Directed Cut Polyhedra with Mining Applications

David Avis and Conor Meagher

September 20, 2011

CCCG 2010

Those ubiquitous cut polyhedra

CCCG 2010 Paul Erdös Lecture

David Avis

Kyoto University and McGill University

July 14, 2010

CCCG 2010

Mining

- Yet another long story so ...
- ... please read Conor's thesis!

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On the Directed Cut Polyhedra and Open Pit Mining

Conor Meagher

Doctor of Philosophy

School of Computer Science

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Montreal,Quebec

2010-08-31

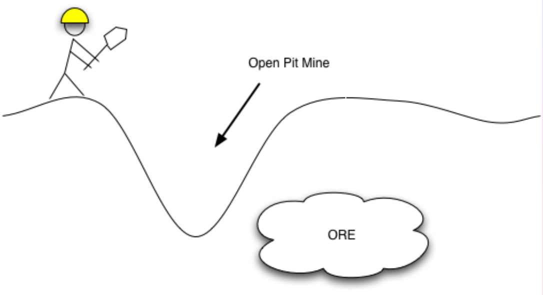
A thesis submitted to McGill University in partial fulfillment of the requirements of the degree of Doctor of Philosophy

Conor’s Acknowledgements

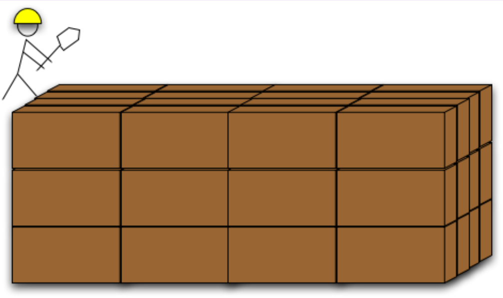
ACKNOWLEDGEMENTS

It is said that after your parents a Ph.D. supervisor has the greatest influence on your life. I had the privilege of having three supervisors during my time at McGill. My three dads were David Avis, Roussos Dimitrakopoulos and Bruce Reed. The combination of discrete optimization, mining and graph theory appearing in this thesis is not an accident and are a direct reflection of their influence and respective areas of expertise. David taught me much of what I know on discrete optimization and that you can take any problem and relate it to the cut polytope. I am greatly indebted to Roussos for introducing me to optimization problems and opportunities in the mining industry. Without Bruce I doubt I would have ever pursued this degree, while my focus drifted away from my original intent to focus on graph colouring Bruce taught me a lot of graph theory and probability before I changed my focus. Much of what I learnt from my supervisors falls into the category of life lessons. I

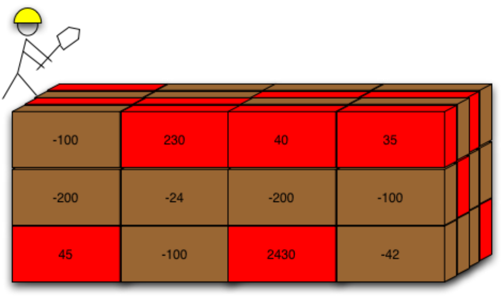
Open pit mine



- The ground is broken up into sections

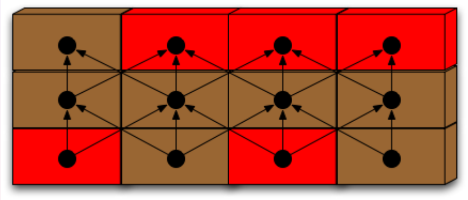


- Using estimation or simulation techniques from drill hole data, economic values are produced for each block



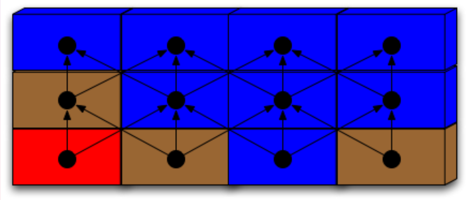
- Ore blocks can return a profit when mined
- Waste blocks cost money to remove

- Each block is considered as a node of a graph
- Arcs are added to represent slope requirements



