Notes on the illumination parameters of convex bodies

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The classical illumination problem

Let $\mathbf{K}$ be a convex body of $\mathbf{E}^d, d \geq 1$ (i.e. a compact convex set of the d-dimensional Euclidean space $\mathbf{E}^d$ with non-empty interior). We call a point $l \in \mathbf{E}^d \setminus \mathbf{K}$ a light-source and say that it illuminates the boundary point p of $\mathbf{K}$ if the half-line starting at l and passing through p intersects the interior of $\mathbf{K}$ somewhere not between l and p. Furthermore, we say that the light-sources $\{l_1, l_2, \dots, l_n\} \subset \mathbf{E}^d \setminus \mathbf{K}$ illuminate $\mathbf{K}$ if each boundary point of $\mathbf{K}$ is illuminated by at least one of the light-sources $l_1, l_2, \dots, l_n$.

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The smallest number of light-sources that can illuminate **K** is called the *illumination number* $I(\mathbf{K})$ *of* **K**.

The illumination conjecture

The illumination conjecture phrased independently by Boltyanski (1960) and Hadwiger (1960), says that any d-dimensional convex body can be illuminated by 2^d light-sources in $\mathbf{E}^d$, that is the inequality

$$I(\mathbf{K}) \leq 2^d$$

holds for any convex body $\mathbf{K} \in \mathbf{E}^d$.

The easy cases

The conjecture has been proved only for $d \leq 2$ (it is quite easy).

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Theorem (several people)

The illumination number of any smooth convex body in $\mathbf{E}^d$ is exactly $d + 1$.

Theorem

For any convex body $\mathbf{K} \in \mathbf{E}^d$, $d \geq 2$ *the inequality*

$$I(\mathbf{K}) \leq$$

$$\leq \min\left\{\binom{2d}{d}(d\ln d + d\ln\ln d), (d+1)^{d-1} - (d-1)(d-2)^{d-1}\right\}$$

holds.

The status of the conjecture II

Theorem (Papadoperakis)

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Theorem (Lassak, Weissbach)

The illumination number of any convex body of constant width in $\mathbf{E}^3$ *is at most 6.*

The status of the conjecture III

Theorem

If $\mathbf{W}$ *is a convex body of constant width in* $\mathbf{E}^d$ *with* $d = 4, 5and6$*, then the illumination number of* $\mathbf{W}$ *is at most 12, 32 and 64, respectively.*

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The illumination number of any almost smooth convex body in $\mathbf{E}^d$*,* $d \geq 3$*, is at most* $2d$*.*

(Almost smooth: at each boundary point the angle of any two supporting hyperplanes is not too big.)

The quantitative version of the illumination numbers of convex bodies was introduced by K. Bezdek. If $\mathbf{K_o}$ is a convex body of $\mathbf{E}^d$ symmetric about the origin o of $\mathbf{E}^d$, then $\mathbf{K_o}$ defines a norm

$$\|x\|_{\mathbf{K_o}} = \inf\{0 < \lambda : \lambda^{-1}x \in \mathbf{K_o}\},$$

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Then let the *illumination parameter* of $\mathbf{K_o}$ be defined as

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(This ensures that far-away light-sources are penalised.)

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In the case of the regular hexagon, the set of optimal light-sources is not unique. There are four different arrangements.

Illumination parameter

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Theorem (Swanepoel)

$IP(d) \leq O(2^d d^2 \log d)$.

Perhaps $IP(d) = O(2^d)$.

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The exact value is known only for $d \leq 3$.
If we illuminate the d-dimensional cross-polytope $\mathbf{X}_d$ by the set of vertices of a slightly enlarged circumscribed cross-polytope, then we get $IP(\mathbf{X}_d) = 2d$.

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Proposition

Let $\mathbf{K_o}$ be a convex body of $\mathbf{E}^d$ symmetric about the origin o of $\mathbf{E}^d$ with $IP(\mathbf{K_o}) = k$. If $\mathbf{C_o}$ is a right cylinder of $\mathbf{E}^{d+1}$ symmetric about the origin of $\mathbf{E}^{d+1}$, whose base is congruent with $\mathbf{K_o}$, then $IP(\mathbf{C_o}) \geq 2k$.

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Theorem

If $d \geq 2$ then $IP(d) \geq 3 \cdot 2^{d-1}$.

