On a discrete isoperimetric inequality

Zsolt Lángi

Department of Geometry, Budapest University of Technology, Hungary

September 16, 2011

Preliminaries

$\mathbb{E}^2$: the Euclidean plane
$\mathbb{H}^2$: the hyperbolic plane
$\mathbb{S}^2$: the sphere

$\mathbb{E}^2$: the Euclidean plane
$\mathbb{H}^2$: the hyperbolic plane
$\mathbb{S}^2$: the sphere

Theorem (Classical Discrete Isoperimetric Inequality)

Among convex polygons of a given perimeter in $\mathbb{E}^2$, $\mathbb{H}^2$ or in $\mathbb{S}^2$, the regular one has maximal area.

A more general approach (Brass, Moser, Pach: Research Problems in Discrete Geometry, 2005):

A more general approach (Brass, Moser, Pach: Research Problems in Discrete Geometry, 2005):

Any inequality relating two geometric quantities (area, perimeter, diameter, width, inradius, circumradius, etc.) of convex (or simple) n-gons is called a "discrete isoperimetric inequality".

A more general approach (Brass, Moser, Pach: Research Problems in Discrete Geometry, 2005):

Any inequality relating two geometric quantities (area, perimeter, diameter, width, inradius, circumradius, etc.) of convex (or simple) n-gons is called a "discrete isoperimetric inequality".

Remark

For most pairs the optimal polygon is the regular n-gon.

A recent question:

A recent question:

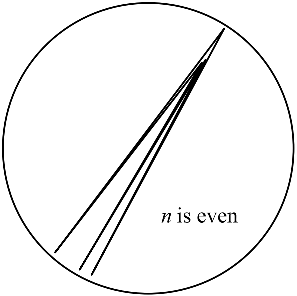
QUESTION (BRASS, 2005)

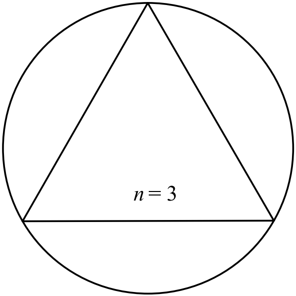
For $n \geq 5$ odd, what is the supremum of the perimeters of the simple n-gons contained in a unit disk of $\mathbb{E}^2$?

A recent question:

QUESTION (BRASS, 2005)

For $n \geq 5$ odd, what is the supremum of the perimeters of the simple n-gons contained in a unit disk of $\mathbb{E}^2$?





THEOREM (AUDET, HANSEN AND MESSINE, 2009)

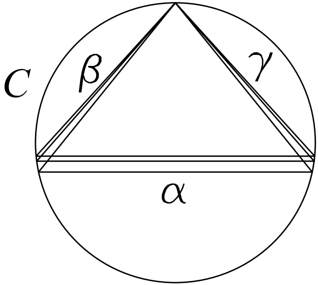
The supremum of the perimeters in the question of Brass is

$$\frac{\left(\sqrt{1+8(n-2)^2}-1\right)^{\frac{1}{2}}\left(\sqrt{1+8(n-2)^2}+3\right)^{\frac{3}{2}}}{4(n-2)}.$$

THEOREM (AUDET, HANSEN AND MESSINE, 2009)

The supremum of the perimeters in the question of Brass is

$$\frac{\left(\sqrt{1+8(n-2)^2}-1\right)^{\frac{1}{2}}\left(\sqrt{1+8(n-2)^2}+3\right)^{\frac{3}{2}}}{4(n-2)}.$$



THEOREM (L., 2011)

Let $\mathbb{M}$ be $\mathbb{E}^2$ or $\mathbb{H}^2$, and let $n \geq 3$ be an odd integer.

THEOREM (L., 2011)

Let $\mathbb{M}$ be $\mathbb{E}^2$ or $\mathbb{H}^2$, and let $n \geq 3$ be an odd integer. If $C \subset \mathbb{M}$ is a disk and P is a simple n-gon contained in C,

THEOREM (L., 2011)

Let $\mathbb{M}$ be $\mathbb{E}^2$ or $\mathbb{H}^2$, and let $n \geq 3$ be an odd integer. If $C \subset \mathbb{M}$ is a disk and P is a simple n-gon contained in C, then there is an isosceles triangle,

THEOREM (L., 2011)

Let $\mathbb{M}$ *be* $\mathbb{E}^2$ *or* $\mathbb{H}^2$*, and let* $n \geq 3$ *be an odd integer. If* $C \subset \mathbb{M}$ *is a disk and* P *is a simple* n*-gon contained in* C*, then there is an isosceles triangle, inscribed in* C *and with side-lengths* $\alpha \geq \beta$*,*

THEOREM (L., 2011)

Let $\mathbb{M}$ *be* $\mathbb{E}^2$ *or* $\mathbb{H}^2$*, and let* $n \geq 3$ *be an odd integer. If* $C \subset \mathbb{M}$ *is a disk and* P *is a simple* n*-gon contained in* C*, then there is an isosceles triangle, inscribed in* C *and with side-lengths* $\alpha \geq \beta$*, such that* $\operatorname{perim} P \leq (n-2)\alpha + 2\beta$*.*

First results

Theorem (L., 2011)

Let $\mathbb{M}$ be $\mathbb{E}^2$ or $\mathbb{H}^2$, and let $n \geq 3$ be an odd integer. If $C \subset \mathbb{M}$ is a disk and P is a simple n-gon contained in C, then there is an isosceles triangle, inscribed in C and with side-lengths $\alpha \geq \beta$, such that $\operatorname{perim} P \leq (n-2)\alpha + 2\beta$.

Remark (L., 2011)

There is a real number $\varepsilon > 0$

THEOREM (L., 2011)

Let $\mathbb{M}$ be $\mathbb{E}^2$ or $\mathbb{H}^2$, and let $n \geq 3$ be an odd integer. If $C \subset \mathbb{M}$ is a disk and P is a simple n-gon contained in C, then there is an isosceles triangle, inscribed in C and with side-lengths $\alpha \geq \beta$, such that $\operatorname{perim} P \leq (n-2)\alpha + 2\beta$.

REMARK (L., 2011)

There is a real number $\varepsilon > 0$ such that for every odd integer $n \geq 3$ and $0 < \rho < \varepsilon$ the following holds.

THEOREM (L., 2011)

Let $\mathbb{M}$ be $\mathbb{E}^2$ or $\mathbb{H}^2$, and let $n \geq 3$ be an odd integer. If $C \subset \mathbb{M}$ is a disk and P is a simple n-gon contained in C, then there is an isosceles triangle, inscribed in C and with side-lengths $\alpha \geq \beta$, such that $\operatorname{perim} P \leq (n-2)\alpha + 2\beta$.

REMARK (L., 2011)

There is a real number $\varepsilon > 0$ such that for every odd integer $n \geq 3$ and $0 < \rho < \varepsilon$ the following holds. If $C \subset \mathbb{S}^2$ is a disk of radius ρ, and P is a simple n-gon contained in C,

Theorem (L., 2011)

Let $\mathbb{M}$ be $\mathbb{E}^2$ or $\mathbb{H}^2$, and let $n \geq 3$ be an odd integer. If $C \subset \mathbb{M}$ is a disk and P is a simple n-gon contained in C, then there is an isosceles triangle, inscribed in C and with side-lengths $\alpha \geq \beta$, such that $\operatorname{perim} P \leq (n-2)\alpha + 2\beta$.

Remark (L., 2011)

There is a real number $\varepsilon > 0$ such that for every odd integer $n \geq 3$ and $0 < \rho < \varepsilon$ the following holds. If $C \subset \mathbb{S}^2$ is a disk of radius ρ, and P is a simple n-gon contained in C, then there is an isosceles triangle, inscribed in C and with side-lengths $\alpha \geq \beta$, such that $\operatorname{perim} P \leq (n-2)\alpha + 2\beta$.

REMARK

Using simple calculus, one can compute the maximum of the quantity $(n-2)\alpha + 2\beta$ for $\mathbb{M} = \mathbb{H}^2$ and $\mathbb{M} = \mathbb{S}^2$, but these expressions are too long to be included here.

REMARK

Using simple calculus, one can compute the maximum of the quantity $(n-2)\alpha + 2\beta$ for $\mathbb{M} = \mathbb{H}^2$ and $\mathbb{M} = \mathbb{S}^2$, but these expressions are too long to be included here.

Notations:

- – If it is not stated otherwise, $\mathbb{M}$ is any of the planes $\mathbb{E}^2$, $\mathbb{H}^2$ and $\mathbb{S}^2$.

REMARK

Using simple calculus, one can compute the maximum of the quantity $(n-2)\alpha + 2\beta$ for $\mathbb{M} = \mathbb{H}^2$ and $\mathbb{M} = \mathbb{S}^2$, but these expressions are too long to be included here.

Notations:

- – If it is not stated otherwise, $\mathbb{M}$ is any of the planes $\mathbb{E}^2$, $\mathbb{H}^2$ and $\mathbb{S}^2$.
  - The closed/open segment with endpoints $p, q \in \mathbb{M}$ is $[p, q]/(p, q)$, respectively.