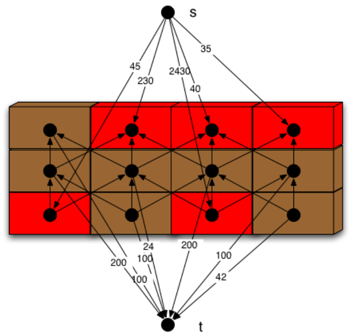
Graph closure

- A graph closure is a subset S of nodes such that no arcs leave S
- A maximum weight graph closure is known as “the ultimate pit”



Maximum network flow

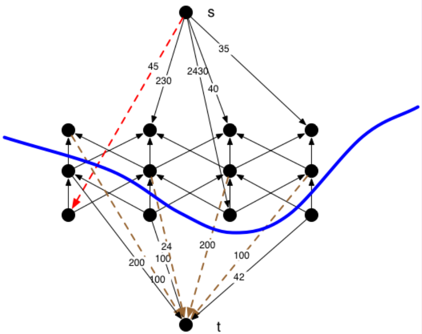
- source node *s* with arcs to each ore node
- sink node *t* with arcs from each waste node



- Capacities on the arcs are the absolute value of the blocks
- Slope arcs have infinite capacity

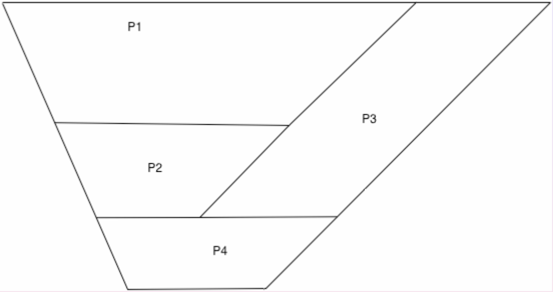
Minimum cut

The minimum cut represents the maximum weight graph closure



- Minimize the waste inside and the ore outside the pit

Pushbacks



- The ultimate pit must be decomposed into a multiperiod schedule
- Let P_i be the pit dug in period i

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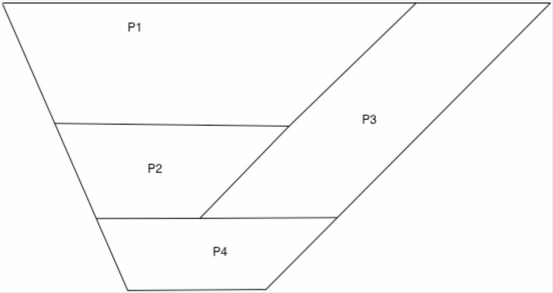

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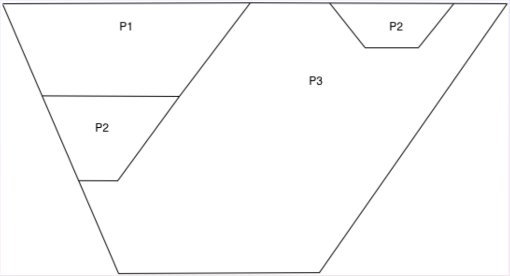
Problems with existing pushback design methods



- The pits have very different sizes and P_2 is not connected.
- We may make a limit the number blocks can be mined per period.
- We may require P_i to be connected
- Either makes the probem NP-hard

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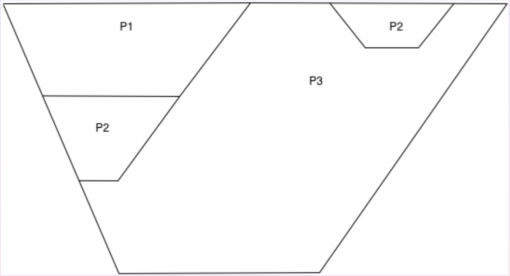

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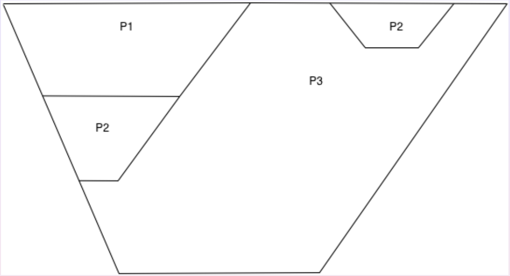

