Dense favourite-distance digraphs

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Workshop on Discrete Geometry

Fields Institute

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Overview

Overview

The unit distance problem

- Definitions and the low-dimensional cases
- High dimensions: the Lenz construction
- Optimality of the Lenz construction
- Stability of the Lenz construction

The favourite distance problem

- Definitions and the two-dimensional case
- The 3-dimensional case
- Stability and optimality in high dimensions

Further work

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High dimensions: the Lenz construction

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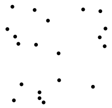
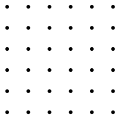
Further work

The unit distance problem

Introduced by Erdős (1946, 1960)

$S \subset \mathbb{R}^d$, $|S| = n$.





Definitions

$$E(S) := \{\boldsymbol{xy} : \boldsymbol{x}, \boldsymbol{y} \in S, |\boldsymbol{xy}| = 1\}$$

$$u(S) := |E(S)|$$

$$u_d(n) := \max\{u(S) : S \subset \mathbb{R}^d, |S| = n\}$$

Problem

Determine $u_d(n)$.

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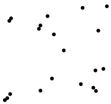

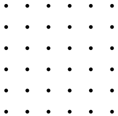

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Theorem (Erdős 1946)

$$e^{1+\frac{c}{\log\log n}} < u_2(n) < cn^{3/2}$$

Lower bound: lattice Upper bound: forbidden $K_{2,3}$

Not dense: $u_2(n) = o(n^2)$

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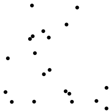

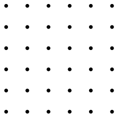

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Theorem (Spencer, Szemerédi, Trotter 1984)

$$u_2(n) < cn^{4/3}$$

Simplest proof due to Székely (1997) — crossing number lemma

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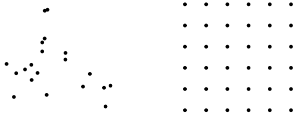

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Theorem (Erdős 1960)

$$cn^{4/3} \log\log n < u_3(n) < cn^{5/3}$$

Lower bound: lattice Upper bound: forbidden $K_{3,3}$

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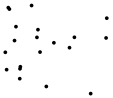

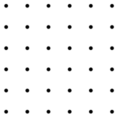

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Problem

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Theorem (Kaplan, Matoušek, Safernová, Sharir; Zahl 2011)

$$u_3(n) < cn^{3/2}$$

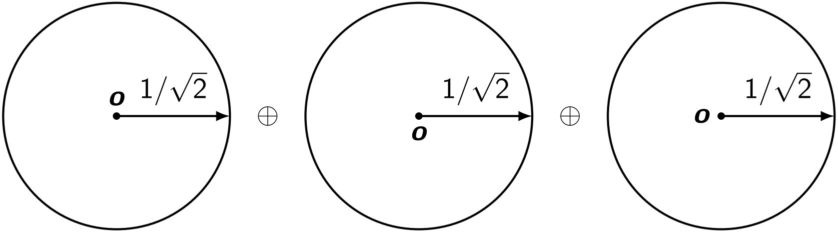
Uses polynomial ham-sandwich theorem

Also not dense: $u_3(n) = o(n^2)$

Overview

The Lenz construction

$\mathbb{R}^6 = \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2$



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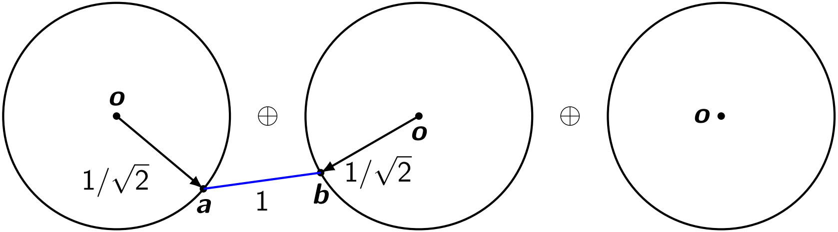

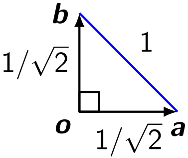

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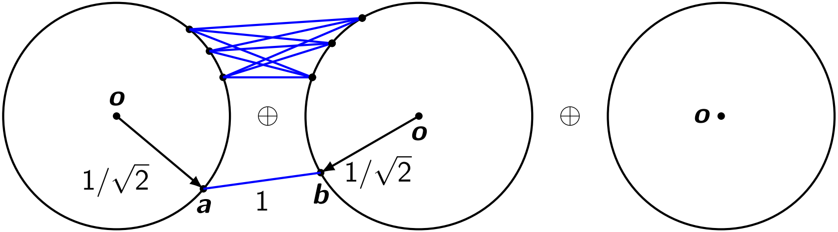

