with anchors in 2-D)



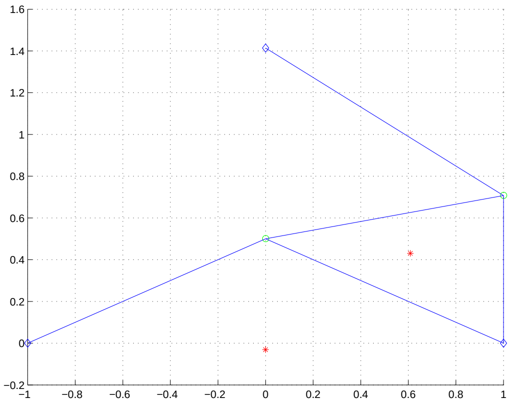
UR and SR



UR but not SR



GR but not UR



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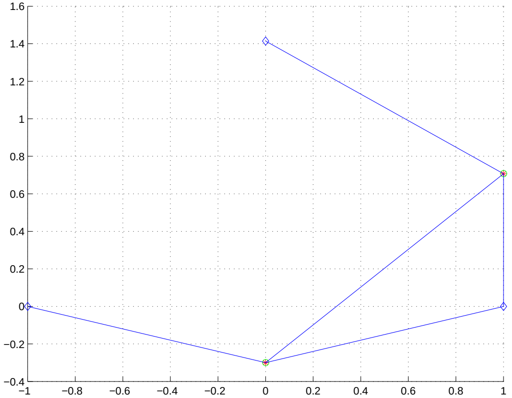

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- NP-hardness related to GR (Saxe 79).
- UR implies GR.
- SR implies UR when P is in generic positions (Connelly 99, also see Alfakih 10).
- It's necessary to have $d + 1$ anchors in general positions to be UR for SNL with anchors, and then SR implies UR (So and Y, 05).

Semidefinite Programming Problem (SDP)

$$\begin{array}{llll} (SDP) & \inf & C \bullet X & \\ & \text{subject to} & A_i \bullet X = b_i & i = 1, \ldots, m, \\ & & X \succeq \mathbf{0}. & \end{array}$$

where $C, A_1, \ldots, A_m$ are given dimension n real symmetric matrices with real scalars $\mathbf{b} = [b_1, \ldots, b_m]$,

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and $\succeq$ represents positive semi-definiteness.

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The dual problem to (SDP):

$$\begin{array}{lll} (SDD) & \sup & \mathbf{b} \cdot \mathbf{y} \\ & \text{subject to} & S = C - \sum_i^m y_i A_i \succeq \mathbf{0}, \end{array}$$

where variables $\mathbf{y} = [y_1, \dots, y_m] \in \mathbf{R}^m$.

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If there is S^* such that $\text{rank}(S^*) \geq n - d$, then the rank of any X^* is bounded above by d.

SDP Computational Complexity and Solution Rank

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- But finding a min-rank SDP solution is strongly NP-Hard.

Matrix Representation of SNL

Find $Y = X^T X$, where $X = [\mathbf{x}_1, \ldots, \mathbf{x}_n]$ is $d \times n$, such that

$$(\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^T \bullet Y \quad = \quad d_{ij}^2, \; \forall \; i,j \in N_x, \;\; i < j,$$

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where $\mathbf{e}_j$ is the vector of all zeros except 1 at the jth position.

SDP Relaxations of SNL, Biswas and Y 04, Biswas, Toh and Y 06

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Then, we face a standard SDP (feasibility) problem for SNL

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or SNL with anchors

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Properties of the SDP Relaxation

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  - It's d if and only if $Y = X^T X$ and then X is a localization.
  - If there exists a dual solution matrix with rank n, then the rank of any primal solution Z must be d.

The Dual of the SDP Relaxations

$$\begin{array}{ll} \text{minimize} & \sum_{i<j\in N_x} w_{ij} d_{ij}^2 \\ \text{subject to} & \sum_{i<j\in N_x} w_{ij}(\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^T = S \succeq 0, \end{array}$$

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or

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\begin{array}{ll}
\text{minimize} & I \bullet V + \sum_{i<j\in N_x} w_{ij} d_{ij}^2 + \sum_{k,j\in N_a} \hat{w}_{kj}\hat{d}_{kj}^2 \\
\text{subject to} & \begin{pmatrix} V & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \sum_{i<j\in N_x} w_{ij}(\mathbf{0};\mathbf{e}_i-\mathbf{e}_j)(\mathbf{0};\mathbf{e}_i-\mathbf{e}_j)^T \\
 & + \sum_{k,j\in N_a} \hat{w}_{kj}(\mathbf{a}_k;-\mathbf{e}_j)(\mathbf{a}_k;-\mathbf{e}_j)^T = S \succeq 0,
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where variable matrix $V \in \mathcal{S}^d$, variable w_{ij} is the (stress) weight on edge between $\mathbf{x}_i$ and $\mathbf{x}_j$, and $\hat{w}_{kj}$ is the (stress) weight on edge between $\mathbf{a}_k$ and $\mathbf{x}_j$.