Platonic solids

The most famous polyhedra are the five Platonic solids. One of them, the regular tetrahedron, is not centrally symmetric. The illumination parameters of the cube and the regular octahedron are known (2^d and $2d$, respectively). Now we calculate the illumination parameters of the remaining two centrally symmetric Platonic solids.

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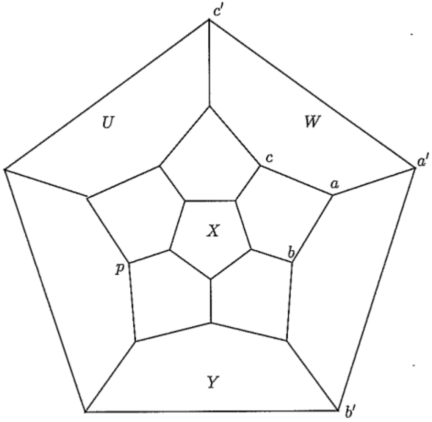
If we would like to illuminate a polytope **P**, then it is enough to illuminate the vertices of **P**. This follows because, if a light-source l illuminates a vertex v which belongs to a k-face F of **P**, then l obviously illuminates each point in the relative interior of F. So when we compute the illumination parameter of **P**, we always compute the sum of norms of a set of light-sources illuminating the vertices of **P**.

Dodecahedral graph



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PROOF. It is easy to see that for any two vertices v and w of $\Gamma_{\mathbf{D}}$ if $d(v,w) \geq 3$ holds, then there are two parallel faces of **D** such that one of them contains v and the other contains w. Hence no light-source can illuminate v and w simultaneously.

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We get the illumination parameter of $\mathbf{I}$ if we cover the vertices of $\Gamma_{\mathbf{D}}$ in the most effective way.

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If l illuminates two vertices, v and w of $\mathbf{D}$ with $d(v, w) = 2$, then without loss of generality we may assume that l illuminates the edges aa' and ab.
If l illuminates each of the five vertices of the face $\mathbf{D} \cap \pi_{aa'b}$, then $e(l) \leq 5/\sqrt{5} = \sqrt{5}$.

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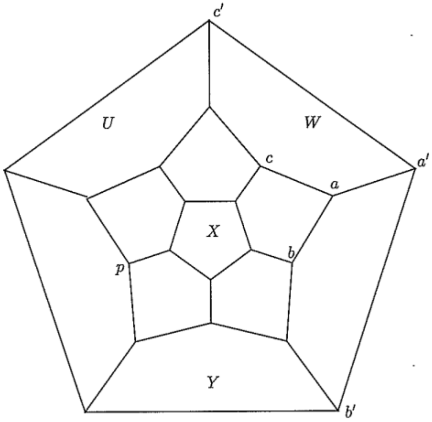
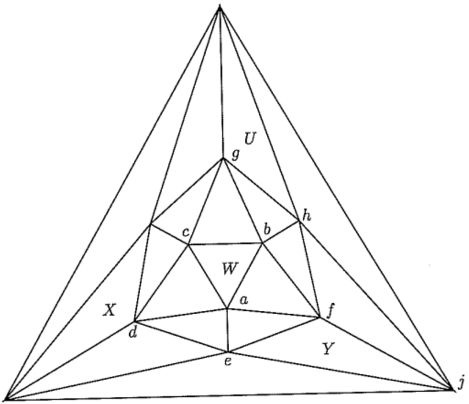
If l illuminates each vertex of the star with centre a, then $e(l) \leq 4/(\sqrt{5}+2) < 1$.

Dodecahedral graph



Icosahedral graph



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PROOF. If $d(v, w) \geq 2$, then there are two parallel faces of $\mathbf{I}$ such that one of them contains v and the other contains w. Hence no light-source can illuminate v and w simultaneously. This implies that no light-source can illuminate more than three vertices of $\mathbf{I}$, because any subset of more than three vertices contains a pair of vertices having distance at least 2.

Computation

We get the illumination parameter of $\mathbf{I}$ if we cover the vertices of $\Gamma_{\mathbf{I}}$ in the most effective way. It follows from the previous computations that the most effective light-sources illuminate either a single vertex or three vertices of a face of $\mathbf{I}$.