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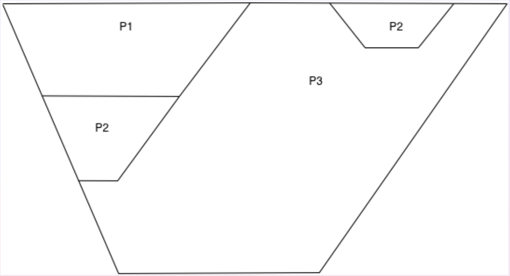

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Partially ordered knapsack

- The block limit introduces a knapsack type constraint.
-

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} c_i x_i \\
s.t. \quad & x_i \le x_j \qquad \textit{for all arcs}(i,j) \\
& \sum_{i=1}^{n} w_i x_i \le b \qquad (1) \\
& x_i \in \{0,1\} \quad \forall i
\end{aligned}
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- Constraint (1) ruins total unimodularity.
- This is the partially ordered knapsack problem and is NP-hard.

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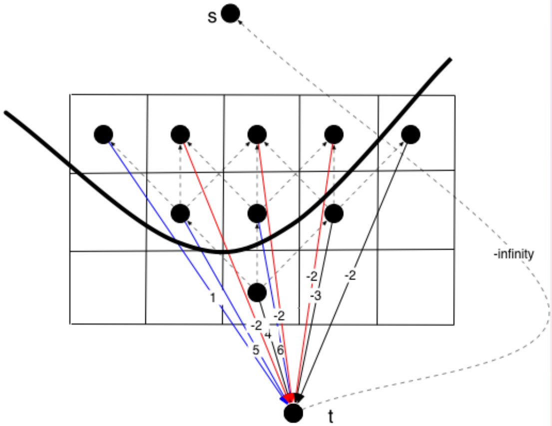
Directed cut approach

- An alternate approach is to optimize over the polytope of all *directed* cuts using cutting planes.
- Not much is known about *directed cut polyhedra*.



Geometry of cuts and metrics

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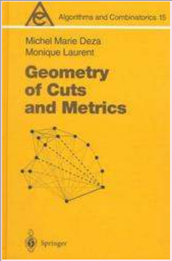
Michel Deza	Monique Laurent
	
and a book	

How about directed cuts?

- Far less studied - surprising because ...
- ... most people learn about directed cuts first: Max-flow Min-cut theorem (1956)
- They appear briefly in early NP-completeness literature, and ...
- ... the work of Goemans-Williamson (1995) on SDP relaxation.

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Weightable oriented multicut quasimetrics

- Very recent general results by M. Deza, E. Deza, J. Vidali and others overlap results in today's talk.
- "It is easy to see that an oriented multicut quasi-semimetric is weightable iff it is oriented cut". (M. Deza)
- See arXiv:1101.0517

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Cut polytope: definition

- Let $S \subseteq \{1, \ldots, n\}$ and $\oplus$ denote exclusive or.
- Define $\delta(S) \in \mathbb{R}^{\binom{n}{2}}$ by

$$\delta(S)ij = \begin{cases} 1 & i \oplus j \in S \\ 0 & otherwise \end{cases} \qquad 1 \leq i < j \leq n$$

- $\delta(S)$ is the edge-incidence vector of the cut $[S, V - S]$ in K_n.
- The cut cone is

$$CUT_n = Cone\{\delta(S) : S \subseteq V(K_n)\}$$

- The cut polytope is

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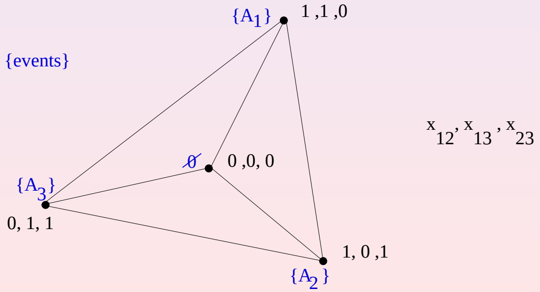
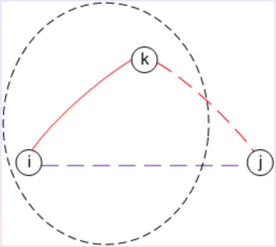
$CUT_3^{\square}$