REMARK

Using simple calculus, one can compute the maximum of the quantity $(n-2)\alpha + 2\beta$ for $\mathbb{M} = \mathbb{H}^2$ and $\mathbb{M} = \mathbb{S}^2$, but these expressions are too long to be included here.

Notations:

- – If it is not stated otherwise, $\mathbb{M}$ is any of the planes $\mathbb{E}^2$, $\mathbb{H}^2$ and $\mathbb{S}^2$.
  - The closed/open segment with endpoints $p, q \in \mathbb{M}$ is $[p, q]/(p, q)$, respectively.
  - The distance of p, q is denoted by $\mathrm{dist}_M(p, q)$, or, if $\mathbb{M} = \mathbb{E}^2$, by $|p - q|$.

REMARK

Using simple calculus, one can compute the maximum of the quantity $(n-2)\alpha + 2\beta$ for $\mathbb{M} = \mathbb{H}^2$ and $\mathbb{M} = \mathbb{S}^2$, but these expressions are too long to be included here.

Notations:

- – If it is not stated otherwise, $\mathbb{M}$ is any of the planes $\mathbb{E}^2$, $\mathbb{H}^2$ and $\mathbb{S}^2$.
  - The closed/open segment with endpoints $p, q \in \mathbb{M}$ is $[p, q]/(p, q)$, respectively.
  - The distance of p, q is denoted by $\text{dist}_M(p, q)$, or, if $\mathbb{M} = \mathbb{E}^2$, by $|p - q|$.

Let P be a simple n-gon contained in the Euclidean disk C.

Let P be a simple n-gon contained in the Euclidean disk C.

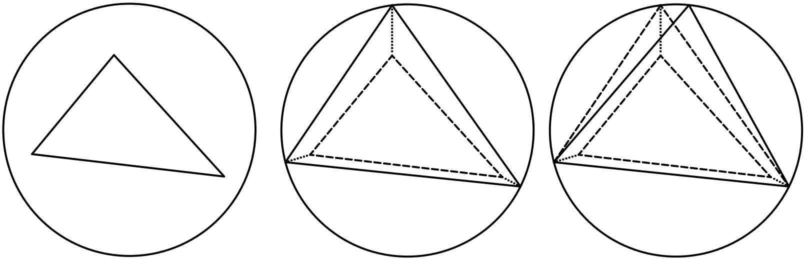
REMARK

It is sufficient to find a triangle contained in C, with side-lengths α, β and γ, that satisfies perim $P \leq (n-2)\alpha + \beta + \gamma$.

Let P be a simple n-gon contained in the Euclidean disk C.

REMARK

It is sufficient to find a triangle contained in C, with side-lengths α, β and γ, that satisfies perim $P \leq (n-2)\alpha + \beta + \gamma$.



Notations:

- The vertices of P are $p_1, p_2, \ldots, p_n = p_0$ in this counterclockwise order.

Notations:

- The vertices of P are $p_1, p_2, \ldots, p_n = p_0$ in this counterclockwise order.
- In a given Descartes coordinate system, the coordinates of a point q are denoted by $q = (\omega_q, \theta_q)$.

THE PROOF FOR $\mathbb{M} = \mathbb{E}^2$

Notations:

- The vertices of P are $p_1, p_2, \ldots, p_n = p_0$ in this counterclockwise order.
- In a given Descartes coordinate system, the coordinates of a point q are denoted by $q = (\omega_q, \theta_q)$.
- The coordinates of p_i are denoted by $p_i = (\omega_i, \theta_i)$.

Notations:

- The vertices of P are $p_1, p_2, \ldots, p_n = p_0$ in this counterclockwise order.
- In a given Descartes coordinate system, the coordinates of a point q are denoted by $q = (\omega_q, \theta_q)$.
- The coordinates of p_i are denoted by $p_i = (\omega_i, \theta_i)$.

DEFINITION

Let $p, q, a, b, c \in \mathbb{M}$.

Notations:

- The vertices of P are $p_1, p_2, \ldots, p_n = p_0$ in this counterclockwise order.
- In a given Descartes coordinate system, the coordinates of a point q are denoted by $q = (\omega_q, \theta_q)$.
- The coordinates of p_i are denoted by $p_i = (\omega_i, \theta_i)$.

DEFINITION

Let $p, q, a, b, c \in \mathbb{M}$. We say that the segment $[p, q]$ and the polygonal curve $[a, b] \cup [b, c]$ do not cross,

Notations:

- The vertices of P are $p_1, p_2, \ldots, p_n = p_0$ in this counterclockwise order.
- In a given Descartes coordinate system, the coordinates of a point q are denoted by $q = (\omega_q, \theta_q)$.
- The coordinates of p_i are denoted by $p_i = (\omega_i, \theta_i)$.

Definition

Let $p, q, a, b, c \in \mathbb{M}$. We say that the segment $[p, q]$ and the polygonal curve $[a, b] \cup [b, c]$ do not cross, if the set $(p, q) \cap \left((a, b] \cup [b, c) \right)$ is either empty or a segment.

LEMMA

Let $[p, q] \in C$ be a vertical segment and set $\delta = |p - q|$.

LEMMA

Let $[p, q] \in C$ be a vertical segment and set $\delta = |p - q|$. Let $a, b, c \in C$ be points such that $\theta_a \leq \theta_b \leq \theta_c$,

LEMMA

Let $[p, q] \in C$ be a vertical segment and set $\delta = |p - q|$. Let $a, b, c \in C$ be points such that $\theta_a \leq \theta_b \leq \theta_c$, and the polygonal curve $[a, b] \cup [b, c]$ and $[p, q]$ do not cross.

LEMMA

Let $[p, q] \in C$ be a vertical segment and set $\delta = |p - q|$. Let $a, b, c \in C$ be points such that $\theta_a \leq \theta_b \leq \theta_c$, and the polygonal curve $[a, b] \cup [b, c]$ and $[p, q]$ do not cross. If $|b - a|, |c - b| \leq \delta$,

LEMMA

Let $[p, q] \in C$ be a vertical segment and set $\delta = |p - q|$. Let $a, b, c \in C$ be points such that $\theta_a \leq \theta_b \leq \theta_c$, and the polygonal curve $[a, b] \cup [b, c]$ and $[p, q]$ do not cross. If $|b - a|, |c - b| \leq \delta$, then there are points $a', b', c' \in C$ such that $|b - a| + |c - b| \leq |b' - a'| + |c' - b'|$,

LEMMA

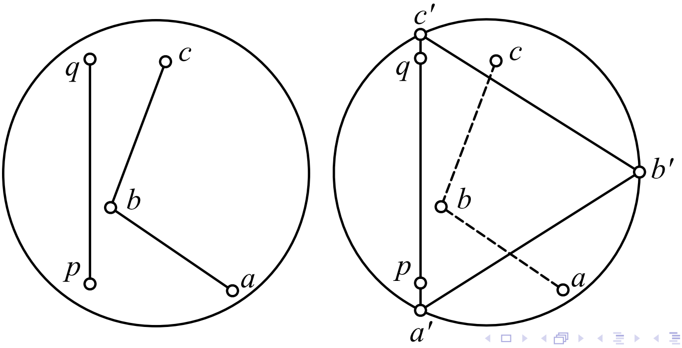
Let $[p, q] \in C$ be a vertical segment and set $\delta = |p - q|$. Let $a, b, c \in C$ be points such that $\theta_a \leq \theta_b \leq \theta_c$, and the polygonal curve $[a, b] \cup [b, c]$ and $[p, q]$ do not cross. If $|b - a|, |c - b| \leq \delta$, then there are points $a', b', c' \in C$ such that $|b - a| + |c - b| \leq |b' - a'| + |c' - b'|$, and $\delta \leq |c' - b'|$.

LEMMA

Let $[p, q] \in C$ be a vertical segment and set $\delta = |p - q|$. Let $a, b, c \in C$ be points such that $\theta_a \leq \theta_b \leq \theta_c$, and the polygonal curve $[a, b] \cup [b, c]$ and $[p, q]$ do not cross. If $|b - a|, |c - b| \leq \delta$, then there are points $a', b', c' \in C$ such that $|b - a| + |c - b| \leq |b' - a'| + |c' - b'|$, and $\delta \leq |c' - b'|$.

LEMMA

Let $[p, q] \in C$ be a vertical segment and set $\delta = |p - q|$. Let $a, b, c \in C$ be points such that $\theta_a \leq \theta_b \leq \theta_c$, and the polygonal curve $[a, b] \cup [b, c]$ and $[p, q]$ do not cross. If $|b - a|, |c - b| \leq \delta$, then there are points $a', b', c' \in C$ such that $|b - a| + |c - b| \leq |b' - a'| + |c' - b'|$, and $\delta \leq |c' - b'|$.



We may assume that

We may assume that
– $[p_0, p_1]$ is a longest side of P,

We may assume that
– $[p_0, p_1]$ is a longest side of P,
– $[p_0, p_1]$ is a vertical segment ($\omega_0 = \omega_1$).

We may assume that
– $[p_0, p_1]$ is a longest side of P,
– $[p_0, p_1]$ is a vertical segment ($\omega_0 = \omega_1$).