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In both cases $e(l) = 1$ holds for any light-source. Hence $IP(\mathbf{I}) \geq 12$. On the other hand, if each vertex v of $\mathbf{I}$ has its own light-source l_v, then l_v could be arbitrarily close to v, hence $IP(\mathbf{I}) = 12$. ■

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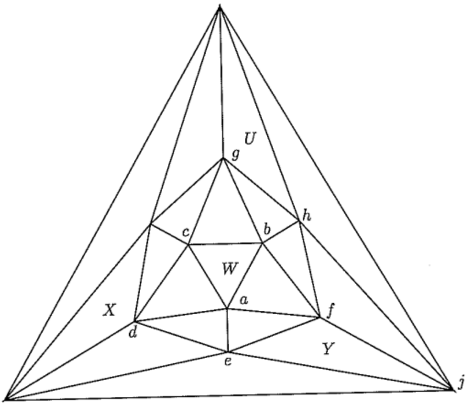
It is easy to see that the vertices of $\Gamma_{\mathbf{I}}$ can be covered by four faces U, W, X and Y. Thus the most effective illumination of $\mathbf{I}$ is not unique; we can mix the two types of light-sources having efficiency 1. Hence the number of light-sources in a set of optimal configuration could be 4, 6, 8, 10 or 12.

Icosahedral graph



Archimedean solids

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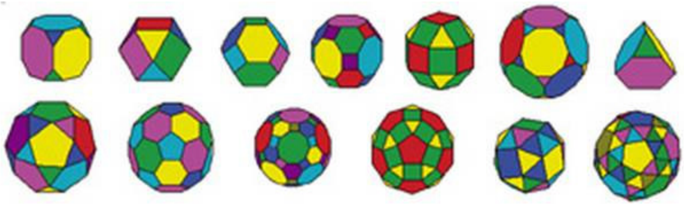

Archimedean solids again

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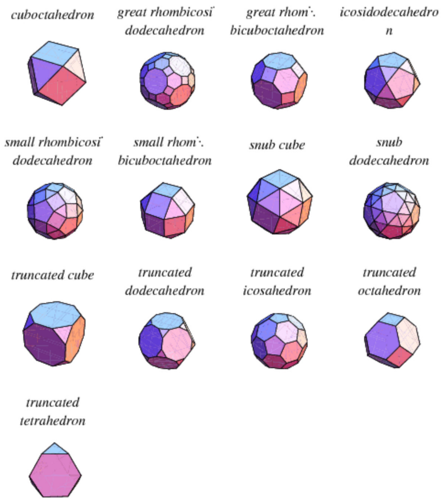

Archimedean solids II

Theorem

There are sets of light-sources which give upper estimates on the illumination parameters of the 10 centrally symmetric Archimedean solids as follows.

Archimedean solids III

Name of the polyhedron	Upper estimate on the illumination parameter	Number of light-sources
truncated cube	$24(\sqrt{2}-1) \approx 9.941$	8
truncated octahedron	9	6
truncated dodecahedron	$(68\sqrt{5}+10)/15 \approx 10.804$	6
truncated icosahedron	12	4
cuboctahedron	12	4 or 12
rhombicuboctahedron	12	4
truncated cuboctahedron	12	4
icosidodecahedron	$3(\sqrt{5}+1) \approx 9.708$	6
truncated icosidodecahedron	$15(3\sqrt{5}+1)/11 \approx 10.511$	6
rhombicosidodecahedron	12	4

Computation

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For the cuboctahedron, rhombicuboctahedron, truncated cuboctahedron and rhombicosidodecahedron, the light-sources are at the vertices of the circumscribed regular tetrahedron.

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For the truncated dodecahedron and icosahedron, the light-sources are at the same positions as the light sources illuminating the corresponding dodecahedron and icosahedron.
For the cuboctahedron, rhombicuboctahedron, truncated cuboctahedron and rhombicosidodecahedron, the light-sources are at the vertices of the circumscribed regular tetrahedron.
For the icosidodecahedron and the truncated icosidodecahedron, the light-sources are at the midpoints of six corresponding edges of the circumscribed regular dodecahedron.

A conjecture

$$IP(d) = 3 \cdot 2^{d-1}.$$

Illumination by subspaces

Let $L \subset \mathbf{E}^d \setminus \mathbf{K}$ be an affine subspace of dimension ℓ, $0 \leq \ell \leq d-1$. Then L illuminates the boundary point p of $\mathbf{K}$ if there exists a point $q \in L$ that illuminates p. Furthermore, we say that the affine subspaces $\{L_1, L_2, \dots, L_n\} \subset \mathbf{E}^d \setminus \mathbf{K}$ illuminate $\mathbf{K}$ if each boundary point of $\mathbf{K}$ is illuminated by at least one of the subspaces $L_1, L_2, \dots, L_n$. Now, the smallest number of affine subspaces of dimension ℓ that are disjoint from $\mathbf{K}$ and can illuminate $\mathbf{K}$ is called the *ℓ-dimensional illumination number* $I_\ell(\mathbf{K})$ of the convex body $\mathbf{K}$ in $\mathbf{E}^d$.