S	x_{12}	x_{13}	x_{23}
$\emptyset$ *or* $\{1,2,3\}$	0	0	0
$\{1\}$ *or* $\{2,3\}$	1	1	0
$\{2\}$ *or* $\{1,3\}$	1	0	1
$\{3\}$ *or* $\{1,2\}$	0	1	1



Simple facets of the cut polyhedra



Triangle Inequalities:

$$x_{i,j} - x_{i,k} - x_{k,j} \quad \leq \quad 0$$

Perimeter Triangle Inequalities:

$$x_{i,j} + x_{j,k} + x_{k,i} \quad \leq \quad 2$$

Semimetric polyhedra

- The semimetric cone is

$$MET_n = \{x \in \mathbb{R}^{\binom{n}{2}} : x_{i,j} - x_{i,k} - x_{k,j} \leq 0 \ \ \forall i, j, k\}$$

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- MET_n is an LP relaxation of CUT_n
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- Define $\delta^+(S) \in \mathbb{R}^{n(n-1)}$ by

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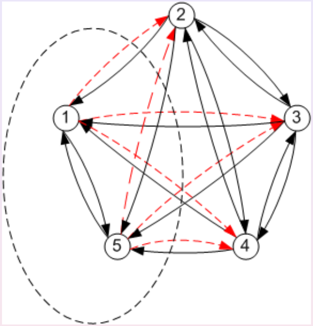

- $S = \{1, 5\}$
- Red edges have value 1, black edges have value 0 in $\delta^+(S)$.

$DCUT_n^{\square}$

S	x_{12}	x_{13}	x_{23}	x_{21}	x_{31}	x_{32}
$\emptyset$ or $\{1,2,3\}$	0	0	0	0	0	0
$\{1\}$	1	1	0	0	0	0
$\{2\}$	0	0	1	1	0	0
$\{3\}$	0	0	0	0	1	1
$\{2,3\}$	0	0	0	1	1	0
$\{1,3\}$	1	0	0	0	0	1
$\{1,2\}$	0	1	1	0	0	0

- There are $2^n - 1$ vertices.
- What do we do next?
- Compute the facets of course!

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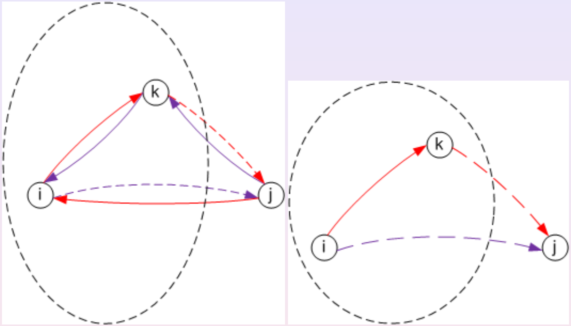
Linearities and facets for directed cuts



- 3 point symmetries: $x_{ij} + x_{jk} + x_{ki} = x_{ji} + x_{kj} + x_{ik}$
- Non-negativity: $x_{ij} \geq 0$
- Triangle inequality: $x_{ij} - x_{ik} - x_{kj} \leq 0$
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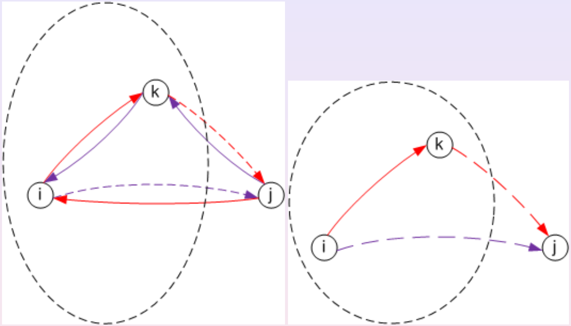
Linearities and facets for directed cuts