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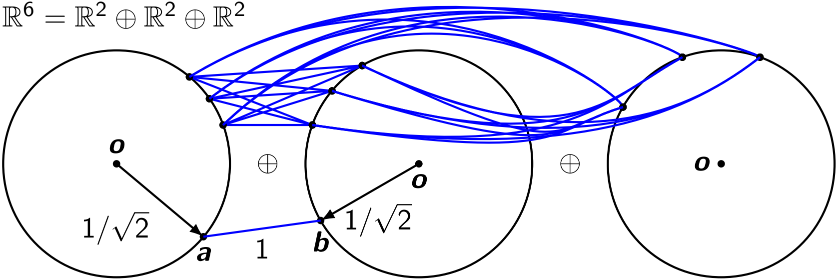

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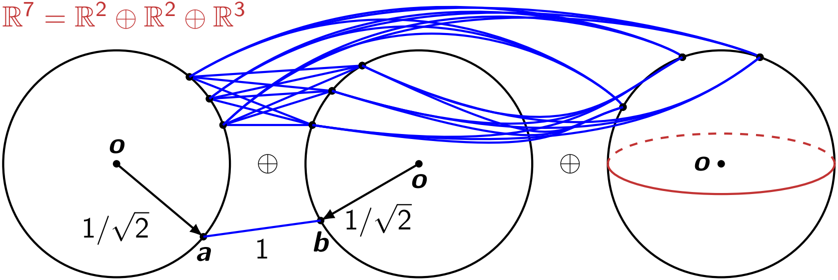

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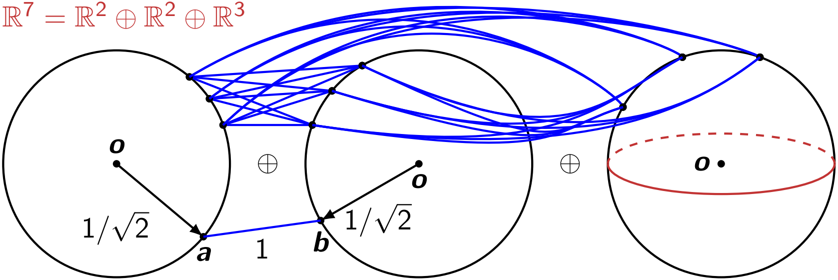

Definition

A Lenz configuration is a set that is congruent with a finite subset of these circles (resp. the circles and the sphere)

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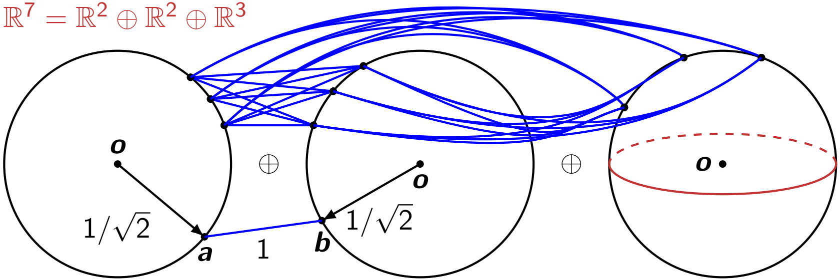

Definition

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There exist complete $\lfloor d/2 \rfloor$-partite unit distance graphs:
Distribute $n/\lfloor d/2 \rfloor$ points on each circle:

$$u_d(n) \geq \qquad \left(\frac{n}{\lfloor d/2 \rfloor}\right)^2$$

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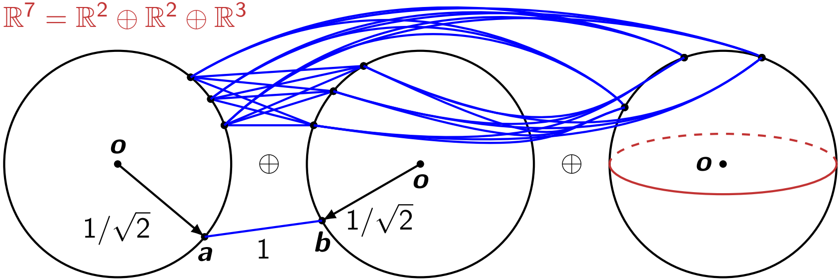