Consider the $(n+1)$-element sequence $\mu_1, \mu_2, \ldots \mu_{n+1}$ defined as follows.

We may assume that
– $[p_0, p_1]$ is a longest side of P,
– $[p_0, p_1]$ is a vertical segment ($\omega_0 = \omega_1$).

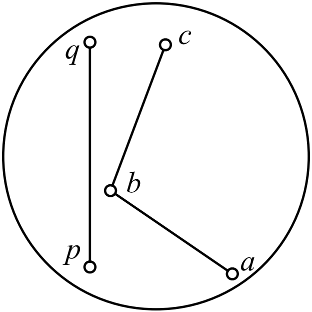
Consider the $(n+1)$-element sequence $\mu_1, \mu_2, \ldots \mu_{n+1}$ defined as follows.
For $i = 1, 2, \ldots, n+1$,

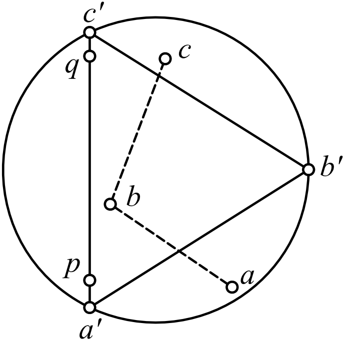
$$\mu_i = \begin{cases} 1, & \text{if } \theta_i < \theta_{i+1}; \\ -1, & \text{if } \theta_i > \theta_{i+1}; \\ 0, & \text{if } \theta_i = \theta_{i+1}. \end{cases}$$

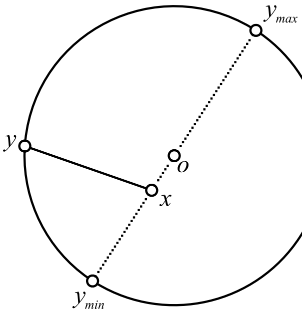
Observe that $\mu_0 = \mu_n \neq 0$.

Since $n+1$ is even, there are two consecutive elements of the sequence that are both nonnegative or both nonpositive.

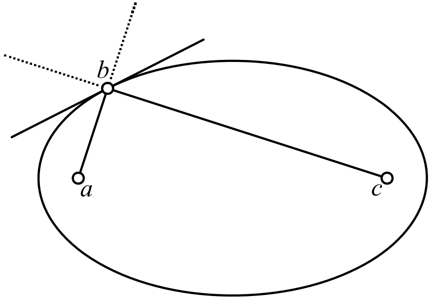
Since $n+1$ is even, there are two consecutive elements of the sequence that are both nonnegative or both nonpositive.Thus, we may apply Lemma.







y_{max}
y
o
x
y_{min}



b
a
c

Notations:

– $L(p,q)$: the line containing $[p,q]$;
– R_p: the connected component of $L(p,q) \setminus (p,q)$, containing p;
– R_q: the connected component of $L(p,q) \setminus (p,q)$, containing q.

Notations:

– $L(p, q)$: the line containing $[p, q]$;

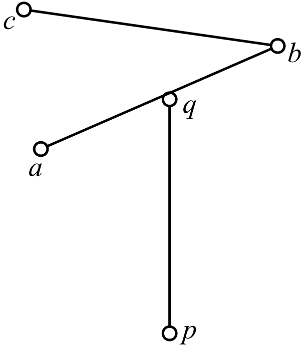
– R_p: the connected component of $L(p, q) \setminus (p, q)$, containing p;

– R_q: the connected component of $L(p, q) \setminus (p, q)$, containing q.

Assumptions: $\theta_p < \theta_q$ and the centre of C is the origin.

Notations:
– $L(p,q)$: the line containing $[p,q]$;
– R_p: the connected component of $L(p,q) \setminus (p,q)$, containing p;
– R_q: the connected component of $L(p,q) \setminus (p,q)$, containing q.

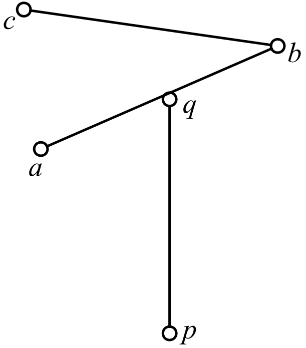
Assumptions: $\theta_p < \theta_q$ and the centre of C is the origin.



Case 1, if $[a,b]$ intersects R_q.

Notations:
– $L(p,q)$: the line containing $[p,q]$;
– R_p: the connected component of $L(p,q) \setminus (p,q)$, containing p;
– R_q: the connected component of $L(p,q) \setminus (p,q)$, containing q.

Assumptions: $\theta_p < \theta_q$ and the centre of C is the origin.

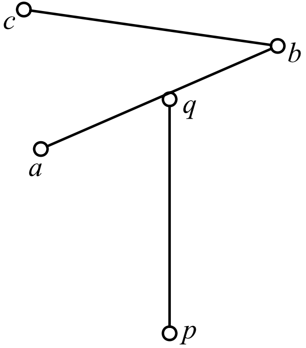


Case 1, if $[a,b]$ intersects R_q.

$|b-a| \leq \delta \leq |b-p|$

Notations:
– $L(p, q)$: the line containing $[p, q]$;
– R_p: the connected component of $L(p, q) \setminus (p, q)$, containing p;
– R_q: the connected component of $L(p, q) \setminus (p, q)$, containing q.

Assumptions: $\theta_p < \theta_q$ and the centre of C is the origin.

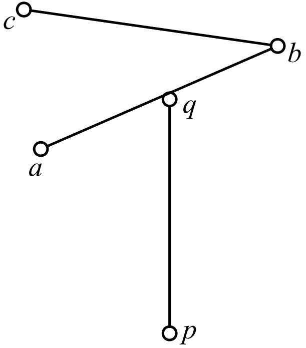


Case 1, if $[a, b]$ intersects R_q.
$|b - a| \leq \delta \leq |b - p|$
$\delta \leq |c - p|$.

Notations:

– $L(p,q)$: the line containing $[p,q]$;
– R_p: the connected component of $L(p,q) \setminus (p,q)$, containing p;
– R_q: the connected component of $L(p,q) \setminus (p,q)$, containing q.

Assumptions: $\theta_p < \theta_q$ and the centre of C is the origin.



Case 1, if $[a,b]$ intersects R_q.

$|b-a| \leq \delta \leq |b-p|$

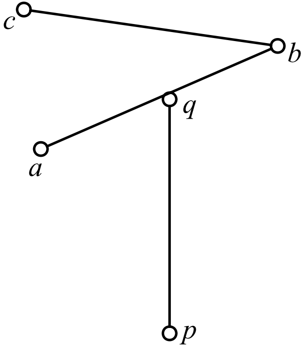
$\delta \leq |c-p|$.

We may choose a', b' and c' as p, b and c, respectively.

Notations:

– $L(p,q)$: the line containing $[p,q]$;
– R_p: the connected component of $L(p,q) \setminus (p,q)$, containing p;
– R_q: the connected component of $L(p,q) \setminus (p,q)$, containing q.

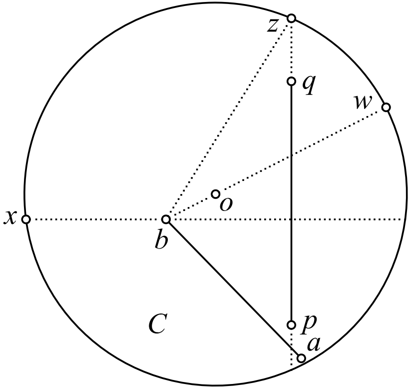
Assumptions: $\theta_p < \theta_q$ and the centre of C is the origin.

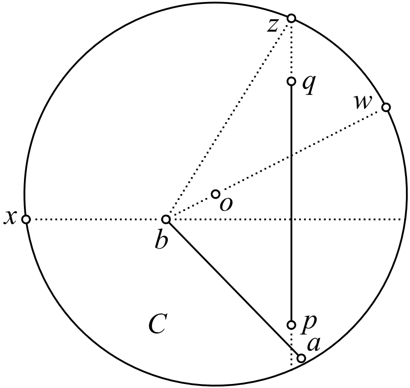


Case 1, if $[a,b]$ intersects R_q.
$|b-a| \leq \delta \leq |b-p|$
$\delta \leq |c-p|$.
We may choose a', b' and c' as p, b and c, respectively.

If $[b,c]$ intersects R_p, we may apply a similar argument.

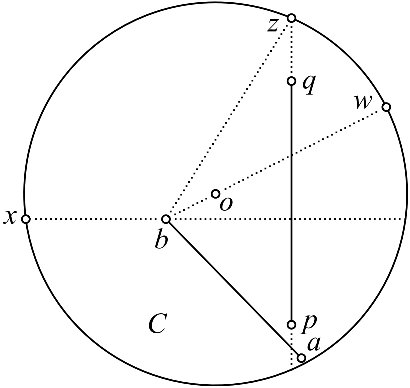
Case 2, if $[a, b]$ intersects R_p.