Benefits of SDP Relaxation, So and Y 05, Biswas, Toh and Y 06

Whether or not a network (G, D) (with or without anchors) is UR can be (numerically) certified in polynomial time.

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http://www.stanford.edu/~yyye/URFrameworkTest.m
http://www.math.nus.edu.sg/~mattohkc/disco.html
http://www.stanford.edu/~yyye/Col.html

An Equivalence Theorem, Biswas and Y 04, So and Y 05

Theorem

The following statements are equivalent for SNL with anchors:

1. *The sensor network is UR;*
2. *The max-rank solution of the SDP relaxation has rank* d*;*
3. *The solution matrix has* $Y = X^T X$ *or* $Trace(Y - X^T X) = 0$ *.*

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Moreover, the localization of a UR instance can be computed approximately in a time polynomial in n, d*, and the accuracy* $\log(1/\epsilon)$.

Identify the Largest UR Subnetwork, So and Y 05

Theorem

If a network with anchors contains a subnetwork that is UR, then the SDP solution submatrix corresponding to the subnetwork has rank d. Thus, the SDP relaxation method finds the localization of the largest UR subnetwork for SNL with anchors.

Certification: Diagonals of the positive semidefinite matrix

$$\hat{Y} - \hat{X}^T \hat{X},$$

can be used to certify the UR subnetwork; that is, $\hat{Y}_{jj} - \|\hat{\mathbf{x}}_j\|^2 = 0$ if any only if the jth sensor point is in the UR subnetwork.

The Dual Matrix Theorem, So and Y 07

Theorem

Any optimal dual solution matrix is a positive semidefinite stress matrix for SNL or SNL with anchors. Therefore, a max-rank positive semidefinite stress matrix can be computed approximately in a time polynomial in n, d, and the accuracy $\log(1/\epsilon)$.

UR Theorems in Generic Position, Gortler and Thurston 09

Theorem

Let the network possess a localization P in generic positions of $\mathbf{R}^d$. Then, the network is UR if and only if there exists a max-rank positive semidefinite stress matrix, that is, the network is SR.

UR Theorems in General Position I

Let the network possess a localization P in general positions of $\mathbf{R}^d$. Then the network is UR if the graph contains a spanning $(d+1)$-lateration graph for SNL with or without anchors (So 06 and Zhu, So and Y 09).

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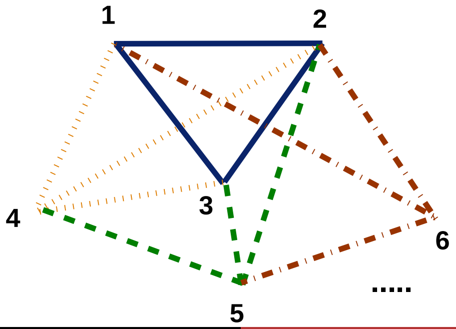
A $(d+1)$-lateration graph:



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- The existence of a max-rank stress matrix implies that the network is UR, that is, SR implies UR (Alfakih and Y 10).
- A network that contains a spanning $(d+1)$-lateration graph is UR if and only if it is SR (Alfakih, Taheri and Y 10), and the same result holds for SNL with anchors.
- Given localization matrix P and the lateration order, such a max-rank stress matrix can be computed exactly in strongly polynomial time (Alfakih, Taheri and Y 10).

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$$\text{orthogonality:} \quad AS = \mathbf{0}, \tag{1}$$

and

$$\text{purity:} \quad S_{ij} = 0, \ \forall (i,j) \notin E. \tag{2}$$

Proof Sketch II

- We start a PSD matrix satisfies orthogonality condition (1), say

$$S^0 = I - A^T(AA^T)^{-1}A,$$

where the columns of A are ordered according to the lateration order, and we call it a "prestress" matrix.

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- In at most n steps, we reach a “prestress” matrix that finally satisfies purity condition (2).

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- SNL based on other metric measurements: angles, path-distances, time-series data, etc.