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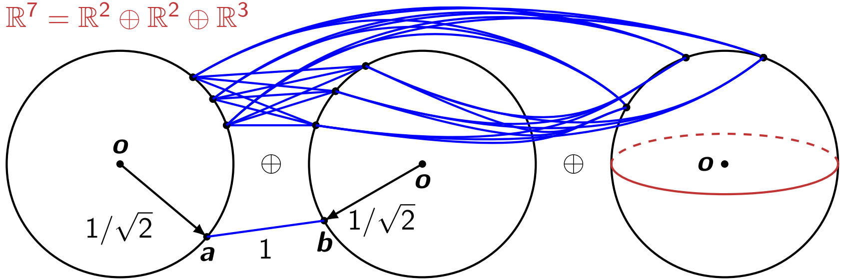

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$$u_d(n) \geq \binom{\lfloor d/2 \rfloor}{2} \left(\frac{n}{\lfloor d/2 \rfloor} \right)^2 = \frac{1}{2} \left(1 - \frac{1}{\lfloor d/2 \rfloor} \right) n^2$$

Dense: $u_d(n) = \Theta(n^2)$

Overview

The Lenz construction is asymptotically optimal

Theorem (Erdős 1960)

For any $d \geq 4$, $\quad u_d(n) = \frac{1}{2}\left(1 - \frac{1}{\lfloor d/2 \rfloor}\right) n^2 + o(n^2)$

Forbidden $K_{\underbrace{3,3,\ldots,3}_{\lfloor d/2 \rfloor + 1 \text{ times}}}$

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Theorem (Erdős and Pach 1990)

For any odd $d \geq 5$, $u_d(n) = \frac{1}{2}\left(1 - \frac{1}{\lfloor d/2 \rfloor}\right) n^2 + \Theta(n^{4/3})$

There exist configurations of n points on a 2-sphere of radius $1/\sqrt{2}$ with $\Omega(n^{4/3})$ unit distance pairs.

Extremal structure

Definition

A finite set $S \subset \mathbb{R}^d$ is an *extremal configuration* if $u(S) = u_d(n)$, where $n = |S|$.

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Theorem (Brass 1997, Van Wamelen 1999)

For $n \geq 5$, $u_4(n) = \begin{cases} \lfloor n^2/4 \rfloor + n & \textit{if } n \textit{ is divisible by } 8 \textit{ or } 10, \\ \lfloor n^2/4 \rfloor + n - 1 & \textit{otherwise.} \end{cases}$

Furthermore, any sufficiently large extremal configuration is a Lenz configuration.

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Furthermore, any sufficiently large extremal configuration is a Lenz configuration.

Theorem (S 2009)

Let $d \geq 5$. *All extremal configurations of size* n *in* $\mathbb{R}^d$ *are Lenz configurations for* $n > n_0(d)$ *sufficiently large.*

Proof is based on a stability result.

Overview

Stability for unit distances

Theorem (S 2009)

Let $d \geq 4$ and $S \subset \mathbb{R}^d$ with $|S| = n$. If

$$u(S) > \frac{1}{2}\left(1 - \frac{1}{p}\right)n^2 - o(n^2)$$

then S is a Lenz configuration with the exception of $o(n)$ points.

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Proof.

Euclidean geometry + extremal graph theory:

- Erdős-Stone theorem
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Similar statements for the graph of diameters.

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Similar statements for the graph of diameters.

This was the motivation to look at favourite distances.

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Further work

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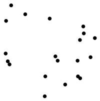
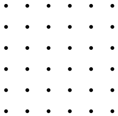
The favourite distance problem

Introduced by Avis, Erdős, Pach (1988)

$S \subset \mathbb{R}^d, \quad |S| = n, \qquad r\colon S \to (0,\infty)$





Definitions

$$\vec{E}_r(S) := \{\boldsymbol{xy} : \boldsymbol{x}, \boldsymbol{y} \in S, |\boldsymbol{xy}| = r(x)\}$$

$$e_r(S) = \left|\vec{E}(S)\right|$$

$$f_d(n) := \max\{e_r(S) : S \subset \mathbb{R}^d, |S| = n, \ \ r\colon S \to (0,\infty)\}$$

Problem

Determine $f_d(n)$.