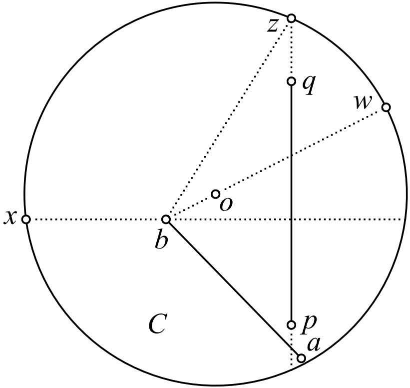
Case 2, if $[a, b]$ intersects R_p.

If $[b, c]$ intersects R_q, then $\delta \leq |c - a|$, and we are done.



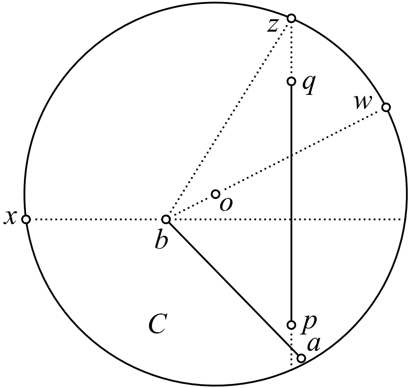
Case 2, if $[a, b]$ intersects R_p.

If $[b, c]$ intersects R_q, then $\delta \leq |c - a|$, and we are done. Let b, c be in the closed half plane $\bar{H}$ bounded by $L(p, q)$.



Case 2, if $[a, b]$ intersects R_p.

If $[b, c]$ intersects R_q, then $\delta \leq |c - a|$, and we are done. Let b, c be in the closed half plane $\bar{H}$ bounded by $L(p, q)$.
$\bar{C}$: the set of points $u \in \bar{H} \cap C$ that satisfy $\theta_u \geq \theta_b$.

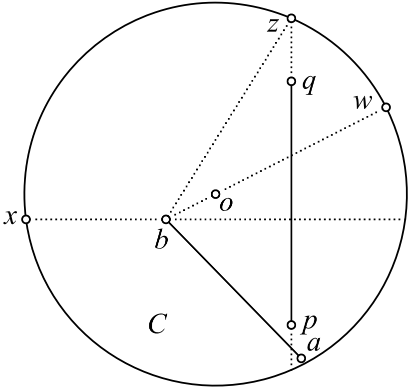


Case 2, if $[a, b]$ intersects R_p.

If $[b, c]$ intersects R_q, then $\delta \leq |c - a|$, and we are done. Let b, c be in the closed half plane $\bar{H}$ bounded by $L(p, q)$.

$\bar{C}$: the set of points $u \in \bar{H} \cap C$ that satisfy $\theta_u \geq \theta_b$.

z: the intersection point of R_q and bd C.



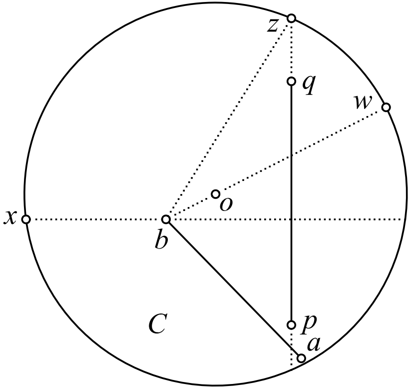
Case 2, if $[a, b]$ intersects R_p.

If $[b, c]$ intersects R_q, then $\delta \leq |c - a|$, and we are done. Let b, c be in the closed half plane $\bar{H}$ bounded by $L(p, q)$.

$\bar{C}$: the set of points $u \in \bar{H} \cap C$ that satisfy $\theta_u \geq \theta_b$.

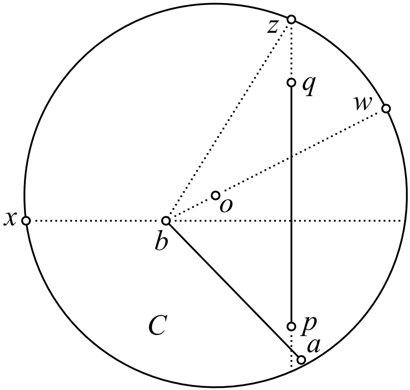
z: the intersection point of R_q and bd C.

x: the point of $\bar{H} \cap$ bd C with $\theta_x = \theta_b$

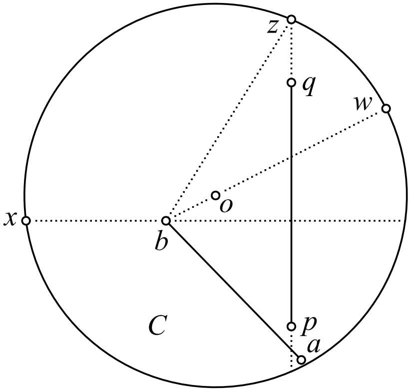


Case 2, if $[a, b]$ intersects R_p.

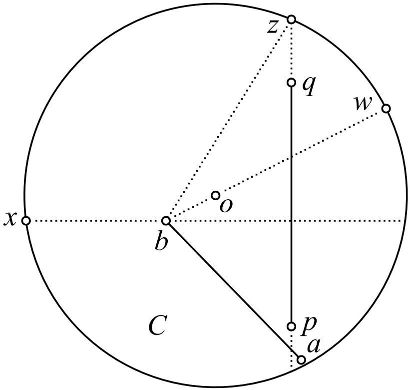
If $[b, c]$ intersects R_q, then $\delta \leq |c - a|$, and we are done. Let b, c be in the closed half plane $\bar{H}$ bounded by $L(p, q)$.
$\bar{C}$: the set of points $u \in \bar{H} \cap C$ that satisfy $\theta_u \geq \theta_b$.
z: the intersection point of R_q and bd C.
x: the point of $\bar{H} \cap$ bd C with $\theta_x = \theta_b$
w: the intersection point of bd C and the ray through o that starts at b, and $w = z$ otherwise.



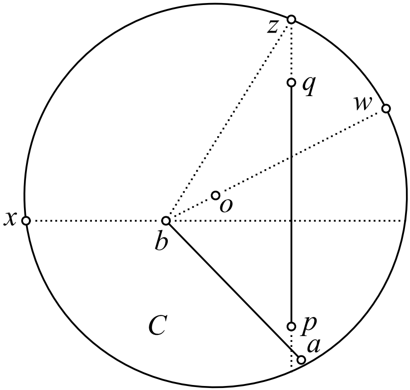
If $w \in \bar{C}$, then $|c - b| \leq |w - b|$, and we may choose a, b and w as a', b' and c', respectively.



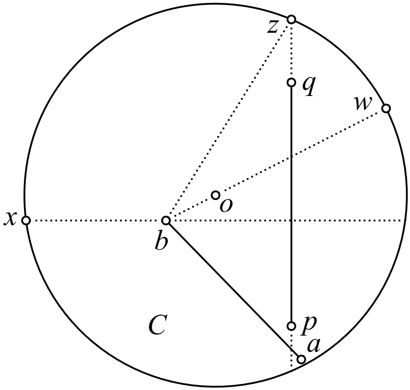
If $w \in \bar{C}$, then $|c-b| \leq |w-b|$, and we may choose a, b and w as a', b' and c', respectively. Assume that $w \notin \bar{C}$. Then $|c-b| \leq max\{|x-b|, |z-b|\}$.



If $w \in \bar{C}$, then $|c-b| \leq |w-b|$, and we may choose a, b and w as a', b' and c', respectively.
Assume that $w \notin \bar{C}$. Then $|c-b| \leq max\{|x-b|, |z-b|\}$. If $|x-b| \leq |z-b|$, then we may choose a, b and z as a', b' and c', respectively.



If $w \in \bar{C}$, then $|c-b| \leq |w-b|$, and we may choose a, b and w as a', b' and c', respectively.
Assume that $w \notin \bar{C}$. Then $|c-b| \leq max\{|x-b|, |z-b|\}$.
If $|x-b| \leq |z-b|$, then we may choose a, b and z as a', b' and c', respectively.
In the opposite case, we may choose a, x and z, respectively.



If $w \in \bar{C}$, then $|c-b| \leq |w-b|$, and we may choose a, b and w as a', b' and c', respectively. Assume that $w \notin \bar{C}$. Then $|c-b| \leq max\{|x-b|, |z-b|\}$. If $|x-b| \leq |z-b|$, then we may choose a, b and z as a', b' and c', respectively. In the opposite case, we may choose a, x and z, respectively.

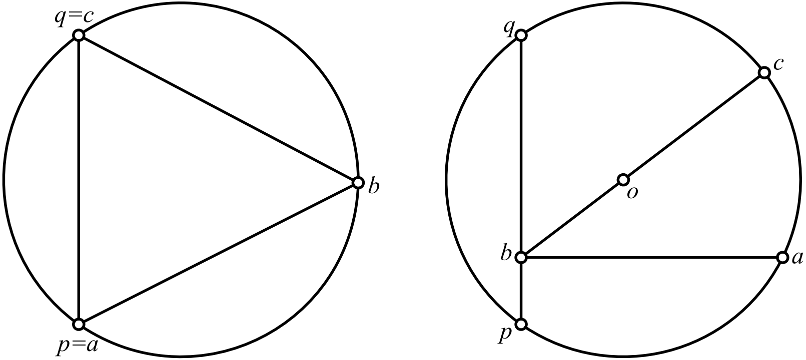
A similar argument proves the assertion in the case that $[b, c]$ intersects R_q.

