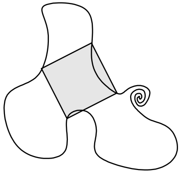
On the square peg problem

Benjamin Matschke



Disputationsvortrag

Plan

The square peg problem

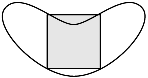
Definition

A **Jordan curve** γ is a continuous simple closed curve in the plane,

$$\gamma : S^1 \hookrightarrow \mathbb{R}^2.$$

Definition

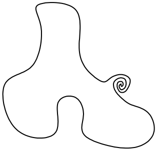
A polygon P is **inscribed in γ** if all vertices of P belong to γ.



The square peg problem

Problem (Otto Toeplitz 1911)

Does every Jordan curve inscribe a square?



- solved for “smooth enough” curves (e.g. C^1),
- open otherwise ($\rightarrow$ why? Because no working approximating argument is known)

The square peg problem

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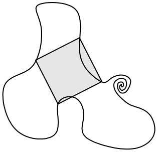

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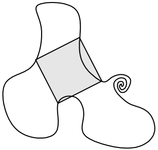

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Known proofs

Many proofs are known for various smoothness conditions:

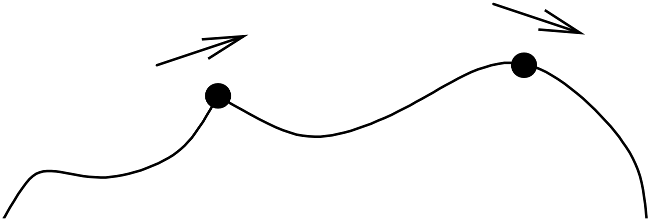
- Toeplitz 1911 (convex curves)?
- Emch 1913, 1915 ("smooth enough" convex curves)
- Schnirel'man 1944 ("a bit less" than C^2)
- Jerrard 1961 (analytic curves)
- Fenn 1970 (convex curves)
- **Stromquist** 1989 ("locally monotone curves")
- Pak 2008 (piecewise linear curves)
- Vrećica–Živaljević 2008 (Stromquist's curves)
- ...

The problem is either due to

- Toeplitz or
- Emch (Kemptner suggested to him the problem).

Stromquist's criterion:

Locally monotone curves:



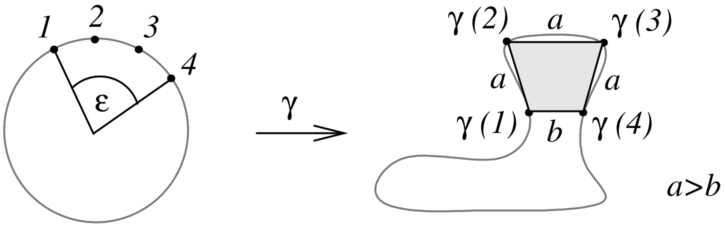
Theorem (Stromquist 2011)

Any locally monotone Jordan curve inscribes a square.

New criterion:

Definition

A **special trapezoid of size ε**...



Theorem (M 2009)

Let $\varepsilon \in (0, 2\pi)$. Any Jordan curve without (or with generically an even number of) special trapezoids of size ε inscribes a square.

New criterion

- This strictly generalizes Stromquist's theorem.
- "Having no inscribed special trapezoid of size ε" is an *open* condition for γ! (w.r.t. compact-open topology; being locally monotone is not an open condition)
- The theorem holds for curves in arbitrary metric spaces.
- Proof based on obstruction theory, first used in this context by Vrećica–Živaljević.

Plan

General proof method for smooth curves

Generically, the number of inscribed squares is odd.

