DISPERSION IN DISKS

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A dispersion problem:

First studied by Cabello, 2007.

> Given n unit disks in the plane, select a point in each disk, such that the minimum pairwise distance among the points is *maximized*.

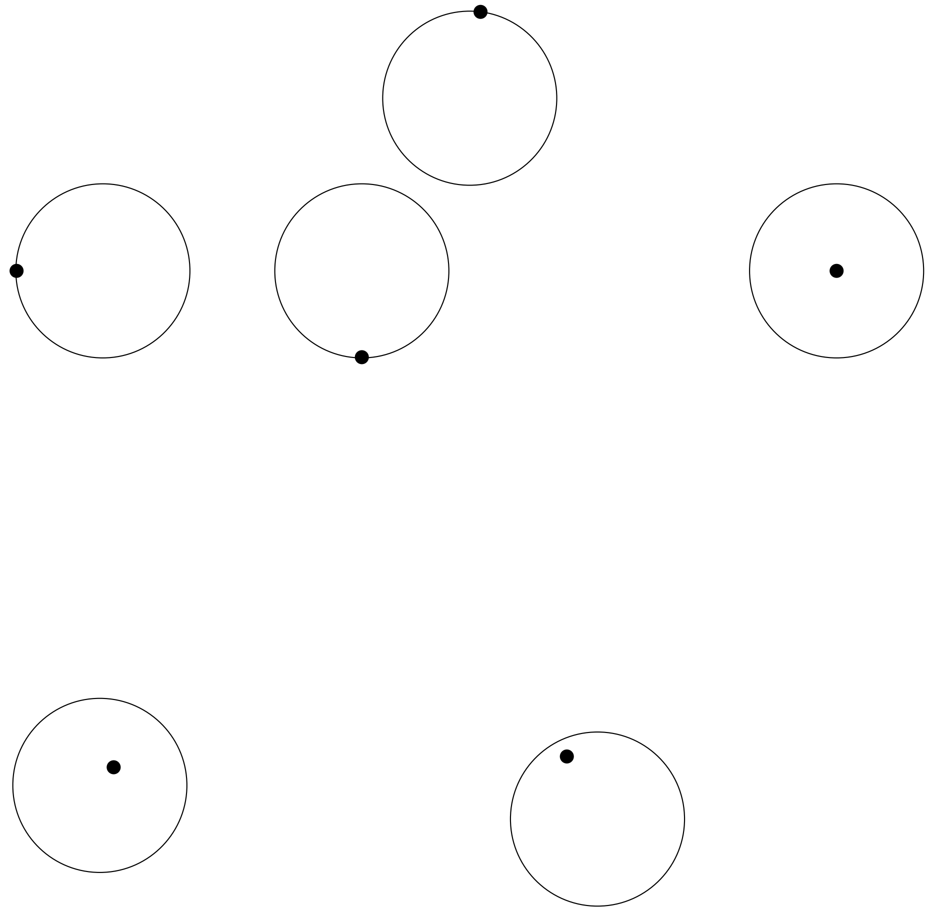
Variants:

- (not necessarily disjoint) disks of arbitrary radii
- disjoint unit disks

First variant: shown to be NP-hard. [Fiala, Kratochvíl, Proskurowski, 2007].

Second variant: also NP-hard; one can modify their reduction.

Dispersion in disjoint unit disks



Background

Let $\mathcal{R}$ be a family of n subsets of a metric space. The problem of *dispersion in* $\mathcal{R}$ is that of selecting n points, one in each subset, such that the minimum inter-point distance is *maximized*.

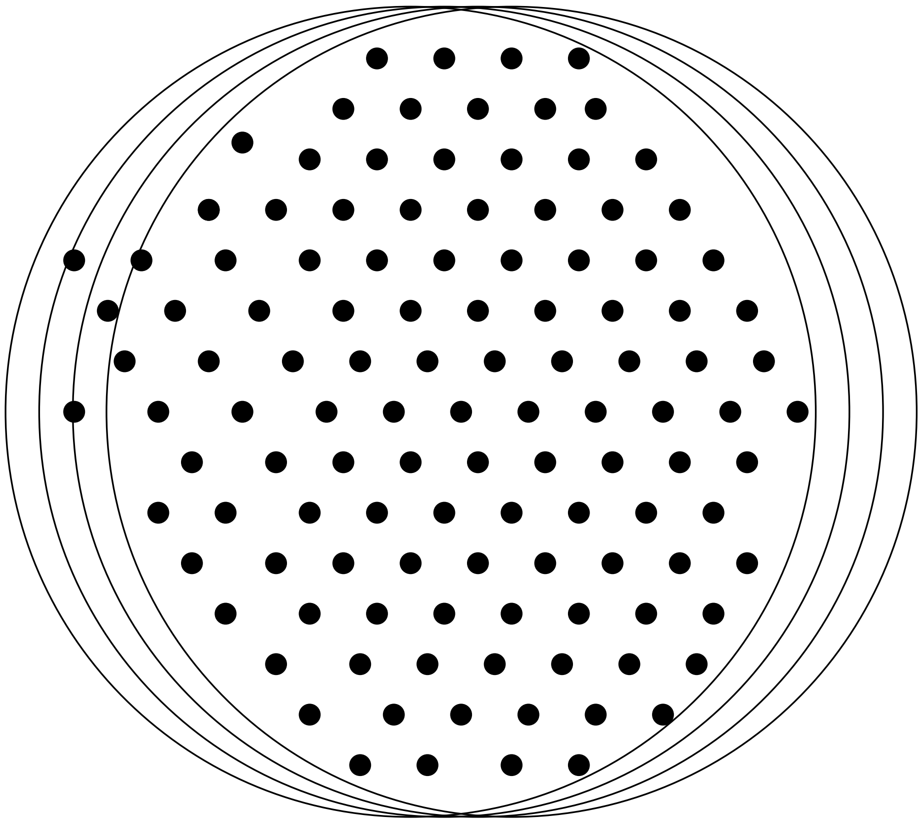
This dispersion problem was introduced by Fiala et al. (2007) as "systems of distant representatives", generalizing the classic problem "systems of distinct representatives".