Case 3, a, b and c are in the same closed half plane H^+ bounded by $L(p, q)$.

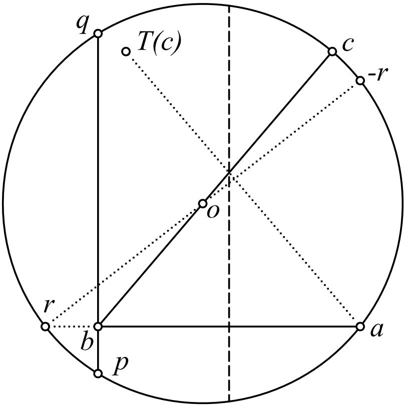
Case 3, a, b and c are in the same closed half plane H^+ bounded by $L(p,q)$.
We may assume that $o \in H^+$, and that $p, q \in \text{bd}\, C$.

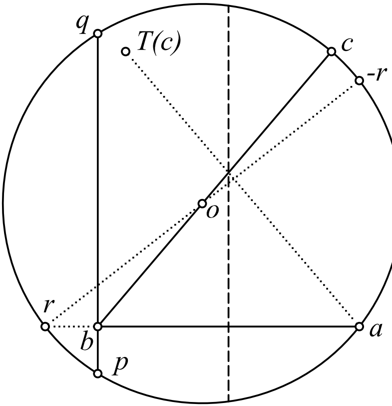
Case 3, a, b and c are in the same closed half plane H^+ bounded by $L(p,q)$.
We may assume that $o \in H^+$, and that $p, q \in \text{bd}\, C$. Now we drop the conditions that $|b-a|, |c-b| \leq \delta$, and maximize $|b-a| + |c-b|$ under the conditions that $a, b, c \in C \cap H^+ = C^+$ and $\theta_a \leq \theta_b \leq \theta_c$.

Case 3, a, b and c are in the same closed half plane H^+ bounded by $L(p,q)$.
We may assume that $o \in H^+$, and that $p, q \in \text{bd}\, C$. Now we drop the conditions that $|b-a|, |c-b| \leq \delta$, and maximize $|b-a| + |c-b|$ under the conditions that $a, b, c \in C \cap H^+ = C^+$ and $\theta_a \leq \theta_b \leq \theta_c$.

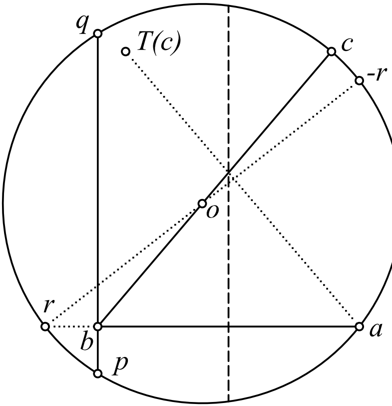


THE PROOF OF THE LEMMA



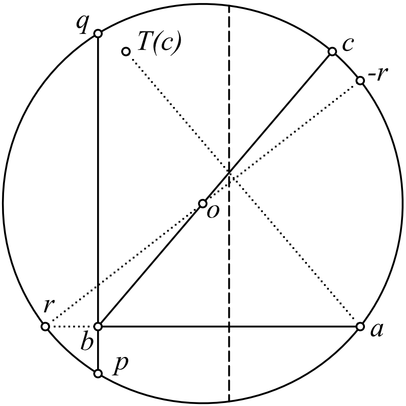


r: the reflected image of a about the y-axis



r: the reflected image of a about the y-axis

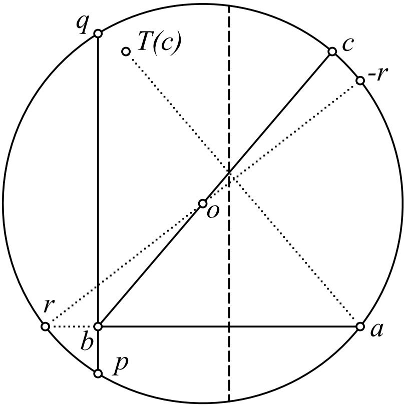
Observation: $\omega_c \leq \omega_{-r} = \omega_a$



r: the reflected image of a about the y-axis

Observation: $\omega_c \leq \omega_{-r} = \omega_a$

T: the reflection about the line bisecting $[a, b]$

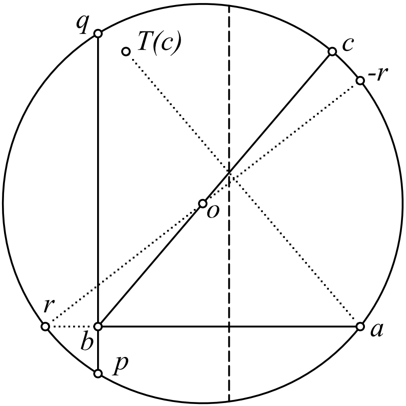


r: the reflected image of a about the y-axis

Observation: $\omega_c \leq \omega_{-r} = \omega_a$

T: the reflection about the line bisecting $[a, b]$

Then: $T(a) = b$, $T(b) = a$ and $T(c) \in C^+$



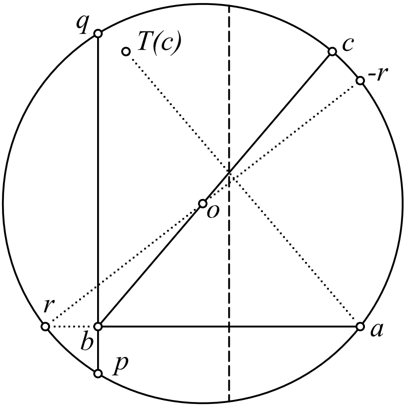
r: the reflected image of a about the y-axis

Observation: $\omega_c \leq \omega_{-r} = \omega_a$

T: the reflection about the line bisecting $[a, b]$

Then: $T(a) = b$, $T(b) = a$ and $T(c) \in C^+$

$|b - a| \leq |p - a|$,



r: the reflected image of a about the y-axis

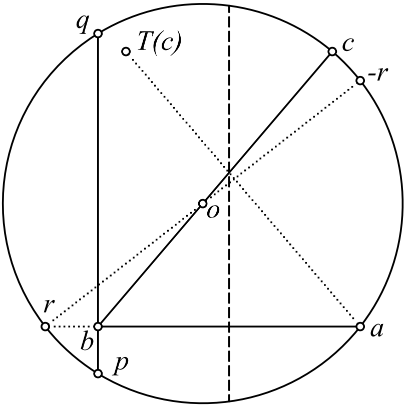
Observation: $\omega_c \leq \omega_{-r} = \omega_a$

T: the reflection about the line bisecting $[a, b]$

Then: $T(a) = b$, $T(b) = a$ and $T(c) \in C^+$

$|b - a| \leq |p - a|$,

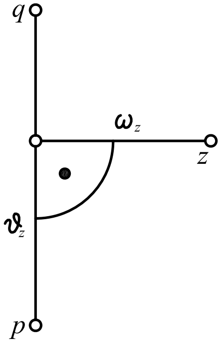
$|c - b| = |T(c) - a| \leq |q - a|$,



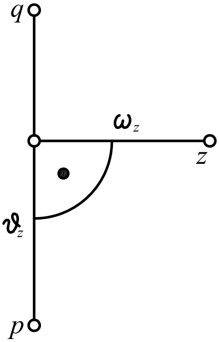
r: the reflected image of a about the y-axis
Observation: $\omega_c \leq \omega_{-r} = \omega_a$
T: the reflection about the line bisecting $[a, b]$
Then: $T(a) = b$, $T(b) = a$ and $T(c) \in C^+$
$|b - a| \leq |p - a|$,
$|c - b| = |T(c) - a| \leq |q - a|$,
we can choose p, a, q as a', b', c'.

What is the coordinate system?

What is the coordinate system?

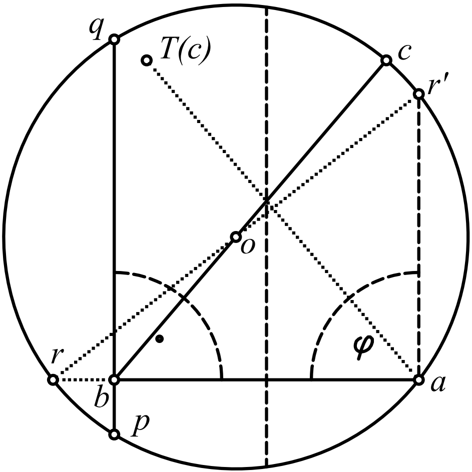


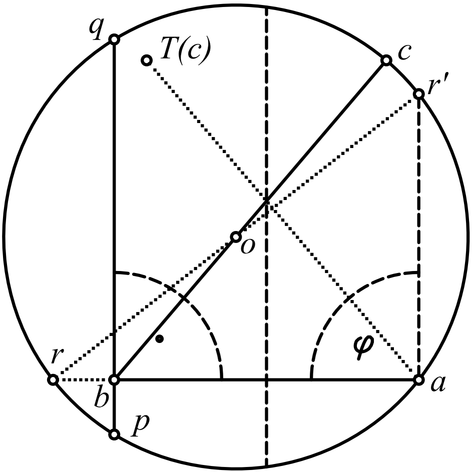
What is the coordinate system?