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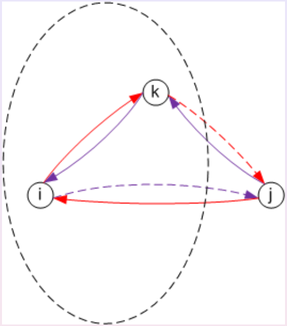

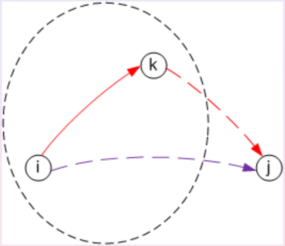

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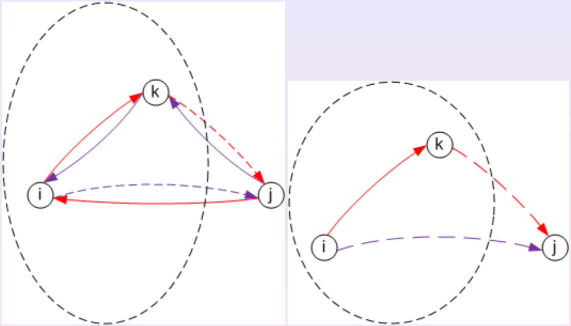

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Directed semimetric polyhedra

- The semimetric cone is

$$DMET_n = \{x \in \mathbb{R}^{n(n-1)}, x_{ij} \geq 0, x_{ij} - x_{ik} - x_{kj} \leq 0$$
$$x_{ij} + x_{jk} + x_{ki} = x_{ji} + x_{kj} + x_{ik} \ \ \forall i, j, k\}$$

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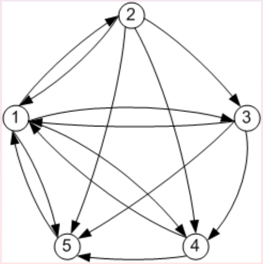
Dimension of the directed cut polytope

Lemma

The dimension of $DCUT_n^{\square}$ *(and* $DMET_n^{\square}$*) is* $\binom{n}{2} + n - 1$

- Upper bound: the weight on edge $ji, j > i$, can be recovered from $ij, 1i, 1j, i1, j1$ and the 3-point symmetries.
- Lower bound: a set of $\binom{n}{2} + n - 1$ linearly independent cut vectors.



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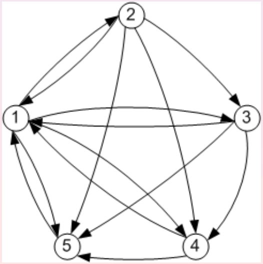
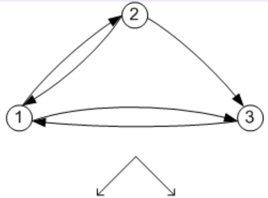


Bijections between the directed and undirected polyhedra

Define the polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$ to be:

- $\mathcal{P}_1 = conv\{\delta^+(S) : 1 \in S \subseteq V(G)\}$.
- $\mathcal{P}_2 = conv\{\delta^+(S) : 1 \notin S \subseteq V(G)\}$.



	12,13,21,23,31
S={1,2,3}	(0, 0, 0, 0, 0)
S={1}	(1, 1, 0, 0, 0)
S={1,2}	(0, 1, 0, 1, 0)
S={1,3}	(1, 0, 0, 0, 0)

	12,13,21,23,31
S={}	(0, 0, 0, 0, 0)
S={2}	(0, 0, 1, 1, 0)
S={3}	(0, 0, 0, 0, 1)
S={2,3}	(0, 0, 1, 0, 1)

Bijection between directed and undirected polyhedra

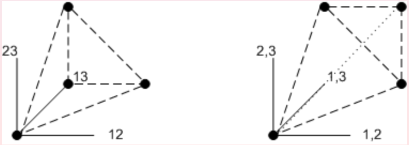
$$\xi_1 : \mathbb{R}^{\binom{n}{2}} \to \mathbb{R}^{\binom{n}{2}+n-1} \textit{ and } \xi_2 : \mathbb{R}^{\binom{n}{2}} \to \mathbb{R}^{\binom{n}{2}+n-1}$$