An especially interesting version of the dispersion problem, (with natural applications to wireless networking and map labeling), is in a geometric setting where:

$\mathcal{R}$ is a set of unit disks in the plane.

Dispersion in disks

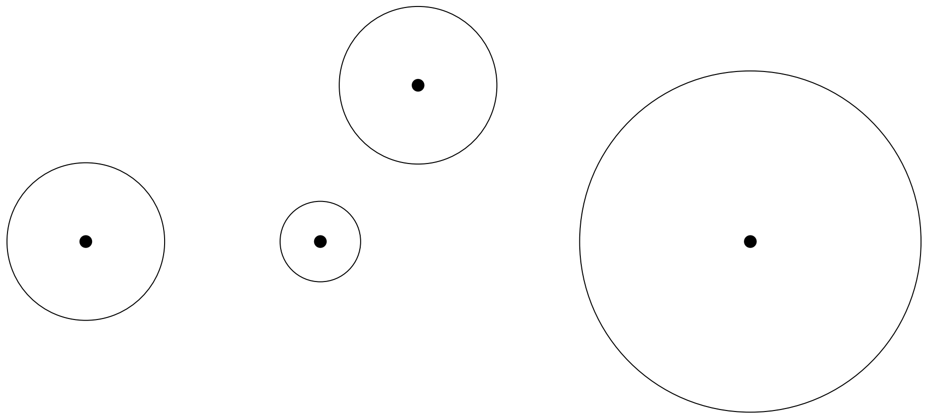
Example: Dispersion a heavily overlapping family of disks: leads to a *packing problem with congruent disks*.



Approximation algorithms (old and new)

Cabello (2007) gave a $O(n^2)$-time approximation algorithm with ratio 0.4465 for dispersion in n not necessarily disjoint unit disks.

For dispersion in disjoint disks, Cabello (2007) noticed that a naive algorithm called CENTERS, which simply selects the centers of the given disks as the points, gives a $\frac{1}{2}$-approximation.



- Arbitrary unit disks: 0.4465 $\rightarrow$ 0.4674
- Disjoint disks (or balls in d-space): 1/2 $\rightarrow$ 0.707

Our results

Theorem 1 *There is an $O(n \log n)$-time approximation algorithm* **A1** *with ratio* 0.5110 *for dispersion in n disjoint unit disks.*

Theorem 2 *There is an LP-based approximation algorithm* **A2***, with $O(n)$ variables and constraints, and running in polynomial time, that achieves approximation ratio* 0.707*, for dispersion in n disjoint disks of arbitrary radii. Moreover, the algorithm can be extended for disjoint balls of arbitrary radii in $\mathbb{R}^d$, for any (fixed) dimension d, while preserving the same features.*

Theorem 3 *In combination with an algorithm of Cabello, the simple $O(n \log n)$-time algorithm in Theorem 1 yields an $O(n^2)$-time algorithm with ratio* 0.4487*, and the LP-based algorithm in Theorem 2 yields a polynomial-time algorithm with ratio* 0.4674*, for dispersion in n (not necessarily disjoint) unit* disks.

Definitions and notations

For two points, $p = (x_p, y_p)$ and $q = (x_q, y_q)$,
$|pq| = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2}$

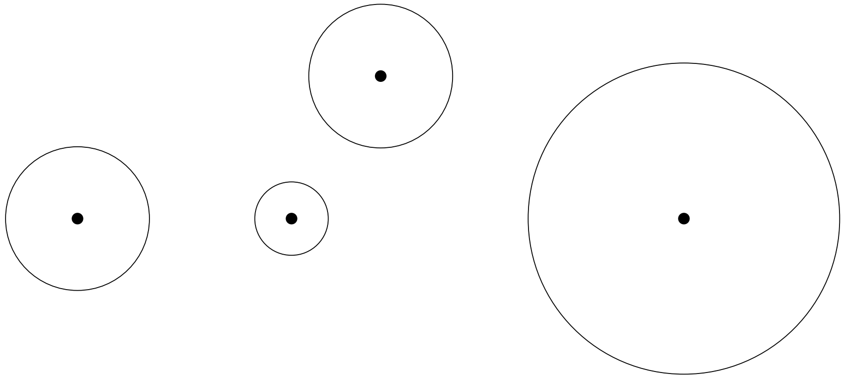
$\mathcal{D} = \{\Omega_1, \ldots, \Omega_n\}$ a set of n disjoint disks of arbitrary radii in the plane; o_i: the center of Ω_i r_i: the radius of Ω_i.