Schnirel'man's proof for smooth curves

Let $\gamma : S^1 \hookrightarrow \mathbb{R}^2$ be smooth (C^∞).
Construct a test map

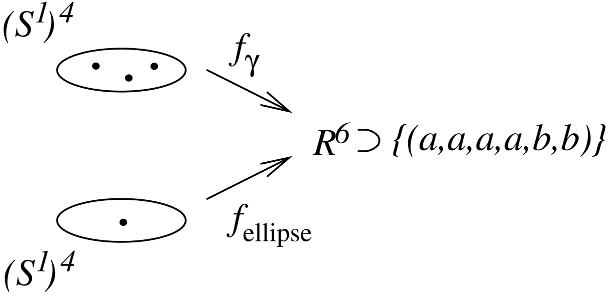
$$f_\gamma : (S^1)^4 \longrightarrow_G \mathbb{R}^4 \times \mathbb{R}^2$$

that measures the four edges and two diagonals of the parametrized quadrilateral. Then

$$Q_\gamma := f_\gamma^{-1}(\{(a, a, a, a, b, b) \in \mathbb{R}^6\})$$

is the set of inscribed squares.
Now deform the given γ to an ellipse.

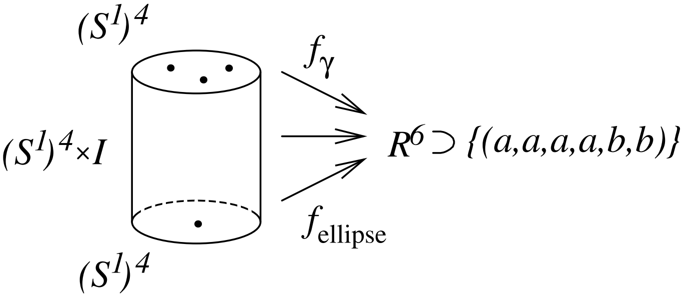
Schnirel'man's proof for smooth curves



- $[Q] \in \mathcal{N}_0((S^1)^4/G) = \mathbb{Z}_2$,
- $[Q] = 1 \Rightarrow Q \neq \emptyset$.

□

Schnirel'man's proof for smooth curves



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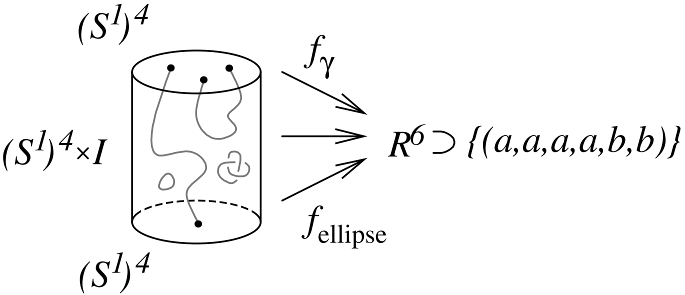

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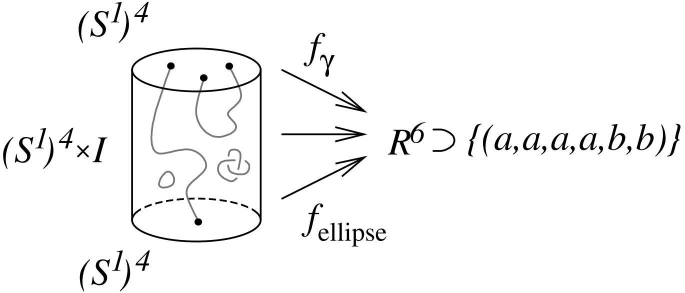

- $[Q] \in \mathcal{N}_0((S^1)^4/G) = \mathbb{Z}_2$,
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$\square$

Plan

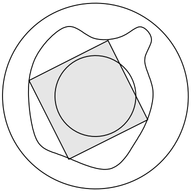
A different, open class of curves

Theorem (M 2011)

Let $\gamma : S^1 \to A$ represent a generator of $\pi_1(A)$, where

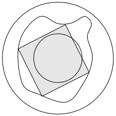
$$A := \{x \in \mathbb{R}^2 \mid 1 \leq ||x|| \leq 1 + \sqrt{2}\}.$$

Then γ inscribes a square with edge length at least $\sqrt{2}$.