An additional assumption for $\mathbb{S}^2$:
the radius ρ of the circle is $\rho \leq \frac{\pi}{4}$.

The proof for $\mathbb{M} = \mathbb{H}^2$ and $\mathbb{M} = \mathbb{S}^2$





In $\mathbb{H}^2$ $\phi < \frac{\pi}{2}$, in $\mathbb{S}^2$ $\phi > \frac{\pi}{2}$.

Lemma does not hold for some spherical disks with radius $\frac{\pi}{4} < \rho < \frac{\pi}{2}$.

EXAMPLE

Let $\varepsilon > 0$ and let $p, q \in \mathbb{S}^2$ be two points with $\text{dist}_S(p, q) = \pi - \varepsilon$.

Lemma does not hold for some spherical disks with radius $\frac{\pi}{4} < \rho < \frac{\pi}{2}$.

Example

Let $\varepsilon > 0$ and let $p, q \in \mathbb{S}^2$ be two points with $\mathrm{dist}_S(p, q) = \pi - \varepsilon$. Let $k, -k$ be the centres of the two open hemispheres bounded by $L(p, q)$.

Lemma does not hold for some spherical disks with radius $\frac{\pi}{4} < \rho < \frac{\pi}{2}$.

Example

Let $\varepsilon > 0$ and let $p, q \in \mathbb{S}^2$ be two points with $\text{dist}_S(p, q) = \pi - \varepsilon$. Let $k, -k$ be the centres of the two open hemispheres bounded by $L(p, q)$. Let $[a, b]$ be a segment of length $\pi - \varepsilon$, perpendicular to $[p, q]$, such that q is the midpoint of $[a, b]$.

Lemma does not hold for some spherical disks with radius $\frac{\pi}{4} < \rho < \frac{\pi}{2}$.

Example

Let $\varepsilon > 0$ and let $p, q \in \mathbb{S}^2$ be two points with $\text{dist}_S(p, q) = \pi - \varepsilon$. Let $k, -k$ be the centres of the two open hemispheres bounded by $L(p, q)$. Let $[a, b]$ be a segment of length $\pi - \varepsilon$, perpendicular to $[p, q]$, such that q is the midpoint of $[a, b]$. Note that $k, -k \in L(a, b) \setminus [a, b]$.

Remark about $\mathbb{S}^2$

Lemma does not hold for some spherical disks with radius $\frac{\pi}{4} < \rho < \frac{\pi}{2}$.

Example

Let $\varepsilon > 0$ and let $p, q \in \mathbb{S}^2$ be two points with $\text{dist}_S(p, q) = \pi - \varepsilon$. Let $k, -k$ be the centres of the two open hemispheres bounded by $L(p, q)$. Let $[a, b]$ be a segment of length $\pi - \varepsilon$, perpendicular to $[p, q]$, such that q is the midpoint of $[a, b]$. Note that $k, -k \in L(a, b) \setminus [a, b]$. Thus, there is a disk C of radius $\rho < \frac{\pi}{2}$ that contains $[p, q]$ and $[a, b]$.

Lemma does not hold for some spherical disks with radius $\frac{\pi}{4} < \rho < \frac{\pi}{2}$.

Example

Let $\varepsilon > 0$ and let $p, q \in \mathbb{S}^2$ be two points with $\text{dist}_S(p, q) = \pi - \varepsilon$. Let $k, -k$ be the centres of the two open hemispheres bounded by $L(p, q)$. Let $[a, b]$ be a segment of length $\pi - \varepsilon$, perpendicular to $[p, q]$, such that q is the midpoint of $[a, b]$. Note that $k, -k \in L(a, b) \setminus [a, b]$. Thus, there is a disk C of radius $\rho < \frac{\pi}{2}$ that contains $[p, q]$ and $[a, b]$. Set $c = a$.

Lemma does not hold for some spherical disks with radius $\frac{\pi}{4} < \rho < \frac{\pi}{2}$.

Example

Let $\varepsilon > 0$ and let $p, q \in \mathbb{S}^2$ be two points with $\mathrm{dist}_S(p, q) = \pi - \varepsilon$. Let $k, -k$ be the centres of the two open hemispheres bounded by $L(p, q)$. Let $[a, b]$ be a segment of length $\pi - \varepsilon$, perpendicular to $[p, q]$, such that q is the midpoint of $[a, b]$. Note that $k, -k \in L(a, b) \setminus [a, b]$. Thus, there is a disk C of radius $\rho < \frac{\pi}{2}$ that contains $[p, q]$ and $[a, b]$. Set $c = a$. If ε is sufficiently small, then $3\pi - 3\varepsilon$ is greater than the perimeter of any triangle inscribed in C.

QUESTION

Let $n \geq 5$ be odd, $0 < \rho < \frac{\pi}{2}$, and $C \subset \mathbb{S}^2$ be a disk of radius ρ. What is the supremum of the perimeters of the simple n-gons contained in C?

Question

Let $n \geq 5$ be odd, $0 < \rho < \frac{\pi}{2}$, and $C \subset \mathbb{S}^2$ be a disk of radius ρ. What is the supremum of the perimeters of the simple n-gons contained in C?

Question

Let $n \geq 5$ be odd, and let $C \subset \mathbb{E}^2$ be a plane convex body. Prove or disprove that if P is a simple n-gon contained in C, then there is a triangle, inscribed in C and with side-lengths α, β and γ, such that $\operatorname{perim} P \leq (n-2)\alpha + \beta + \gamma$. Is it true for plane convex bodies in the hyperbolic plane or on the sphere?

QUESTION

Let $n \geq 5$ be odd, and let $\mathbb{M}$ be a Minkowski plane with the unit disk C. What is the supremum of the perimeters of the simple n-gons contained in C? In particular, can Theorem be generalized for Minkowski planes? Can it be generalized for an arbitrary plane convex body of $\mathbb{M}$ instead of the unit disk of $\mathbb{M}$?

Question

Let $n \geq 5$ be odd, and let $\mathbb{M}$ be a Minkowski plane with the unit disk C. What is the supremum of the perimeters of the simple n-gons contained in C? In particular, can Theorem be generalized for Minkowski planes? Can it be generalized for an arbitrary plane convex body of $\mathbb{M}$ instead of the unit disk of $\mathbb{M}$?

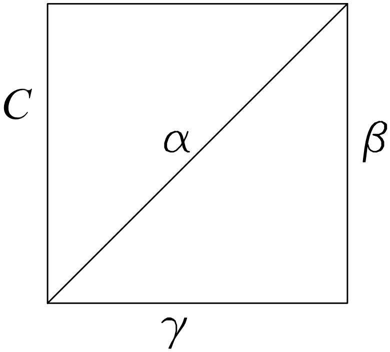
In the last two questions the optimal triangle inscribed in C is not necessarily isosceles.

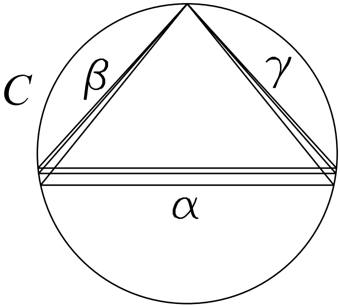
THEOREM

Let $n \geq 3$ be an odd integer, and let C be a plane convex body in $\mathbb{E}^2$ or in $\mathbb{H}^2$. For every simple n-gon P contained in C there is a triangle, inscribed in C and with side-lengths $\alpha \geq \beta \geq \gamma$, such that $\operatorname{perim} P \leq (n-2)\alpha + \beta + \gamma$.

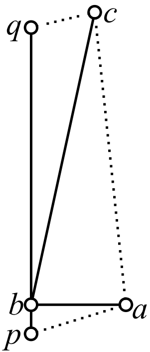
THEOREM

Let $n \geq 3$ be an odd integer, and let C be a plane convex body in $\mathbb{E}^2$ or in $\mathbb{H}^2$. For every simple n-gon P contained in C there is a triangle, inscribed in C and with side-lengths $\alpha \geq \beta \geq \gamma$, such that $\operatorname{perim} P \leq (n-2)\alpha + \beta + \gamma$.





Our lemma fails:



$$p = (0,0), \quad q = (0,1),$$
$$a = (0.31, 0.095), \quad b = (0, 0.095),$$
$$c = (0.208, 1.05),$$
$$C = \operatorname{conv}\{p, q, a, b, c\},$$

$$|b - a| = 0.3100\ldots,$$
$$||c - b| = 0.9773\ldots,$$
$$|c - a| = 0.9604\ldots,$$

$$|b - a| + |c - b| = 1.2873\ldots,$$
$$|c - p| + |q - c| = 1.2843\ldots,$$
$$|a - p| + |q - a| = 1.2808\ldots,$$
$$|a - p| + |c - a| = 1.2846\ldots.$$

The idea of the proof

Step 1: To examine under what conditions does the assertion of the lemma fail

Step 2: To prove the theorem in this case using a different method.