Here: the *distance between two disks* is the distance between their centers; e.g., the distance between two tangent unit disks is 2.

δ_{ij}: the distance between Ω_i and Ω_j.

δ: the minimum pairwise distance of the disks in $\mathcal{D}$, i.e.,
$\delta = \min_{i \neq j} \delta_{ij}$.

The algorithm Centers:



Let OPT denote an optimal solution and CEN denote the solution returned by Centers. Clearly: CEN $= \delta$.

Since the disks are disjoint, $r_i + r_j \leq \delta_{ij}, \; i \neq j$.

It follows that

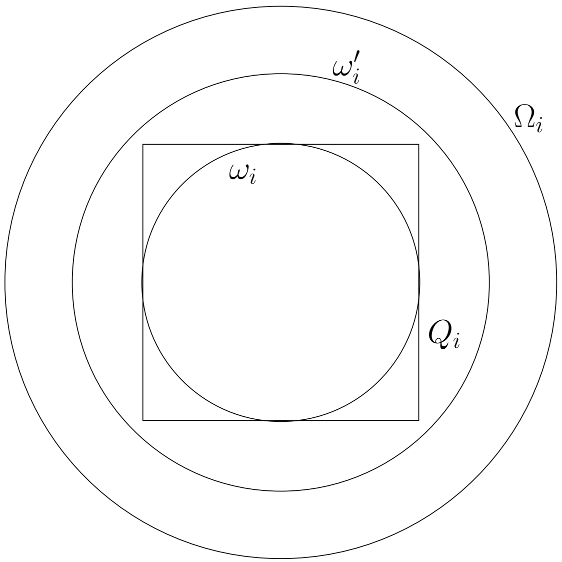
$$\text{OPT} \leq \min_{i \neq j} (\delta_{ij} + r_i + r_j) \leq 2 \min_{i \neq j} \delta_{ij} = 2\delta.$$

Consequently, the algorithm Centers achieves an approximation ratio of $\frac{\text{CEN}}{\text{OPT}} \geq \frac{\delta}{2\delta} = \frac{1}{2}$ for disjoint disks.

Next: An LP-based approx. alg. for disjoint disks

$\Omega_1, \ldots, \Omega_n$ be n pairwise disjoint disks of radii $r_1, \ldots, r_n$, and centers $o_1, \ldots, o_n$. We set two parameters $\lambda = 1/2$ and $\lambda' = 3/4$.

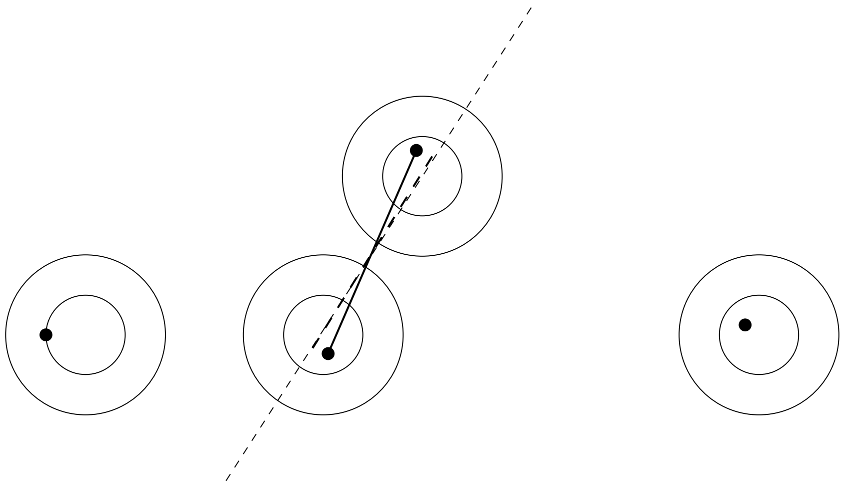
For $i = 1, \ldots, n$, let ω_i and ω_i' be two disks of radii $\lambda \cdot r_i$ and $\lambda' \cdot r_i$, respectively, that are concentric with Ω_i. Conveniently select disjoint convex polygons Q_i, $i = 1, \ldots, n$, such that $\omega_i \subset Q_i \subset \omega_i' \subset \Omega_i$, for each $i = 1, \ldots, n$. E.g., Q_i is an axis-aligned square of side length r_i concentric with ω_i.