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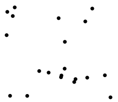

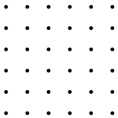

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Theorem (Avis, Erdős, Pach 1988)

$$f_2(n) < \sqrt{2}\, n^{3/2} + n/2.$$

Forbidden $\vec{K}_{2,3}$

Not dense: $u_2(n) = o(n^2)$

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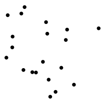

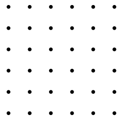

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Theorem (Aronov, Sharir 2002)

$$f_2(n) < cn^{15/11}$$

Bound actually holds for point-circle incidences

Conjecture (Brass, Moser, Pach 2005)

$f_2(n) = \Theta(n^{4/3})$

Sharp if true: $f_2(n) = \Omega(n^{4/3})$ is known.

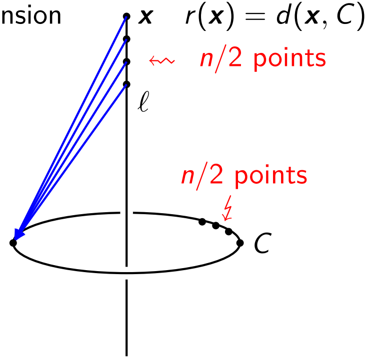
Overview

The favourite distance problem in $\mathbb{R}^3$

Theorem (Avis, Erdős, Pach 1988)

$$\frac{n^2}{4} + \frac{n}{2} \leq f_3(n) \leq \frac{n^2}{4} + cn^{2-\varepsilon_0}.$$

Lower bound: Suspension



Upper bound: Forbidden $\vec{K}_{3,3,3}$: extremal digraph theory

Dense unlike unit distances: $f_3(n) = \Theta(n^2)$

The favourite distance problem in $\mathbb{R}^3$

Theorem (S)

For sufficiently large n,

$$\frac{n^2}{4} + \frac{5n}{2} - 6 \le f_3(n) < \frac{n^2}{4} + \frac{5n}{2} + 6.$$

Uses a structural result for extremal configurations.

Definitions

A spindle configuration is a finite subset of a circle and its axis of symmetry. A pair (S, r) is an extremal configuration if $e_r(S) = f_d(n)$, where $n = |S|$.

Theorem (S)

If (S, r) is an extremal configuration on a sufficiently large set $S \subset \mathbb{R}^3$, then S is a spindle configuration up to two points, and for all points $\boldsymbol{x} \in S$ on the line of the spindle, $r(\boldsymbol{x})$ equals the distance from $\boldsymbol{x}$ to the circle of the spindle.

Proof is again based on a stability result.

I conjecture that the two points are unnecessary.

Stability

Theorem (S)

If a set S of n points in $\mathbb{R}^3$ and $r\colon S \to (0,\infty)$ are such that $e_r(S) > n^2/4 - o(n^2)$, then S is a suspension up to $o(n)$ points.

Proof follows in the steps of Avis, Erdős, Pach (1988).

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Proof follows in the steps of Avis, Erdős, Pach (1988).

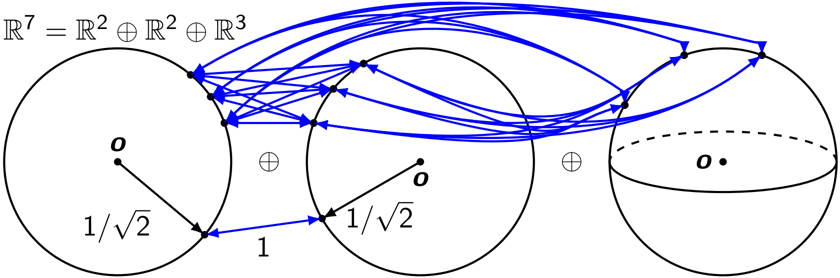
There is a similar statement for the furthest neighbour digraph of a set of points in $\mathbb{R}^3$: fix

$$r(\boldsymbol{x}) := \max\{|\boldsymbol{x}\boldsymbol{y}| : \boldsymbol{y} \in S\}.$$

The furthest neighbour digraph has been considered by Csizmadia (1996), who determined extremal configurations for sufficiently large n.