The idea of the proof

Step 1: To examine under what conditions does the assertion of the lemma fail

Step 2: To prove the theorem in this case using a different method.

DEFINITION

If a', b', c' satisfy $|p - q| \leq |c' - a'|$ and $|b - a| + |c - b| \leq |b' - a'| + |c' - b'|$, we say that a', b' *and* c' *satisfy Property (*)*.

Assumptions:

1. $p = (0, 0)$ and $q = (0, 1)$.

Assumptions:

1. $p = (0, 0)$ and $q = (0, 1)$.
2. $\theta_a \leq \theta_b \leq \theta_c$,

Assumptions:

1. $p = (0, 0)$ and $q = (0, 1)$.
2. $\theta_a \leq \theta_b \leq \theta_c$,
3. a is not farther from the bisector of $[p, q]$ than c (or in other words, $\theta_a + \theta_c \geq 1$),

Assumptions:

1. $p = (0, 0)$ and $q = (0, 1)$.
2. $\theta_a \leq \theta_b \leq \theta_c$,
3. a is not farther from the bisector of $[p, q]$ than c (or in other words, $\theta_a + \theta_c \geq 1$),
4. at least one of ω_a, ω_b and ω_c is positive.

Assumptions:

1. $p = (0, 0)$ and $q = (0, 1)$.
2. $\theta_a \leq \theta_b \leq \theta_c$,
3. a is not farther from the bisector of $[p, q]$ than c (or in other words, $\theta_a + \theta_c \geq 1$),
4. at least one of ω_a, ω_b and ω_c is positive.

Notation: $p_a = (0, \theta_a)$, $p_b = (0, \theta_b)$, $p_c = (0, \theta_c)$.

THE PROOF FOR $\mathbb{E}^2$

Assumptions:

1. $p = (0, 0)$ and $q = (0, 1)$.
2. $\theta_a \leq \theta_b \leq \theta_c$,
3. a is not farther from the bisector of $[p, q]$ than c (or in other words, $\theta_a + \theta_c \geq 1$),
4. at least one of ω_a, ω_b and ω_c is positive.

Notation: $p_a = (0, \theta_a)$, $p_b = (0, \theta_b)$, $p_c = (0, \theta_c)$.

LEMMA

If there are no points $a', b', c' \in \operatorname{conv}\{p, q, a, b, c\}$ satisfying Property (*), then the following hold.

LEMMA

If there are no points $a', b', c' \in \text{conv}\{p, q, a, b, c\}$ *satisfying Property (*), then the following hold.*

(A) $|c - a| < 1$.

Lemma

If there are no points $a', b', c' \in \text{conv}\{p, q, a, b, c\}$ satisfying Property (*), then the following hold.

(A) $|c - a| < 1$.

(B) a, b *and* c *are in the same closed half-plane bounded by* $L(p, q)$.

LEMMA

If there are no points $a', b', c' \in \text{conv}\{p, q, a, b, c\}$ satisfying Property (*), then the following hold.

(A) $|c - a| < 1$.

(B) *a, b and c are in the same closed half-plane bounded by $L(p, q)$.*

(B) $\theta_c > 1$ *and* $0 < \theta_a < \frac{1}{2}$.

LEMMA

If there are no points $a', b', c' \in \operatorname{conv}\{p, q, a, b, c\}$ satisfying Property (*), then the following hold.

(A) $|c - a| < 1$.

(B) a, b and c are in the same closed half-plane bounded by $L(p, q)$.

(B) $\theta_c > 1$ and $0 < \theta_a < \frac{1}{2}$.

(D) $b \in \operatorname{conv}\{p_a, p_c, a, c\}$.

LEMMA

If there are no points $a', b', c' \in \text{conv}\{p, q, a, b, c\}$ satisfying Property (*), then the following hold.

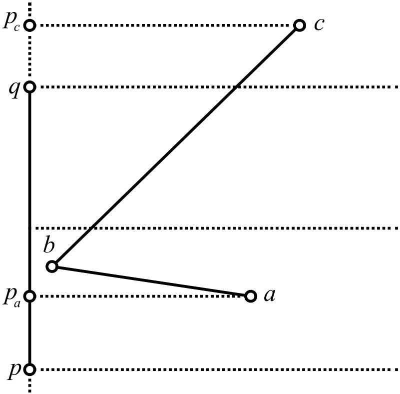
(A) $|c - a| < 1$.

(B) a, b *and* c *are in the same closed half-plane bounded by* $L(p, q)$.

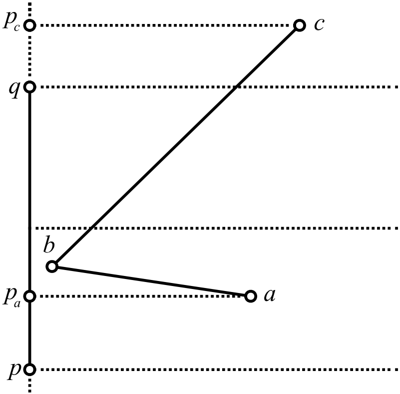
(B) $\theta_c > 1$ *and* $0 < \theta_a < \frac{1}{2}$.

(D) $b \in \text{conv}\{p_a, p_c, a, c\}$.

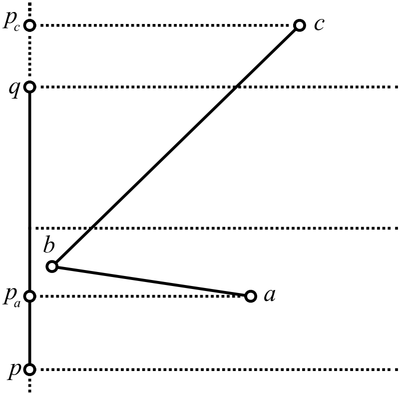
(E) $|b - a| + |c - b| \leq |p_a - a| + |c - p_a|$.



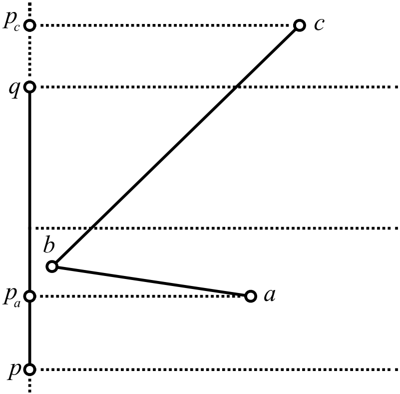
p_c
c
q
b
p_a
a
p



We may assume that a, b, c, p, q does not satisfy Property (*).



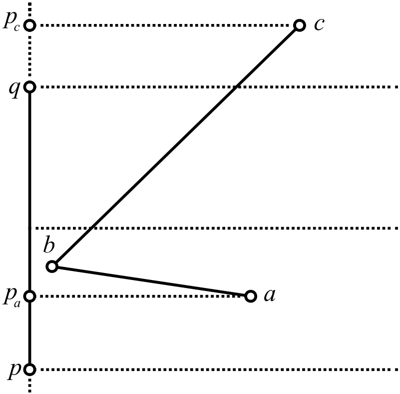
We may assume that a, b, c, p, q does not satisfy Property (*).
Observation: $\theta_c - \theta_a > \frac{\theta_c}{2}$.



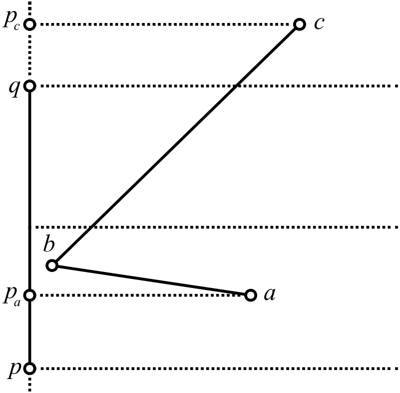
We may assume that a, b, c, p, q does not satisfy Property (*).

Observation: $\theta_c - \theta_a > \frac{\theta_c}{2}$.

Case 1, $n = 5$.



We may assume that a, b, c, p, q does not satisfy Property (*).
Observation: $\theta_c - \theta_a > \frac{\theta_c}{2}$.
Case 1, $n = 5$.
Then the remaining two edges of P are $[p, a]$ and $[c, q]$,

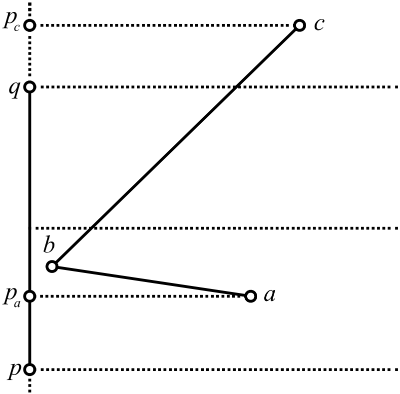


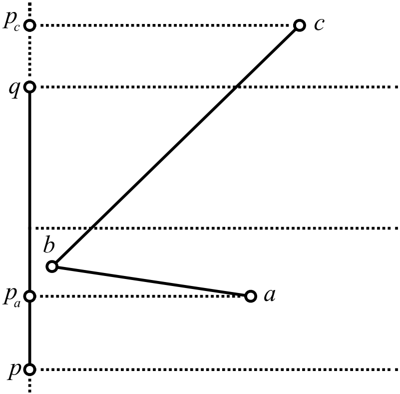
We may assume that a, b, c, p, q does not satisfy Property (*).