Ideas for the algorithm

1. Suppose we restrict the feasible region of each point p_i from the given disk Ω_i to the smaller concentric disk ω_i of radius $\lambda \cdot r_i$. The centers of the original disks Ω_i are still in the feasible regions for each of the n points. So the $\frac{1}{2}$-approximation that we could easily achieve earlier, is still attainable. E.g., setting $\lambda = 0$ yields the algorithm CENTERS.

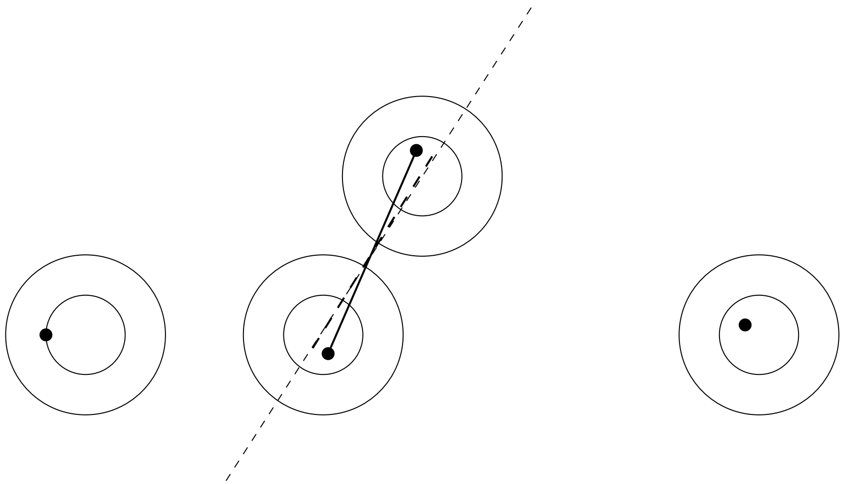
We then show the existence of a good approximation for the dispersion problem constrained to the smaller disks.



Ideas for the algorithm

2. If λ is small, then the distance between two points (in two smaller disks) can be well approximated by the projection of the connecting segment onto the line connecting the disk centers.

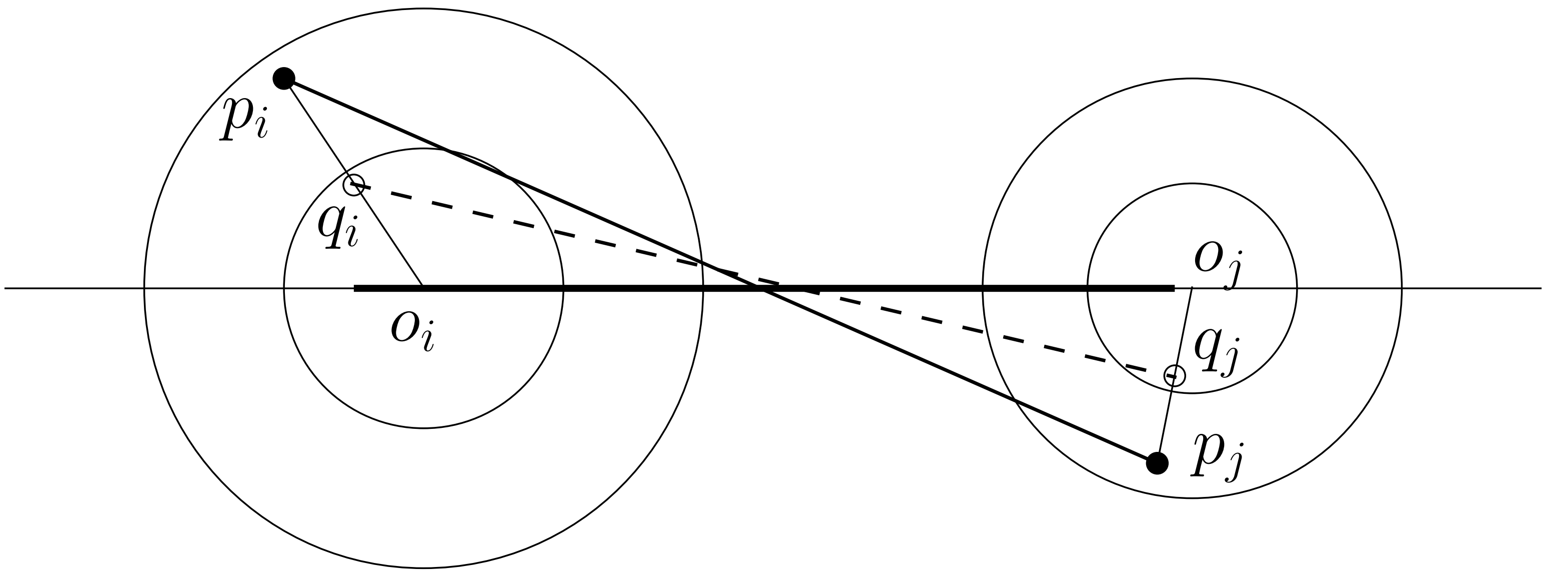
Enclose each smaller disk ω_i in a suitable convex polygon Q_i, where $\omega_i \subset Q_i \subset \omega'_i \subset \Omega_i$. The length of each such projection can be expressed as a linear combination of the coordinates of the two points. Use linear programming to maximize the smallest projection of an inter-point distance.