- γ needs to be only continuous, not even injective.
- This is the first known *open* class of curves $S^1 \to \mathbb{R}^2$ that inscribe squares.

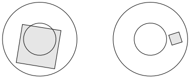
A different, open class of curves



Proof idea:

Let S be the set of squares with all vertices in A. Then,

$$S = \{\text{big squares}\} \uplus \{\text{small squares}\}.$$



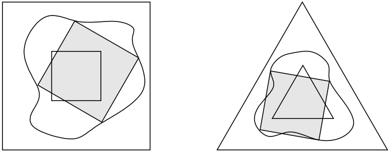
Now,

- an ellipse in A inscribes *one* big square, and
- bordisms of squares stay in their component.

□

A different, open class of curves

Similar theorems for other shapes:



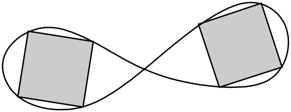
Question

Can this approach be made more general in order to solve the square peg problem completely?

Plan

Immersed curves

Example



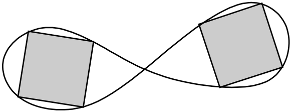
(2 =even)

Conjecture (Cantarella 2008)

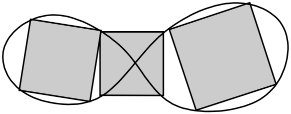
Modulo 2, the number of inscribed squares of "generic" curves is the following ambient isotopy invariant of the curve: ...

Immersed curves

Counter-example:

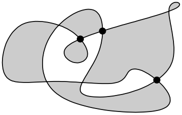


and



Immersed curves

Let $\gamma : S^1 \looparrowright \mathbb{R}^2$ be "generic".
Chequerboard coloring associated to γ:

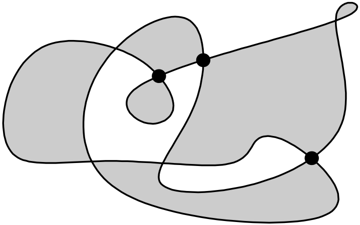


Crossings are called **fat** if the black angles are $> 90°$.
Dots mark the fat crossings.

Theorem (M 2011)

$$\#\{\text{inscribed squares}\} = \#\{\text{fat crossings}\} + \#\{\text{black components}\} \mod 2.$$

Proof



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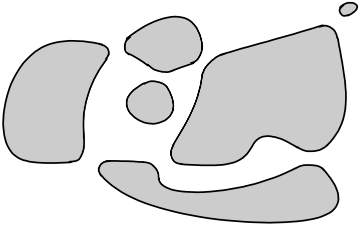

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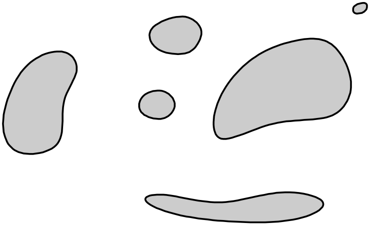

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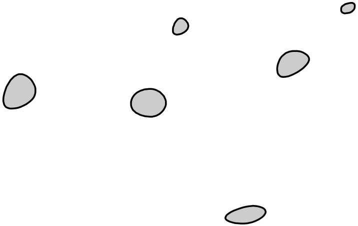

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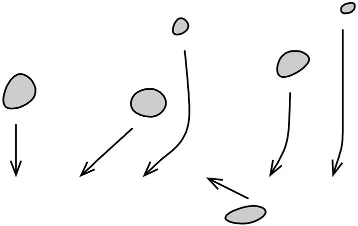

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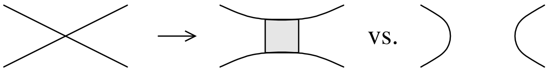

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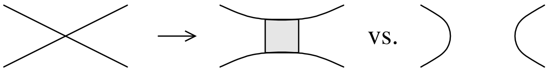

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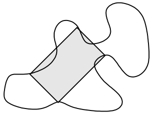
Plan

What about rectangles

Problem

Does every smooth Jordan curve inscribe a rectangle of a given aspect ratio $r : 1$?