Overview

The Lenz construction again



Set r identically 1 to obtain

$$f_d(n) \geq 2u_d(n) \geq \left(1 - \frac{1}{\lfloor d/2 \rfloor}\right) n^2.$$

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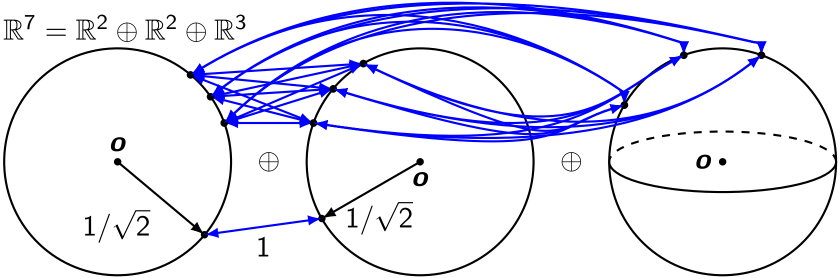

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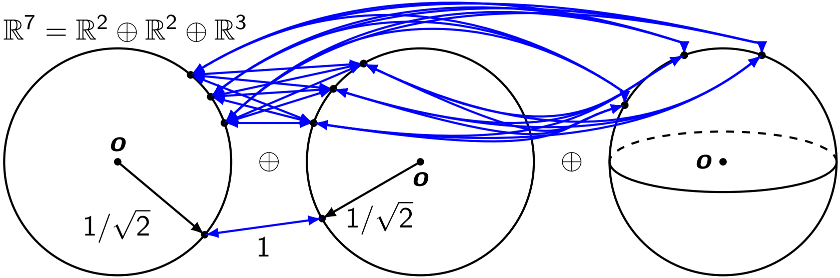
Theorem (Avis, Erdős, Pach 1988)

$$f_d(n) = \left(1 - \frac{2}{d}\right) n^2 + O(n^{2-\varepsilon_0}) \qquad \textit{for even } d \geq 4.$$

The Lenz construction again



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Theorem (Erdős, Pach 1990)

$$f_d(n) = \left(1 - \frac{1}{\lfloor d/2 \rfloor}\right) n^2 + o(n^2) \qquad \textit{for odd } d \geq 5.$$

Extremal digraph theory

If unit distance graphs are dense, then favourite distance digraphs are equally dense

Observation (S)

For any even $d \geq 4$, $f_d(n) = \left(1 - \frac{2}{d}\right) n^2 + \Theta(n)$.

For any odd $d \geq 5$, $f_d(n) = \left(1 - \frac{1}{\lfloor d/2 \rfloor}\right) n^2 + \Theta(n^{4/3})$

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Actually follows easily from previously mentioned theorems:

Theorem (Erdős 1967)

For any even $d \geq 4$, $u_d(n) = \frac{1}{2}\left(1 - \frac{2}{d}\right) n^2 + \Theta(n)$

Theorem (Erdős and Pach 1990)

For any odd $d \geq 5$, $u_d(n) = \frac{1}{2}\left(1 - \frac{1}{\lfloor d/2 \rfloor}\right) n^2 + \Theta(n^{4/3})$

Stability and extremality in dimensions $d \geq 4$

Theorem (S)

If a set S of n points in $\mathbb{R}^d$, $d \geq 4$, and $r\colon S \to (0, \infty)$ are such that $e_r(S) > (1 - \frac{1}{\lfloor d/2 \rfloor})n^2 - o(n^2)$, then S is a Lenz configuration on which r is contant, up to $o(n)$ points.

Very easy if $d \geq 6$. $d = 5$ much harder.

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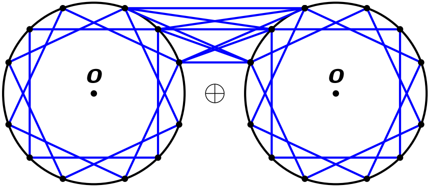
Theorem (S)

Let $d \geq 4$ and let n be sufficiently large. If a set S of n points in $\mathbb{R}^d$, and $r\colon S \to (0,\infty)$ satisfy $e_r(S) = f_d(n)$, then r is constant and S is a Lenz configuration, except when $d = 4$ and a second extremal configuration exists when $n \equiv 1 \pmod 8$:



$r(\boldsymbol{x}) = 1$ if $\boldsymbol{x} \neq \boldsymbol{o}$

$r(\boldsymbol{o}) = 1/\sqrt{2}$

Overview

Further work

- The equally dense argument generalises to metric spaces.
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- Stability and extremal results for normed planes that are not strictly convex: density is $1/2$.
- Normed spaces? Difficult case is when density is $1/2$: there exist arbitrarily large $K_{s,s}$ but not $K_{s,s,s}$ as unit distance graphs.

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THANK YOU FOR YOUR ATTENTION.