Observation: $\theta_c - \theta_a > \frac{\theta_c}{2}$.

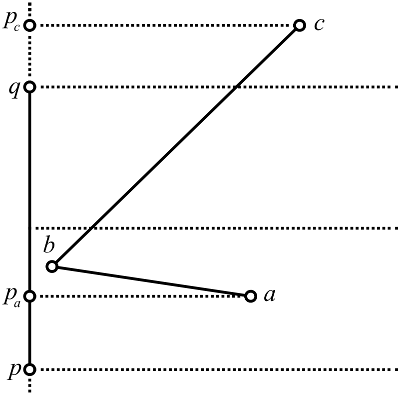
Case 1, $n = 5$.

Then the remaining two edges of P are $[p, a]$ and $[c, q]$, and perim $P \leq 3|p - c| + |p - q| + |q - c|$.

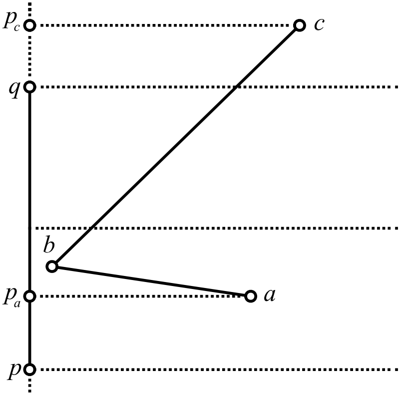

p_c
c
q
b
p_a
a
p



Case 2, $n \geq 7$.

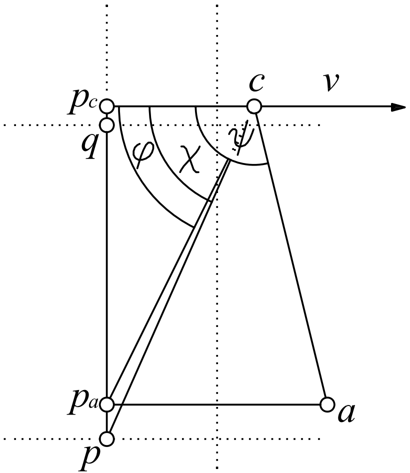


Case 2, $n \geq 7$.If $|c-a| \geq |c-p_a|$, then p, a, c satisfies Property (*). Thus, we assume that $\omega_c \geq \frac{\omega_a}{2}$.



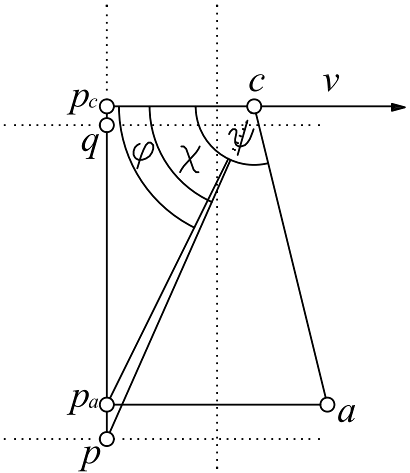
Case 2, $n \geq 7$.If $|c-a| \geq |c-p_a|$, then p, a, c satisfies Property (*). Thus, we assume that $\omega_c \geq \frac{\omega_a}{2}$.

Since $|q-p| = 1$, it is sufficient to prove that $5 + |p_a - a| + |c - p_a| \leq 5|c - p| + |a-p| + |c-a|$.

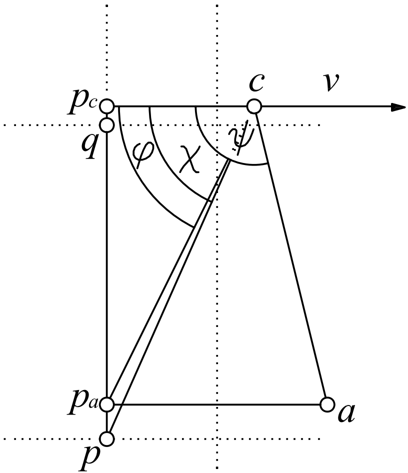


Notations:

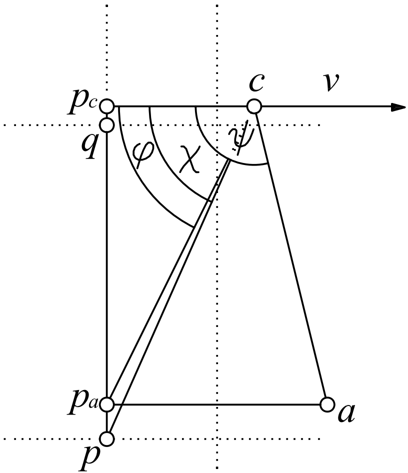
$M(c) = 5 + |p_a - a| + |c - p_a|$,
$N(c) = 5|c-p|+|a-p|+|c-a|$,
$v = (1, 0)$, $w = \left(\frac{\omega_a}{2}, \theta_c\right)$.



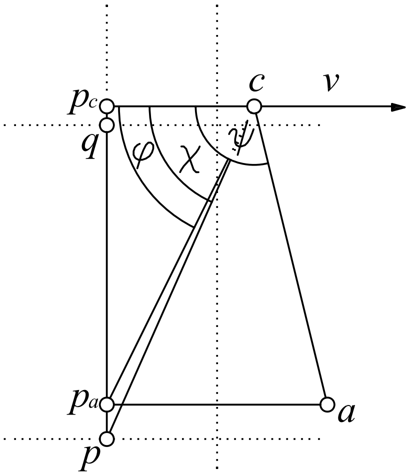
Notations:
$M(c) = 5 + |p_a - a| + |c - p_a|$,
$N(c) = 5|c-p| + |a-p| + |c-a|$,
$v = (1,0)$, $w = \left(\frac{\omega_a}{2}, \theta_c\right)$.
Observation: $M(w) \leq N(w)$



Notations:
$M(c) = 5 + |p_a - a| + |c - p_a|$,
$N(c) = 5|c - p| + |a - p| + |c - a|$,
$v = (1, 0)$, $w = \left(\frac{\omega_a}{2}, \theta_c\right)$.
Observation: $M(w) \leq N(w)$
We need to show that $v(M) \leq v(N)$.



Notations:
$M(c) = 5 + |p_a - a| + |c - p_a|$,
$N(c) = 5|c-p| + |a-p| + |c-a|$,
$v = (1, 0)$, $w = \left(\frac{\omega_a}{2}, \theta_c\right)$.
Observation: $M(w) \leq N(w)$
We need to show that $v(M) \leq v(N)$.
Observations: $0 < \phi \leq \pi - \psi < \pi$
$\cos\phi \geq -\cos\psi$.



$M(c) = 5 + |p_a - a| + |c - p_a|$,
$N(c) = 5|c-p|+|a-p|+|c-a|$,
$v(M) = \cos\phi$,
$v(N) = 5\cos\chi + \cos\psi \geq 5\cos\chi - \cos\phi$.
We set:
$I = 5\cos\chi - 2\cos\phi \leq v(N) - v(M)$, and need to show that $I \geq 0$.

The proof for $\mathbb{E}^2$

$$I = \frac{5\omega_c}{\sqrt{\omega_c^2+\theta_c^2}} - \frac{2\omega_c}{\sqrt{\omega_c^2+(\theta_c-\theta_a)^2}} \geq \frac{5\omega_c}{\sqrt{\omega_c^2+\theta_c^2}} - \frac{2\omega_c}{\sqrt{\omega_c^2+(\theta_c/2)^2}} =$$

$$= \frac{\omega_c\left(21\omega_c^2+\frac{9}{4}\theta_c^2\right)}{\sqrt{\omega_c^2+\theta_c^2}\sqrt{\omega_c^2+(\theta_c/2)^2}\left(5\sqrt{\omega_c^2+(\theta_c/2)^2}+2\sqrt{\omega_c^2+\theta_c^2}\right)} \geq 0.$$

The proof for $\mathbb{E}^2$

$$I = \frac{5\omega_c}{\sqrt{\omega_c^2+\theta_c^2}} - \frac{2\omega_c}{\sqrt{\omega_c^2+(\theta_c-\theta_a)^2}} \geq \frac{5\omega_c}{\sqrt{\omega_c^2+\theta_c^2}} - \frac{2\omega_c}{\sqrt{\omega_c^2+(\theta_c/2)^2}} =$$

$$= \frac{\omega_c\left(21\omega_c^2+\frac{9}{4}\theta_c^2\right)}{\sqrt{\omega_c^2+\theta_c^2}\sqrt{\omega_c^2+(\theta_c/2)^2}\left(5\sqrt{\omega_c^2+(\theta_c/2)^2}+2\sqrt{\omega_c^2+\theta_c^2}\right)} \geq 0.$$

For $\mathbb{H}^2$ a similar proof works.

And finally . . .

The End