A key fact relating projections to distances

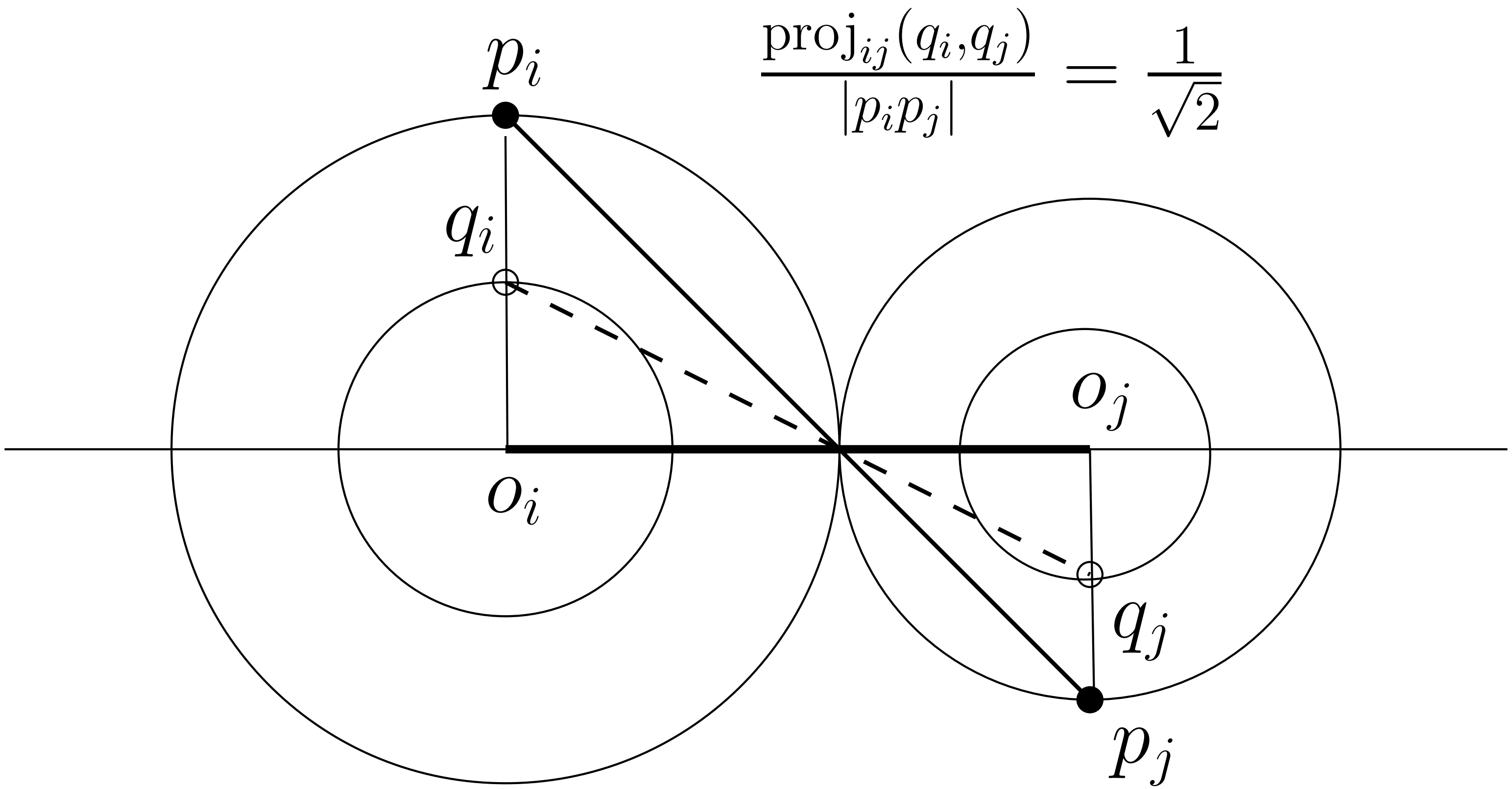
Lemma 1 *Let $\lambda = 1/2$. Consider two disjoint disks Ω_i and Ω_j at distance $\delta_{ij} = |o_i o_j|$. Let $p_i \in \Omega_i$ and $p_j \in \Omega_j$ be two points. Let $q_i \in \omega_i$ be the point on $o_i p_i$ at distance $\lambda |o_i p_i|$ from o_i. Similarly define $q_j \in \omega_j$ as the point on $o_j p_j$ at distance $\lambda |o_j p_j|$ from o_j. Then*

$$\frac{\mathrm{proj}_{ij}(q_i, q_j)}{|p_i p_j|} \geq \frac{1}{\sqrt{2}}.$$



A key fact relating projections to distances

This bound is tight:



Linear Program

A set $\{q_1, \ldots, q_n\}$ of n points is sought, where $q_i = (x_i, y_i) \in Q_i$, for $i = 1, \ldots, n$. LP2 maximizes the minimum pairwise projection on the line connecting the corresponding centers of the disks; that is, for each pair (i, j), the length of the projection of the segment connecting the two points q_i and q_j, on the line connecting the corresponding disk centers o_i and o_j.