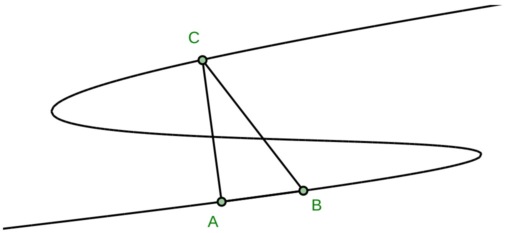


- This is open! (for $r \neq 1$).

We have

- Partial results for $r = \sqrt{3}$.
- There is a mod-2 formula for immersed curves.

Plan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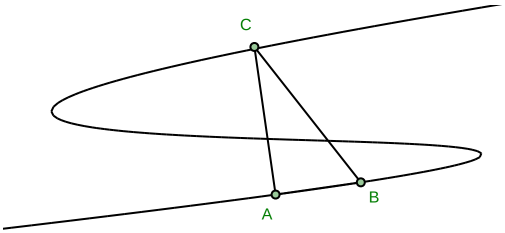
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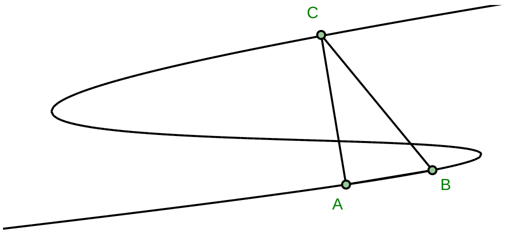
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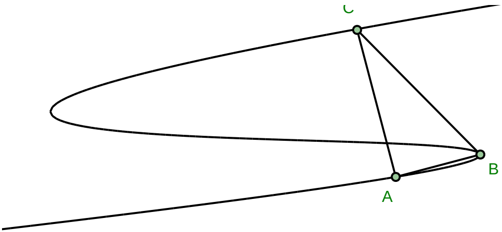
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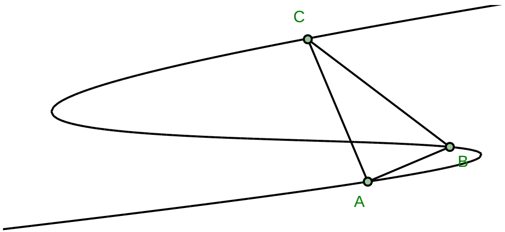
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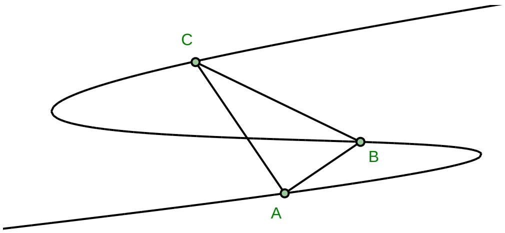
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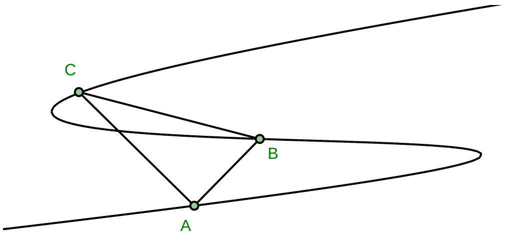
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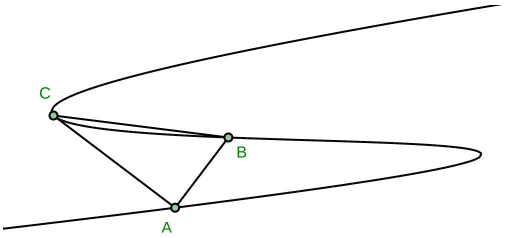