The mapping ξ_1 between $\mathrm{CUT}_n^{\square}$ and $\mathcal{P}_1$ is defined by,

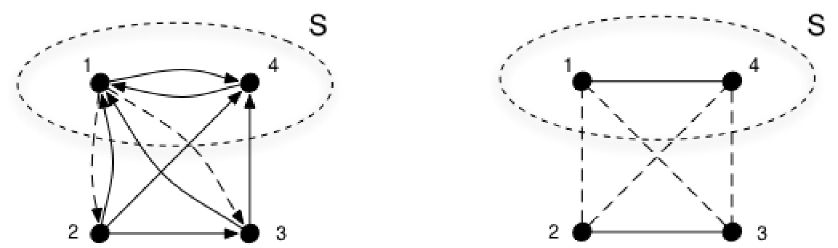
$$\begin{cases} x_{i1} = 0 & \textit{for } 2 \le i \le n \\ x_{1i} = x_{1,i} & \textit{for } 2 \le i \le n \\ x_{ij} = \frac{1}{2}(x_{i,j} + x_{1,j} - x_{1,i}) & \textit{for } 2 \le i < j \le n. \end{cases}$$

equivalently,

$$\begin{cases} x_{1,i} = x_{1i} & \textit{for } 2 \le i \le n \\ x_{i,j} = x_{ij} + x_{ji} = x_{1i} - x_{1j} + 2x_{ij} & \textit{for } 2 \le i < j \le n \end{cases}$$



Bijection example



The above figure is an example for $S = \{1, 4\}$.

$$
\begin{aligned}
\xi_1(\delta(S)) &= \xi_1(x_{1,2}, x_{1,3}, x_{1,4}, x_{2,3}, x_{2,4}, x_{3,4}) \\
&= \xi_1((1,1,0,0,1,1)) \\
&= (1,1,0,0,0,0,0,0) \\
&= (x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{31}, x_{34}, x_{41}) = \delta^+(S)
\end{aligned}
$$

We can define a similar mapping between P_2 and $\mathrm{CUT}_n^{\square}$ and use these bijections to show that.

Theorem

The directed cut polytope is the convex hull of two cut polytopes that only intersect at the origin.

and...

Theorem

If $a^T x \leq 0$ is a facet of the undirected cut cone then:

$$\sum_{2 \leq i < j \leq n} 2a_{i,j} x_{ij} + \sum_{i=2}^{n} c_{i1} x_{i1} + \sum_{i=2}^{n} b_{1i} x_{1i} \leq 0$$

is a facet of the directed cut cone. Where,

$$\begin{cases} b_{1i} = 0 & \text{for } 2 \leq i \leq n \\ b_{i1} = a_{1,i} + \sum_{k=2}^{i-1} a_{k,i} - \sum_{j=i+1}^{n} a_{i,j} & \text{for } 2 \leq i \leq n \end{cases}$$

and

$$\begin{cases} c_{1i} = a_{1,i} - \sum_{k=2}^{i-1} a_{k,i} + \sum_{j=i+1}^{n} a_{i,j} & \text{for } 2 \leq i \leq n \\ c_{i1} = 0 & \text{for } 2 \leq i \leq n. \end{cases}$$

The proof uses the following lemma:

Lemma (YB, Lemma 26.5.2)

Let $v^T x \leq 0$ be facet inducing for CUT_n and let $R(v)$ denote its set of roots. Let F be a subset of E_n.
If $v_{\bar{F}} \neq 0$, then $rank(R(v)_F) = |F|$.