We are lead to the following symbolic LP:

$$
\begin{aligned}
&\text{maximize} \quad z && \text{(LP2)}\\
&\text{subject to} \quad \begin{cases} q_i \in Q_i, & 1 \le i \le n \\ \mathrm{proj}_{ij}(q_i, q_j) \ge z, & 1 \le i < j \le n \end{cases}
\end{aligned}
$$

Writing the linear constraints

$o_i = (\xi_i, \nu_i)$, $p_i = (x_i, y_i)$, for $i = 1, \ldots, n$

For simplicity, assume $\xi_1 \leq \xi_2 \leq \ldots \leq \xi_n$. Consider a pair (i, j), where $i < j$. $\alpha_{ij} \in [-\pi/2, \pi/2)$ is the angle of the line determined by o_i and o_j. Assuming that $\xi_1 \leq \xi_2 \leq \ldots \leq \xi_n$, we have

$$\cos \alpha_{ij} = \frac{\xi_j - \xi_i}{|o_i o_j|}, \quad \sin \alpha_{ij} = \frac{\nu_j - \nu_i}{|o_i o_j|}.$$

Let $\overline{a_{ij}} = (\cos \alpha_{ij}, \sin \alpha_{ij})$, so that $|\overline{a_{ij}}| = 1$. Let $\overline{s_{ij}} = (x_j - x_i, y_j - y_i)$. Hence:

$\text{proj}_{\alpha_{ij}}(p_i, p_j) = \langle \overline{a_{ij}} \cdot \overline{s_{ij}} \rangle = (x_j - x_i) \cos \alpha_{ij} + (y_j - y_i) \sin \alpha_{ij}.$

- For each pair i, j, where $i < j$, generate the constraint:
$(x_j - x_i) \cos \alpha_{ij} + (y_j - y_i) \sin \alpha_{ij} \geq z$;

Solving the linear program

Lemma 2 *For any given $\varepsilon > 0$, a $(1 - \varepsilon)$-approximation of the solution of* LP2 *can be obtained in polynomial time.*

The constraints of the linear program LP2 involve irrational numbers, and hence it cannot be claimed that the original LP is solvable in polynomial time. However, it is enough to solve the LP up to some precision. For this, it is enough to approximate the numbers involved in the constraints up to some precision, which is polynomial in the error of the output.

Establishing the approximation ratio

Lemma 3 *For any given $\varepsilon > 0$, the approximation algorithm* **A2** *can achieve a ratio at least $\frac{1-\varepsilon}{\sqrt{2}}$ for pairwise disjoint disks.*

Reducing the number of constraints to $O(n)$

Recall that $\text{OPT} \leq 2\delta$. The LP solution, z^*, is bounded from above as (recall that $\lambda' = 3/4$)

$$z^* \leq \delta + \frac{3(r_i + r_j)}{4} \leq \frac{7\delta}{4},$$

where (i, j) are a closest pair of disks.

It follows that there is no need to write any constraints for pairs of disks at distance larger than 7δ. Indeed, if now (i, j) is such a pair, the distance between two points, one in Q_i and one in Q_j, is at least

$$\delta_{ij} - \frac{3(r_i + r_j)}{4} \geq \delta_{ij} - \frac{3\delta_{ij}}{4} = \frac{\delta_{ij}}{4} > \frac{7\delta}{4} > z^*.$$

An easy *packing argument* shows that the number of pairs of disks at distance at most 7δ is only $O(n)$.

Extension to any (fixed) dimension d

Differences:

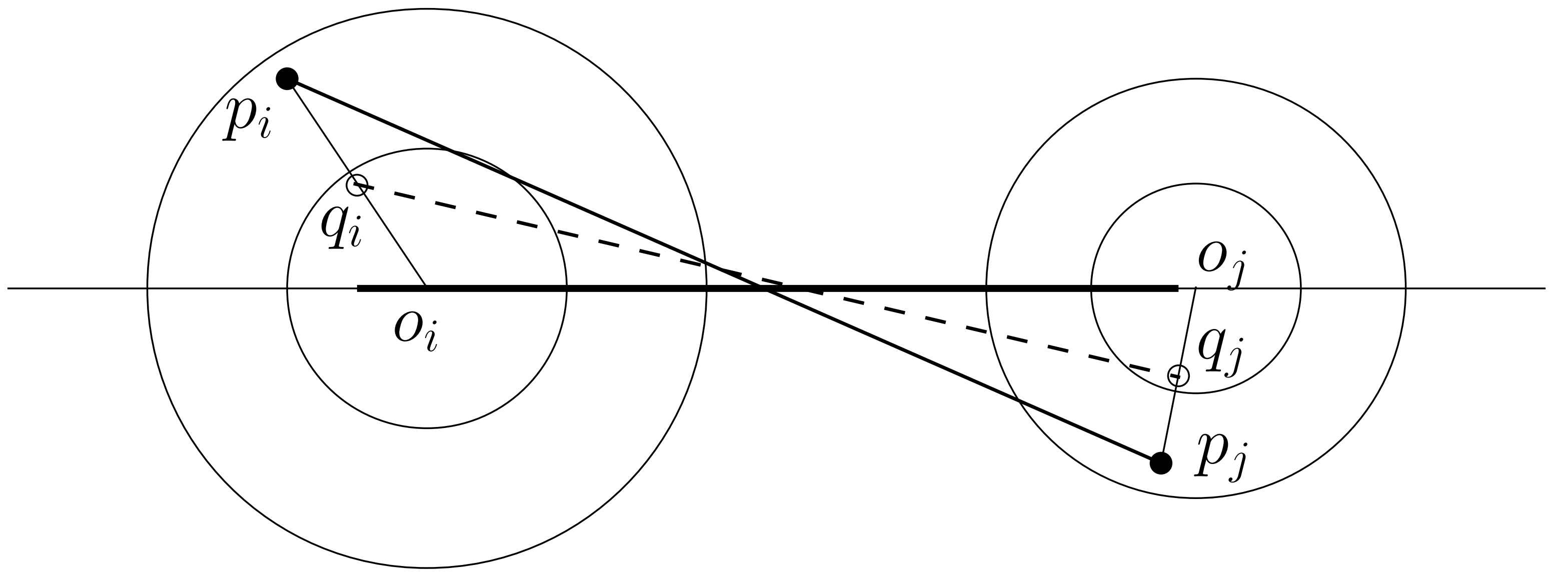
- The balls ω_i and ω_i' are two smaller balls of radii $\lambda \cdot r_i$ and $\lambda' \cdot r_i$ concentric with Ω_i, where $\lambda = 1/2$ and $\lambda' = 3/4$.
- Q_i is any suitable convex polytope in $\mathbb{R}^d$ such that $\omega_i \subset Q_i \subset \omega_i' \subset \Omega_i$.

A similar lemma (and proof) with the planar case:

Extension to any (fixed) dimension d

Lemma 4 *Let $\lambda = 1/2$. Consider two disjoint balls Ω_i and Ω_j at distance $\delta_{ij} = |o_i o_j|$. Let $p_i \in \Omega_i$ and $p_j \in \Omega_j$ be two points. Let $q_i \in \omega_i$ be the point on $o_i p_i$ at distance $\lambda |o_i p_i|$ from o_i. Similarly define $q_j \in \omega_j$ as the point on $o_j p_j$ at distance $\lambda |o_j p_j|$ from o_j. Then*

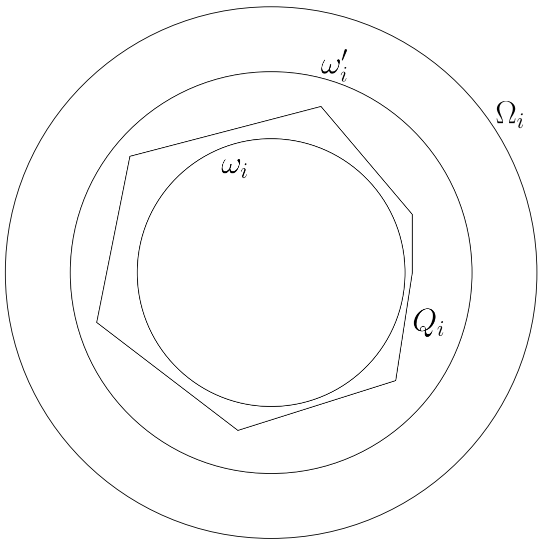
$$\frac{\mathrm{proj}_{ij}(q_i, q_j)}{|p_i p_j|} \geq \frac{1}{\sqrt{2}}.$$



Implementation in d-space

There exists a convex polytope $Q \subset \mathbb{R}^d$ and a function $f(d)$ such that $\omega \subset Q \subset \omega' \subset \Omega$, where Q has $f(d)$ facets, and ω, ω' and Ω are concentric balls of radii 1/2, 3/4 and 1, respectively. For $d \geq 5$, a concentric unit hyper-cube is *not* contained in the unit ball!

The polytope Q_i is a translate of $r_i Q$ placed at o_i, so that $\omega_i \subset Q_i \subset \omega'_i \subset \Omega_i$. Each symbolic constraint $q_i \in Q_i$ is implemented as $f(d)$ linear inequalities, one for each facet of Q_i.



Implementation in d-space

Each symbolic constraint $\text{proj}_{ij}(q_i, q_j) \geq z$ implements the dot products. Again (as in the planar case) there is no need to write any constraints for pairs of balls at distance larger than 7δ, and the number of such pairs is linear in n for fixed d. The total number of constraints is therefore $O(n)$.

The approximation ratio remains the same as for the planar case, namely $\frac{1-\varepsilon}{\sqrt{2}}$, for any given $\varepsilon > 0$, e.g., 0.707 for $\varepsilon = 10^{-4}$.