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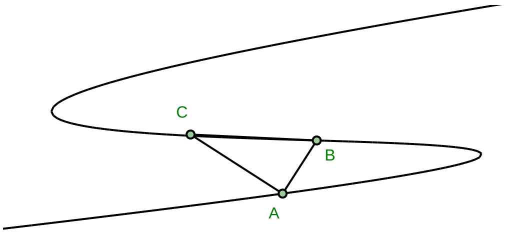
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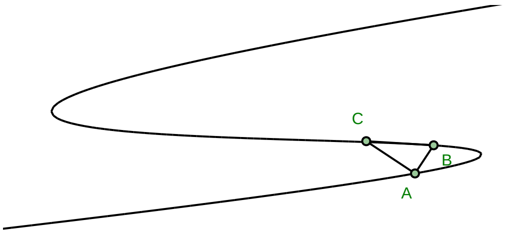
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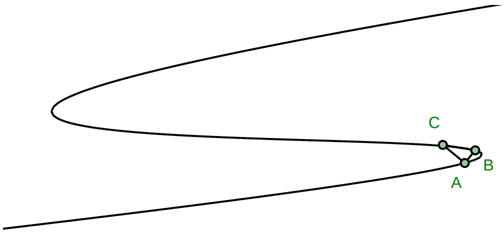
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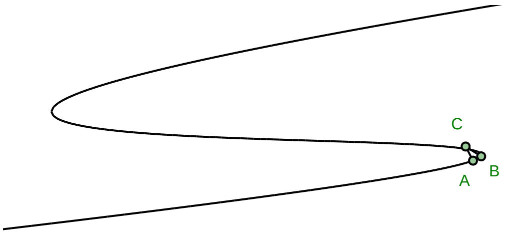
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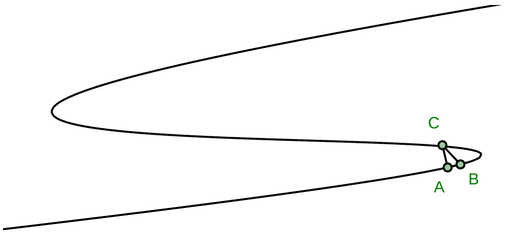
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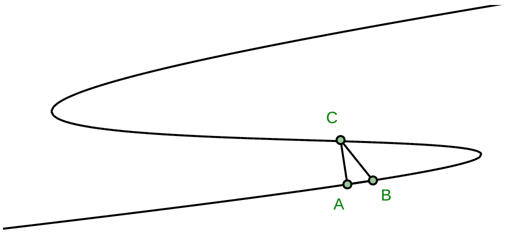
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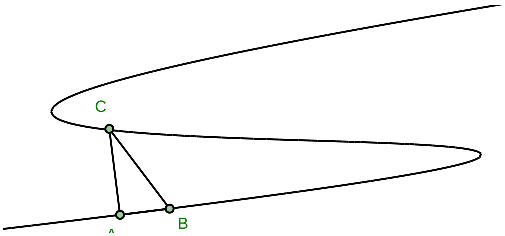
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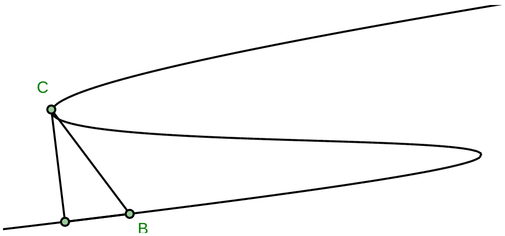
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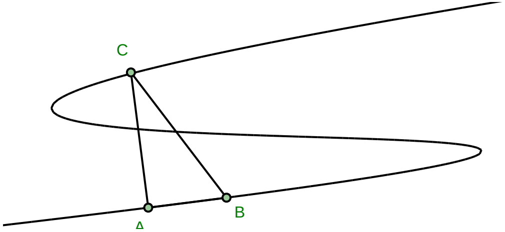