Proof of theorem

- The $k = 1, ..., \binom{n}{2} - 1$ roots $\delta(S_j)$ of $v^T x \leq 0$ extend to roots $\delta^+(S_j)$ of the cut polytope.
- We may assume $1 \in S_j$ for all j.
- Let $F = \{1i : 2 \leq i \leq n\}$. From lemma there are $\delta(T_i)$ $(1 \leq i \leq n-1)$ linearly independent roots of $v^T x \leq 0$ whose projections on F are linearly independent.
- We may assume $1 \notin T_i$ for all i.
- $\delta^+(T_i) \cup \delta^+(S_j)$ form $\binom{n}{2} + n - 2$ linearly independent roots for the new inequality.

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Corollary

- *The triangle inequalities are facet defining for $DCUT_n$.*
- *Let $b_1, \cdots, b_n$ be an integers that sum to one. The inequality:*

$$\sum_{1 \le i < j \le n} b_i b_j x_{i,j} \le 0$$

is known as a hypermetric inequality.

- *Hypermetric facets of the cut cone give facets of the dicut cone:*

$$\sum_{2 \le i < j \le n} b_i b_j x_{ij} + \sum_{i=1}^{n} (b_1 - \sum_{k=2}^{i-1} b_k + \sum_{j=i+1}^{n} b_j) b_i x_{1i} + \sum_{i=1}^{n} (b_1 + \sum_{k=2}^{i-1} b_k - \sum_{j=i+1}^{n} b_j) b_i x_{i1} \le 0$$

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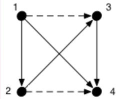
$\mathrm{DCUT}_4^{\square}$

- 31 vertices and 40 facets
- 12 non-negativity constraints
- 16 triangle inequalities
- Six new homogeneous inequalities



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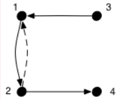

$$x_{31} + x_{12} + x_{24} \leq 1 + x_{21}$$

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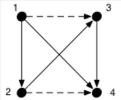

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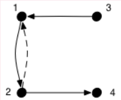

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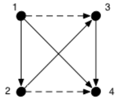

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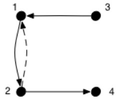

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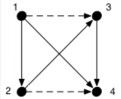

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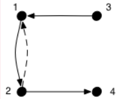

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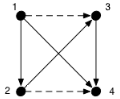

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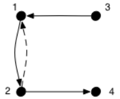

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Relaxations for CUT(G) and $\text{CUT}^{\square}(G)$

- Let G be an undirected graph.
- We can optimize over $\text{MET}(G)$ and $\text{MET}^{\square}(G)$ in polynomial time by setting $c_{ij} = 0$ when $ij \notin E(G)$.

$$\max \sum_{(i,j)} c_{ij} x_{ij}$$
$$s.t.\ x \in DMET_n^{\square}$$

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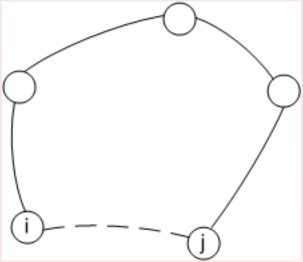
Projecting the triangle inequalities

Theorem (Barahona, Mahjoub, 1986)

Given an arbitrary graph G, the projection of MET_n onto the edge set of G is:

$$MET(G) = \{x \in \mathbb{R}_+^{E(G)} \quad : \quad x_e - x(C \setminus \{e\}) \leq 0 \quad \textit{for each chordless cycle } C \textit{ of } G, e \in C\}$$



Integer hull

- Theorem (Seymour 1981, Barahona, Mahjoub, 1986)
  $CUT(G) = MET(G)$ or, equivalently, $CUT^{\square}(G) = MET^{\square}(G)$

  if and only if

  G has no K_5-minor.
  - Do these results generalize to directed graphs?

Integer hull

- Theorem (Seymour 1981, Barahona, Mahjoub, 1986)
  $CUT(G) = MET(G)$ *or, equivalently,* $CUT^{\square}(G) = MET^{\square}(G)$

  if and only if

  G has no K_5*-minor.*
  - Do these results generalize to directed graphs?