A hybrid algorithm for unit disks

For dispersion in (not necessarily disjoint) unit disks, Cabello presented a hybrid algorithm that applies two different algorithms PLACEMENT and CENTERS and then returns the better solution.

We present an improved hybrid algorithm that uses the algorithm PLACEMENT in combination with either the simple $O(n \log n)$-time algorithm or the LP-based algorithm.

A hybrid algorithm for unit disks

Write OPT $= 2x$ and $\delta = 2\mu$.

We can assume w.l.o.g. that $\delta \leq 2$, as otherwise the unit disks are disjoint. We also record the obvious inequalities:

$$\delta \leq \text{OPT} \leq \delta + 2 \leq 4 \quad \Leftrightarrow \quad \mu \leq x \leq 1 + \mu \leq 2. \tag{1}$$

The algorithm Placement, which runs in $O(n^2)$ time, achieves a ratio of

$$c_1(x) = \frac{-\sqrt{3} + \sqrt{3}x + \sqrt{3 + 2x - x^2}}{4x}, \quad \text{for } 1 \leq x \leq 2, \tag{2}$$

and a ratio of at least $\frac{1}{2}$ for $0 \leq x \leq 1$.

A hybrid algorithm for unit disks

We discuss the hybrid algorithm that runs PLACEMENT and **A2**.

Obviously the n disks of radius $\mu \leq 1$ concentric with the n input unit disks are pairwise-disjoint.

The hybrid algorithm runs PLACEMENT on the given unit disks and **A2** on the disks of radius μ and then returns the better solution.

Clearly the solution is valid, and it remains to analyze the approximation ratio.

For any given $\varepsilon > 0$, it achieves a ratio at least $\frac{(1-\varepsilon)\sqrt{2}}{1+\sqrt{9-2\sqrt{6}}} = (1-\varepsilon) \cdot 0.46749\ldots$.

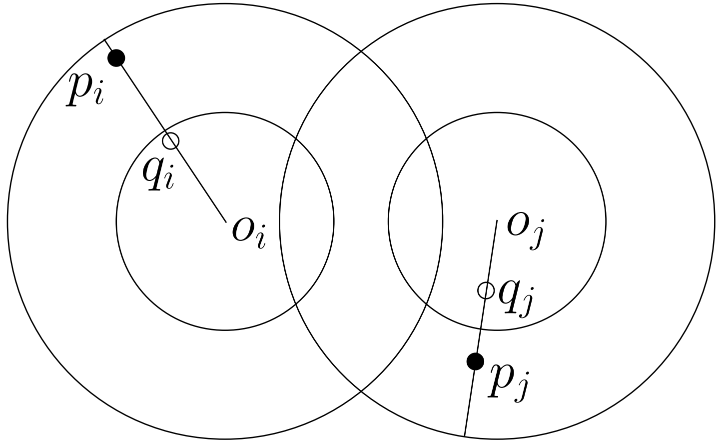
For $\varepsilon = 10^{-4}$, we get a 0.4674-approximation.

A hybrid algorithm for unit disks; a key lemma

Relate the optimal solution OPT for the unit disks to the optimal solution OPT_μ for the smaller disjoint disks of radius μ:

Lemma 5 *For a problem instance with $\mu \in [0,1]$, we have $\mathrm{OPT}_\mu \geq \mathrm{OPT} - 2(1-\mu)$.*

Proof of key lemma:$\mathrm{OPT}_\mu \geq \mathrm{OPT} - 2(1-\mu)$



Consider an optimal solution given by n points $p_1, \ldots, p_n$, where $p_i \in \Omega_i$, such that $|p_i p_j| \geq \mathrm{OPT}$, for all $i \neq j$, and $|p_i p_j| = \mathrm{OPT}$ for at least one pair (i, j). For each i, let $q_i \in \Omega_i$ be the point on $o_i p_i$ at distance $\mu |o_i p_i|$ from o_i. Obviously, the set $\{q_i \ : \ i = 1, \ldots, n\}$ is a valid solution for dispersion in the disks of radius μ concentric with the unit disks $\Omega_1, \ldots, \Omega_n$. Moreover, since $|p_i q_i| \leq 1 - \mu$, for any i, by the triangle inequality we have $|q_i q_j| \geq \mathrm{OPT} - 2(1-\mu)$, for any $i \neq j$. Consequently, $\mathrm{OPT}_\mu \geq \mathrm{OPT} - 2(1-\mu)$.

Summary of current best approximation ratios for the three variants of dispersion in disks

Recall our two algorithms **A1** and **A2** and the two algorithms Placement and Centers by Cabello.

- Arbitrary (not necessarily unit or disjoint): $3/8 = 0.375$ by Placement.
- Unit (not necessarily disjoint): 0.4674 by **A2** plus Placement, which improves 0.4465 by Centers plus Placement.
- Disjoint (not necessarily unit): 0.707 by **A2**, which improves 0.5 by Centers.

Conclusion

- Other applications of using projections for approximating distances?
- The dispersion problem in other domains instead of disks?