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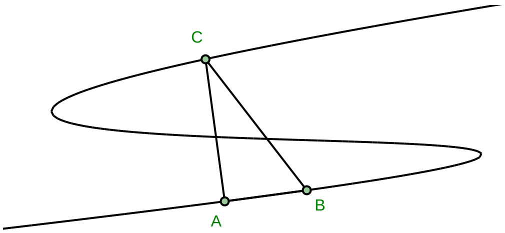
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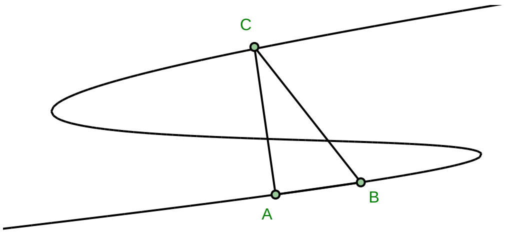
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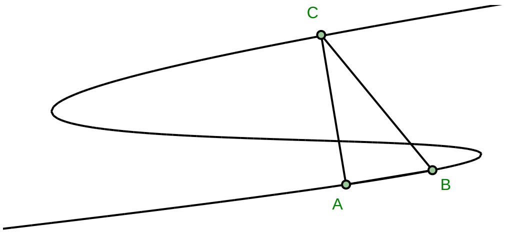
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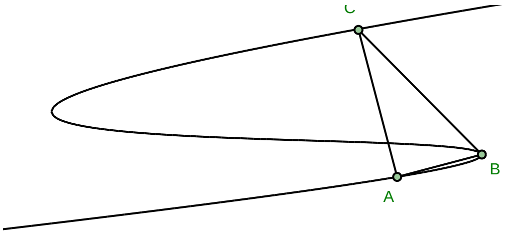
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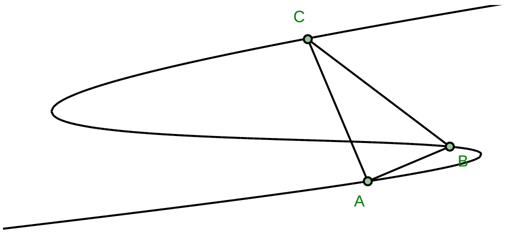
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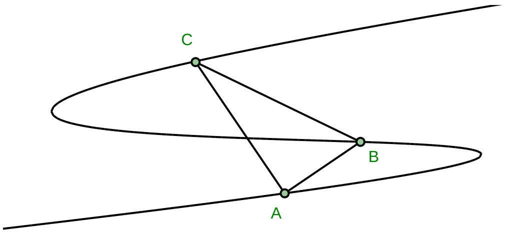
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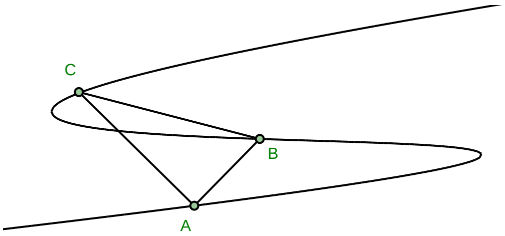
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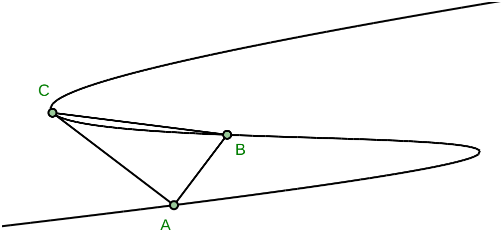
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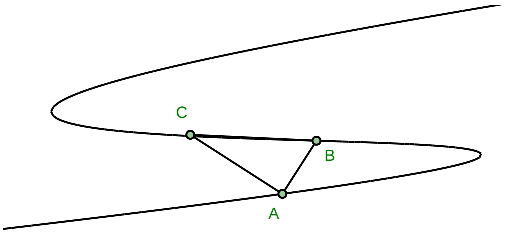
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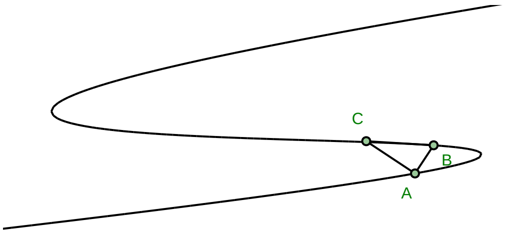