Relaxations for DCUT(G) and DCUT(G)$^\square$

- We can optimize over DMET(G) and DMET$^\square$(G) in polynomial time by setting $c_{ij} = 0$ when $ij \notin E(G)$.

$$\max \sum_{(i,j)} c_{ij} x_{ij}$$
$$s.t.\ x \in DMET_n^\square$$

- Is there a compact linear description for DMET(G)?

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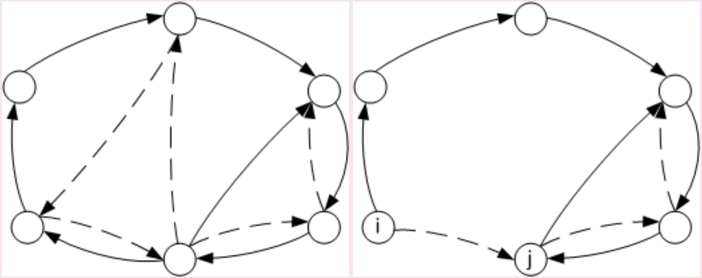
Projecting the triangle and cycle inequalities

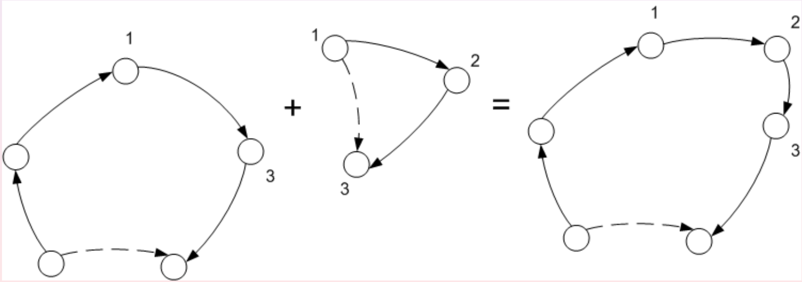
The projection of DMET_n onto an arbitrary digraph G is more complex:

$$DMET(G) = \{x \in \mathbb{R}^{A(G)} : x_e \geq 0, ...?$$



Triangular elimination

- Triangular elimination is a method of zero lifting and Fourier-Motzkin elimination using triangle inequalities [Avis, Imai, Ito, Sasaki '05]
- Can prove large families of inequalities are facet inducing by directed version triangular elimination.

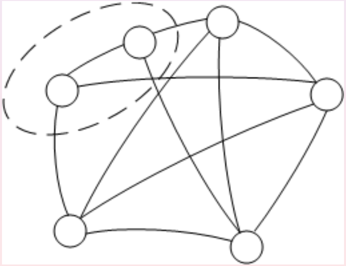


Forbidden graph minors

For a graph G not containing a K_5 minor [Seymour '81]:

$$MET(G) = CUT(G)$$

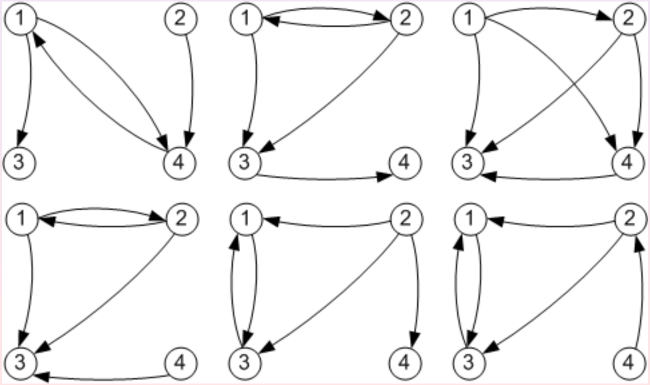
and $MET^{\square}(G) = CUT^{\square}(G)$ was proved using switching [Barahona, Mahjoub '86]



Forbidden graph minors (directed case)

If G contains any of the following 6 graphs as a "directed minor" then:

$$DMET(G) \neq DCUT(G) \text{ and } DMET^{\square}(G) \neq DCUT^{\square}(G)$$



Open problems

- Find compact descriptions for $DMET(G)$ and $DMET^{\Box}(G)$
- Generalize Seymour's Theorem to directed graphs
- Solve the open pit mining problem with geometric constraints

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