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K. Bezdek conjectures that

$$I_\ell(\mathbf{K}) \leq I_\ell(C)$$

holds for any convex body $\mathbf{K}$ in $\mathbf{E}^d$ where C denotes a d-dimensional affine cube of $\mathbf{E}^d$.

Successive illumination parameters

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Let $\mathbf{K_o}$ be a convex body in $\mathbf{E}^d$ symmetric about the origin o of $\mathbf{E}^d$. We say that the convex sets $\{S_1, S_2, \ldots, S_n\} \subset \mathbf{E}^d \setminus \mathbf{K_o}$ of dimension ℓ, $0 \leq \ell \leq d-1$ illuminate $\mathbf{K_o}$ if each boundary point of $\mathbf{K_o}$ is illuminated by at least one point of $S_1 \cup S_2 \cup \cdots \cup S_n$. Let

$$\|S\|_{\mathbf{K_o}} = \inf\{\lambda : \lambda^{-1} S \subset \mathbf{K_o}\}.$$

Then the *ℓ-dimensional illumination parameter* of $\mathbf{K_o}$ is defined as

$$IP_\ell(\mathbf{K_o}) = \inf\left\{\sum_i \alpha_\ell \cdot \|S_i\|_{\mathbf{K_o}} :\right.$$

$$\left. : S_i \text{ is convex and has dimension } \ell,\ \cup S_i \text{ illuminates } \mathbf{K_o}\right\}.$$

Successive parameters II

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If $\alpha_0 = 1$, then $IP_0(\mathbf{K_o}) = IP(\mathbf{K_o})$. The natural choices for α_ℓ are the following

$$\alpha_\ell = \begin{cases} \ell + 1 & \text{or} \\ 2^\ell. \end{cases}$$

In both cases $\alpha_0 = 1$ and $\alpha_1 = 2$.

Theorem

If $\mathbf{K_o}$ *is a convex body of* $\mathbf{E}^2$ *symmetric about the origin* o *of* $\mathbf{E}^2$*, then* $IP_1(\mathbf{K_o}) \leq \sqrt{28/3}\alpha_1 < 3.056\alpha_1$.

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If H is an affine-regular convex hexagon in $\mathbf{E}^2$, then $IP_1(H) = 3\alpha_1$.

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Conjecture:
If $\mathbf{K_o}$ is a convex body of $\mathbf{E}^2$ symmetric about the origin o of $\mathbf{E}^2$, then $IP_1(\mathbf{K_o}) \leq 3\alpha_1$.

Propositions

We assume that $\alpha_0 = 1$.

Proposition

If $\mathbf{K_o}$ *is a smooth o-symmetric convex body in* $\mathbf{E}^2$, $d \geq 2$ *then*

$$\frac{\alpha_1}{2} IP_0(\mathbf{K_o}) \leq IP_1(\mathbf{K_o}) \leq \alpha_1 IP_0(\mathbf{K_o}).$$

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Proposition

For any o-symmetric convex body $\mathbf{K_o}$ *in* $\mathbf{E}^d$, $d \geq 2$, *we have that*

$$2\alpha_{d-1} \leq IP_{d-1}(\mathbf{K_o}) \leq \alpha_{d-1} IP_0(\mathbf{K_o}).$$

On the successive illumination parameters of spheres

Let C_d, B_d and X_d be the d-dimensional cube, ball and cross-polytope, respectively.

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On the successive illumination parameters of cubes and cross-polytopes

Theorem

For any ℓ with $0 \leq \ell \leq d-1$, $d \geq 2$ we have that

$$IP_\ell(C_d) \leq 2^{d-\ell}\alpha_\ell.$$

Equality holds for $\ell = 0, 1$ and $d-1$.

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Equality holds for $\ell = 0, 1$ *and* $d-1$.

Theorem

$IP_0(X_d) = 2d$ *for all* d, *and*

$$IP_1(X_2) = 2\alpha_1,\ \ IP_1(X_3) = 3\alpha_1,\ \ IP_2(X_3) = 2\alpha_2.$$

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