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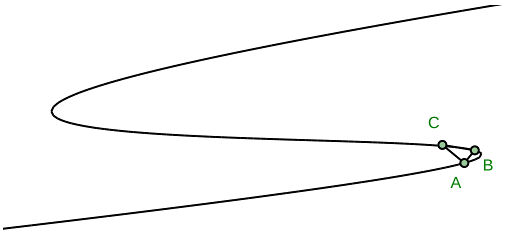
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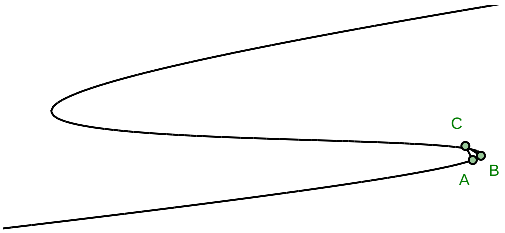
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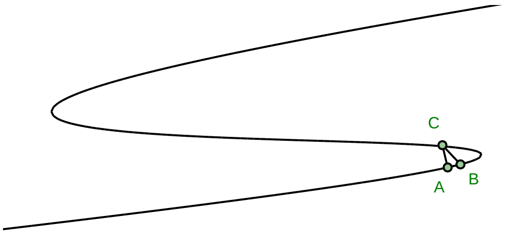
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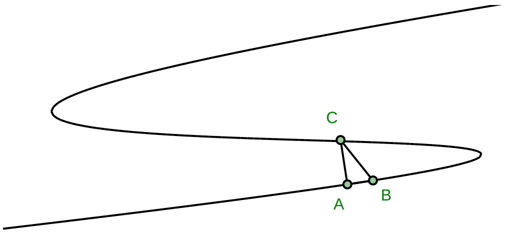
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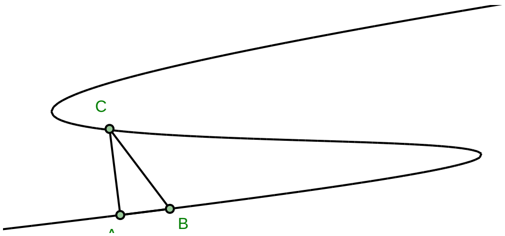
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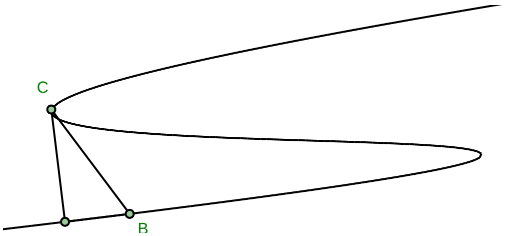
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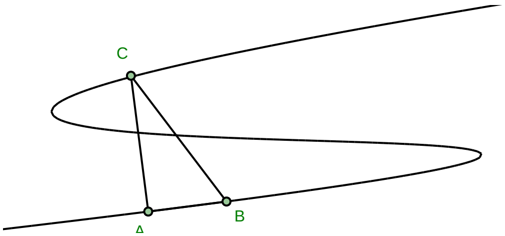
C
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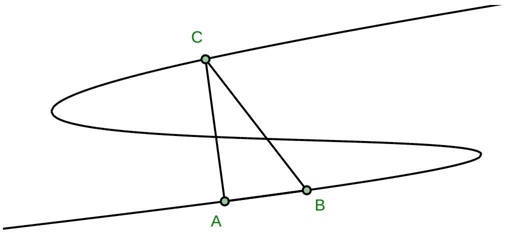
C
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Thank you!

Discussion